Analyticity of the percolation density θ in all dimensions

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Abstract

We prove that for Bernoulli bond percolation on $\mathbb{Z}^d$, $d \geq 2$ the percolation density is an analytic function of the parameter in the supercritical interval $(p_c, 1]$. This answers a question of Kesten from 1981.

1 Introduction

Perhaps the first occurrence of questions of smoothness in percolation theory dates back to the work of Sykes & Essam [29]. Trying to compute the value of p_c for bond percolation on the square lattice $\mathbb{Z}^2$, Sykes & Essam obtained that the free energy (aka. mean number of clusters per vertex) $\kappa(p) := \mathbb{E}_p(|C_o|^{-1})$, where C_o denotes the cluster of the origin, satisfies the functional equation $\kappa(p) = \kappa(1-p) + \phi(p)$ for some polynomial $\phi(p)$. Under the assumption of smoothness of κ for every value of the parameter p other than p_c, at which it is conjectured that κ has a singularity, they obtained that $p_c = 1/2$ due to the symmetry of the functional equation around $1/2$. Their work generated considerable interest, and a lot of the early work in percolation was focused on the smoothness of functions like κ and $\chi := \mathbb{E}_p(|C_o|)$ that describe the macroscopic behaviour of its clusters. Kunz & Souillard [24] proved that κ is analytic for small enough p. Grimmett [12] proved that κ is C^∞ for $p \neq p_c$ in the case $d = 2$. A breakthrough was made by Kesten [21], who proved that κ and χ are analytic on $[0, p_c)$ for all $d \geq 2$. (Despite all the efforts, the argument of Sykes & Essam has never been made rigorous, and all proofs of the fact that $p_c = 1/2$ when $d = 2$ use different methods, see e.g. [4, 20].)

Except for the special case of κ on $\mathbb{Z}^2$ (and other planar lattices), smoothness results are harder to obtain in the supercritical interval $(p_c, 1]$, partly because the cluster size distribution $P_n := \mathbb{P}_p(|C(o)| = n)$ has an exponential tail below p_c [1, 26] but not above p_c [2]. Still, it is known that κ and $\theta := \mathbb{P}(|C_o| = \infty)$ are infinitely differentiable for $p \in (p_c, 1]$ (see [8] or [15, §8.7] and references therein). It was a well-known open question, dating back to [21] at least, and appearing in several textbooks ([23, Problem 6],[13, 15]), whether θ is analytic for $p \in (p_c, 1]$ for percolation on the hypercubic lattice $\mathbb{Z}^d, d \geq 2$. This paper answers this question in the affirmative.

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Part of the interest for this question comes form Griffiths' [11] discovery of models, constructed by applying the Ising model on 2-dimensional percolation clusters, in which the free energy is infinitely differentiable but not analytic. This phenomenon is since called a *Griffiths singularity*, see [31] for an overview and further references.

The study of the analytical properties of the free energy is a common theme in several models of Statistical Mechanics. Perhaps the most famous such example is Onsager's exact calculation of the free energy of the square-lattice Ising model [27]. A corollary of this calculation is the computation of the critical temperature, as well as the analyticity of the free energy for all temperatures other than the critical one. See also [19] for an alternative proof of the latter result. The analytical properties of the free energy have also been studied for the q-Potts model, which generalizes the Ising model. For this model, the analyticity of the free energy has been proved for $d = 2$ and all supercritical temperatures when q is large enough [32].

Before our result, partial progress on the analyticity of the percolation density had been made by Braga et.al. [5, 6], who showed that θ is analytic for p close enough to 1. We recently settled the 2-dimensional case [9] by introducing a notion of *interfaces* that has already found further applications [17]. Shortly after our paper [9] was released, Hermon and Hutchcroft [18] proved that θ is analytic above p_c for every non-amenable transitive graph, by establishing that the cluster size distribution P_n has an exponential tail in the whole supercritical interval.

