MATHEMATICAL TRIPOS **Part IV**

Sunday 32 May 2003 9 to 12

PAPER 1

TOPICS IN PURE MATHEMATICS

Attempt **THREE** *questions*

There are **four** *questions in total*

The questions carry equal weight

Do not start reading the questions printed on the subsequent pages until instructed to do so by the invigilator.

[TURN OVER

1 State and prove the Riemann Hypothesis.

2 (a) Construct from first principles the group cohomology functors H^i, and show that they are unique up to canonical equivalence.
[*You may assume that for a group G the category of G-modules has enough injectives.*]

(b) For an elliptic curve E over $\mathbf{Q}$ the *Tate-Shafarevich group* $Ш(E/\mathbf{Q})$ is defined by the exact sequence

$$0 \to Ш(E/\mathbf{Q}) \to H^1(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}), E(\bar{\mathbf{Q}})) \to \prod_v H^1(\mathrm{Gal}(\bar{\mathbf{Q}}_v/\mathbf{Q}_v), E(\bar{\mathbf{Q}}_v)),$$

where the last map is the product of the natural maps

$$H^1(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}), E(\bar{\mathbf{Q}})) \to H^1(\mathrm{Gal}(\bar{\mathbf{Q}}_v/\mathbf{Q}_v), E(\bar{\mathbf{Q}}_v)),$$

where v runs over all the places of $\mathbf{Q}$, including the infinite one.
Is $Ш(E/\mathbf{Q})$ always finite? Give a proof or counter-example as appropriate.

3 Explain what is meant by the *Ramsey number* $R(n)$, and prove its existence for all n. Compute $R(124)$.

4 Let G be a finite group. Does there necessarily exist a finite Galois extension L of the field K with $\mathrm{Gal}(L/K) \simeq G$, when K is
(i) $\mathbf{C}(t)$,
(ii) $\mathbf{F}_p(t)$,
(iii) $\mathbf{Q}$?
Justify your answers.
[*If you use the classification of finite simple groups you should prove it.*]