Random walks on graphs, and the Kirchhoff and Wiener Index

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Which problem is harder?

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Clearly, $CT/n < cc < CT$.
How much larger than cc can CT be?

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Theorem (G & P. Winkler '11)

The cover time of a graph on L edges is at most $2L^2$. The cover time for Brownian motion on graph of total length L is at most $2L^2$.

Cover Cost and the Wiener Index

Theorem (G & S. Wagner '12+)

For every tree we have
$\sum_{y \in V(T)} \left(H_{ry} + d(r,y) \right) = 2W(T) := \sum_{x,y \in V(T)} d(x,y).$

in other words:

$$\boxed{CC(r) + D(r) = 2W(T)}$$

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Corollary

The extremal rooted trees on n vertices for $CC(r)$ are the path rooted at a midpoint (maximum) and the star rooted at a leaf (minimum).

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For every tree T, the quantity

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Proof based on $H_{xy} + H_{yx} = T_{x \leftrightarrow y} = 2mr(x,y)$ (by the commute time formula of Chandra et. al.)

Vertex orderings - Trees

Theorem (classic)

the vertices of any graph can be put in a linear preorder so that for random walk on the graph vertices appearing earlier in the preorder are "easier to reach but more difficult to get out of" and the other way round.

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Theorem (G & Wagner '12+)

For every tree T, and every pair of vertices $x, y \in V(T)$, TFAE:

1. $D(x) \leq D(y)$;
2. $D_\pi(x) \leq D_\pi(y)$;
3. $H_{yx} \leq H_{xy}$;
4. $RC_\pi(x) \leq RC_\pi(y)$;
5. $RC(x) \leq RC(y)$;
6. $CC(x) \geq CC(y)$.

Vertex orderings - General graphs

The Kirchhoff index (or *quasi-Wiener index*) is defined as

$$K(G) := \sum_{\{x,y\} \subseteq V(G)} r(x,y) = \frac{1}{2} \sum_{x \in V(G)} \sum_{y \in V(G)} r(x,y).$$

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Theorem (G & Wagner '12+)

For every graph G, and every vertex $x \in V(G)$, we have

$$CC(x) = mR(x) - \frac{n}{2} R_\pi(x) + K_\pi(G),$$
$$RC(x) = mR(x) + \frac{n}{2} R_\pi(x) - K_\pi(G),$$
$$RC_\pi(x) = 2mR_\pi(x) - K_{\pi^2}(G), \text{ and}$$
$$CC_\pi(x) = K_{\pi^2}(G).$$

Eigenvalue formulas

The fact that $CC_\pi(x)$ is constant was already known; moreover, it can be expressed in terms of the eigenvalues of the matrix M of transition probabilities of G as $CC_\pi(x) = 2m \sum_{k=2}^{n} \frac{1}{1-\lambda_k}$

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Interestingly, a similar formula applies to the Kirchhoff index:

$$K(G) = n \sum_{\lambda \neq 0} \frac{1}{\lambda},$$

the sum being over all nonzero *Laplacian* eigenvalues of G

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- What are the extremal rooted n-vertex graphs for $CC(r)$?

- In a large graph, how can you change $R_\pi(x)$ a lot by attaching few new edges to x?