Our proof of the analyticity of $\theta(p)$ on the supercritical interval involves expressing the function as an infinite sum of polynomials $f_n(p)$, and then extending p to the complex plane. To show that this sum converges to an analytic function, we need suitable upper bounds for $|f_n(z)|$ inside regions of the complex plane. These bounds can be obtained once f_n decays to 0 fast enough. Possible candidates for f_n are the probabilities P_n, since one can write $\theta(p) = 1 - \sum_n P_n(p)$. However, as P_n decays slower than exponentially for $p > p_c$ [15, 24], the bounds we obtain for $|P_n(z)|$ do not provide the desired convergence. Instead of working with the whole of C_o, an alternative approach is to work with the 'perimeter' of its boundary. As we observed in [9], in the planar case, the suitable notion of perimeter turns out to be the *interface* of C_o. An interface consists of a set of closed edges that we call the *boundary* of the interface, and separate C_o from infinity, and a set of open edges that is part of C_o and incident to the boundary (Figure 1). With this definition we obtain that $1-\theta(p)$ coincides with the probability $\mathbb{P}_p$(at least one interface occurs), which can be expanded as a sum over all interfaces, i.e. over all subgraphs of the lattice that could potentially coincide with the interface of C_o. Since several interfaces might occur simultaneously, we have to apply the inclusion-exclusion principle. Thus we obtain

$$1 - \theta(p) = \sum (-1)^{c(M)+1} \mathbb{P}_p(\text{M occurs}),$$

where the sum ranges over all finite collections of edge-disjoint interfaces, called *multi-interfaces*, and $c(M)$ denotes the number of interfaces of the collection.

For any plausible multi-interface M, the probability $\mathbb{P}_p(\text{M occurs})$ is just $P_M(p) := p^{|M|}(1-p)^{|\partial M|}$ by the definitions, where $|M|$ and $|\partial M|$ denote the number of edges of the multi-interface and its boundary, respectively (in Figure 1, these are depicted in dark solid lines and dashed lines, respectively). We

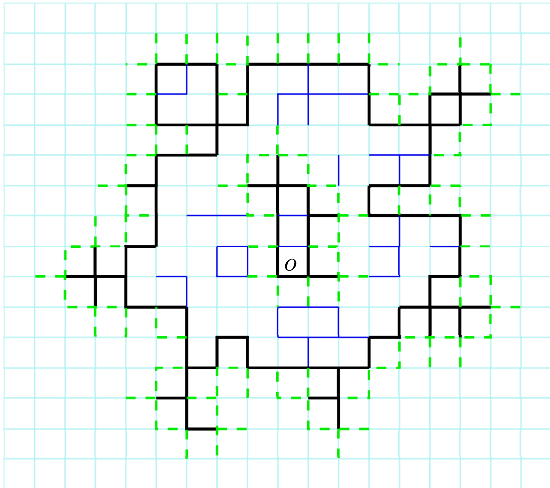

Figure 1: An example of two nested interfaces, depicted in dark solid lines. Dashed lines depict the boundary of the interface.

can extend this polynomial expression to $\mathbb{C}$ hoping to obtain strong enough upper bounds for $|P_M(z)|$. In the special case where M comprises a single interface, these bounds are obtained by combining the well-known coupling between supercritical bond percolation on $\mathbb{Z}^2$ and subcritical bond percolation on its dual with the exponential tail of P_n on the subcritical interval. In the general case, the bounds are obtained by some combinatorial arguments and the BK inequality. See [9] for details.

Our notion of interfaces can be generalised to higher dimensions in such a way that a unique interface is associated to any cluster. A slight modification of the above method still yields the analyticity of θ for the values of p close to 1, but not in the whole supercritical interval. The main obstacle is that for values of p in the interval $(p_c, 1-p_c)$, the distribution of the size of the interface of C_o has only a stretched exponential tail, which follows from the work of Kesten and Zhang [22]. (As we observed in [10], this behaviour holds for $p = 1-p_c$ as well.)

In the same paper [22], Kesten and Zhang introduced some variants of the standard boundary of C_o that are obtained by dividing the lattice $\mathbb{Z}^d$ into large boxes, and proved that these variants satisfy the desired exponential tail on the whole supercritical interval.[1] It is natural to try to apply our method to those variants, however, their occurrence does not prevent the origin from being connected to infinity. Instead, we expand these variants into larger objects that we call *separating components*. In Section 3 (Lemma 3.4) we prove that whenever a separating component S occurs, we can find inside S and its boundary $\partial_{\boxtimes} S$ an edge cut $\partial^b \mathcal{S}_o$ separating the origin from infinity. Conversely, some separating component occurs whenever C_o is finite (Lemma 3.2). Thus we can express θ

[1]The threshold $p_c(H^d)$ in Kesten's and Zhang's original formulation was proved later to coincide with $p_c(\mathbb{Z}^d)$ by Grimmett and Marstrand [14].

in terms of the occurrence of separating components (see (5) in Section 3.3). In contrast to the behaviour of the boundary of C_o which has only a stretched exponential tail on the interval $(p_c, 1-p_c]$, $\partial^b \mathcal{S}_o$ has an exponential tail in the whole supercritical interval. We plug this exponential decay into a general tool from [9] (Theorem 2.1), which rests on an application of the Weierstrass M-test to polynomials of the form $p^m(1-p)^n$, to obtain the analyticity of θ above p_c in Section 3.4. In Section 4 we use similar arguments to prove the analyticity of the k-point function τ and its truncation τ^f, as well as of χ and κ.

Typically, $\partial^b \mathcal{S}_o$ has size of smaller magnitude than the boundary of C_o, and it is obtained from the latter by 'smoothening' some of its parts with 'fractal' structure. As a corollary, we re-obtain, in Section 3.5, a result of Pete [28] about the exponential decay of the probability that C_o is finite but sends a lot of closed edges to the infinite component.

2 Preliminaries

2.1 Graph theory

Consider an infinite connected graph $G = (V, E)$. Given a finite subgraph H of G, we define its *internal boundary* ∂H to be the set of vertices of H that are incident with an infinite component of $G \setminus H$. We define the *vertex boundary* $\partial^V H$ of H as the set of vertices in $V \setminus V(H)$ that have a neighbour in H. The *edge boundary* $\partial^E H$ is the set of edges in $E \setminus E(H)$ that are incident to H.

Consider now a vertex x of G. We say that a set S of edges of G is an *edge cut* of x if x belongs to a finite component of $G - S$. We say that S is a *minimal edge cut* of x if it is minimal with respect to inclusion. For a finite connected subgraph H of G, its minimal edge cut is the set of edges with one endvertex in H and one in an infinite component of $G \setminus H$.

The *diameter* $diam(H)$ of H is defined as $\max_{x,y \in V(H)}\{d_G(x, y)\}$ where $d_G(x, y)$ denotes the graph-theoretic distance between x and y.

2.2 Percolation

We recall some standard definitions of percolation theory in order to fix our notation. For more details the reader can consult e.g. [15, 25].

Consider the hypercubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, E(\mathbb{Z}^d))$, the vertices of which are the points in $\mathbb{R}^d$ with integer coordinates, and two vertices are connected with an edge when they have distance 1. We let $\Omega := \{0, 1\}^{E(\mathbb{Z}^d)}$ be the set of *percolation configurations* on $\mathbb{L}^d$. We say that an edge e is *closed* (respectively, *open*) in a percolation configuration $\omega \in \Omega$, if $\omega(e) = 0$ (resp. $\omega(e) = 1$).

By Bernoulli, bond *percolation* on $\mathbb{L}^d$ with parameter $p \in [0, 1]$ we mean the random subgraph of $\mathbb{L}_d$ obtained by keeping each edge with probability p and deleting it with probability $1-p$, with these decisions being independent of each other. The corresponding probability measure on the configurations of open and closed edges is denoted by $\mathbb{P}_p$. We also denote by $\mathbb{E}_p$ the expectation with respect to $\mathbb{P}_p$.

The *percolation threshold* $p_c(\mathbb{L}^d)$ is defined by

$$p_c(\mathbb{L}^d) := \sup\{p \mid \mathbb{P}_p(|C_o| = \infty) = 0\},$$

where o denotes the origin $(0, \ldots, 0) \in \mathbb{Z}^d$, and its *cluster* C_o is the component of o in the subgraph of $\mathbb{L}^d$ spanned by the open edges. We will write $C_o(\omega)$ when we want to emphasize the dependence of the cluster on the (random) percolation instance ω.

2.3 Analyticity

In order to prove that θ and the other functions describing the macroscopic behaviour of our model are analytic we will utilize some results proved in [9]. The first result provides sufficient conditions for analyticity. The second result will be used when estimates for the analytic extensions of those functions are needed.

We say that an event E has *complexity* k, if it is a disjoint union of a family of events $(F_n)_{n\in\mathbb{N}}$ where each F_n is measurable with respect to a set of edges of G of cardinality at most k.

Theorem 2.1 ([9])**.** *Let $I \subset [0,1]$ be an interval and $f(p) : I \to \mathbb{R}$ a function that can be expressed as a sum*

$$f(p) = \sum_{n\in\mathbb{N}} \sum_{i\in L_n} a_i \mathbb{P}_p(E_{n,i})$$

where $a_n \in \mathbb{R}$, L_n is a finite index set, and each $E_{n,i}$ is an event measurable with respect to $\mathbb{P}_p$ (in particular, the above sum converges absolutely for every $p \in I$). Suppose that

(i) each $E_{n,i}$ has complexity of order $\Theta(n)$, and

(ii) for each open subinterval $J \subset I$ there is a constant $0 < c_J < 1$ such that $\sum_{i\in L_n} a_i \mathbb{P}_p(E_{n,i}) = O({c_J}^n)$.

Then $f(p)$ is analytic in I.

In the following lemma $D(p, \delta)$ denotes the open disk of radius δ centred at p.

Lemma 2.2 ([9])**.** *For every finite subgraph S of G and every $o \in V(G)$, the function $P(p) := \mathbb{P}_p(C_o = S)$ admits an entire extension $P(z), z \in \mathbb{C}$, such that for every $0 < \delta < 1$, every $0 \le p < 1$ with $p + \delta < 1$ and every $z \in D(p, \delta)$, we have*

$$|P(z)| \le c^{|\partial^E S|} P(p + \delta),$$

where $c = c_{\delta,p} := \frac{1-p+\delta}{1-p-\delta}$. Moreover, $|P(z)| \le c_\delta^{|\partial^E S|} P(1 - \delta)$ for every $z \in D(1, \delta)$, where $c_\delta := \frac{1+\delta}{1-\delta}$.

3 Analyticity of θ

In this section we will prove that θ is analytic on the supercritical interval for every $d \ge 3$. (The case $d = 2$ was handled in [9].)

Theorem 3.1. *For Bernoulli bond percolation on $\mathbb{L}^d, d \ge 3$, the percolation density $\theta(p)$ is analytic on $(p_c, 1]$.*

3.1 Setting up the renormalisation

We start by introducing some necessary definitions. Consider a positive integer N. For every vertex x of $\mathbb{Z}^d$, we let $B(x) = B(x, N)$ denote the box $\{y \in \mathbb{Z}^d : \|y - Nx\|_\infty \leq 3N/4\}$. With a slight abuse, we will use the same notation $B(x)$ to also denote the corresponding subset of $\mathbb{R}^d$, namely $\{y \in \mathbb{R}^d : \|y - Nx\|_\infty \leq 3N/4\}$.

The collection of all these boxes can be thought of as the vertex set of graph canonically isomorphic to $\mathbb{Z}^d$. We will denote this graph by $N\mathbb{L}^d$. Whenever we talk about percolation (clusters) from now on, we will be referring to percolation, with a fixed parameter $p > p_c$, on $\mathbb{L}^d$ and not on $N\mathbb{L}^d$; we will never percolate the latter.

For any percolation cluster C, we denote by $C(N)$ the set of boxes B such that the subgraph of C induced by its vertices lying in B has a component of diameter at least $N/5$. The boxes with this property will be called *C-substantial*. Notice that $C(N)$ is a connected subgraph of $N\mathbb{L}^d$. The internal boundary of $C(N)$ is denoted by $\partial C(N)$ following the terminology of Section 2.1. Notice that $\partial C(N)$ is not necessarily connected. For technical reasons, we would like it to be, and therefore we modify our lattice by adding the diagonals: we introduce a new graph $N\mathbb{L}^d_\boxtimes$, the vertices of which are the boxes $B(x), x \in \mathbb{Z}^d$, and we connect two boxes with an edge of $N\mathbb{L}^d_\boxtimes$ whenever they have non-empty intersection. When $N = 1$, the vertex set of $\mathbb{L}^d_\boxtimes$ is simply $\mathbb{Z}^d$. It is not too hard to show (see [30, Theorem 5.1]) that

If C is finite then $\partial C(N)$ is a connected subgraph of $N\mathbb{L}^d_\boxtimes$. (1)

Given two diagonally opposite neighbours x, y of $\mathbb{L}^d$, we will write $B(x, y)$ for the intersection $B(x) \cap B(y)$. A percolation cluster C is a *crossing cluster* for some box $B(x)$ or $B(x, y)$, if C contains a vertex from each of the $(d-1)$-dimensional faces of that box. We say that a box $B(x)$ is *good* in a percolation configuration ω if it has a crossing cluster C with the property that the intersection of C with each of the boxes $B(x, y)$ contains a crossing cluster (of $B(x, y)$), and every other cluster of $B(x)$ has diameter less than $N/5$. A box that is not good will be called *bad*. It is known [15, Theorem 7.61] that, for every $p > p_c$, the probability of having a crossing cluster and no other cluster of diameter greater than $N/5$ converges to 1 as $N \to \infty$. Combining this with a union bound we easily deduce that

for every $p > p_c$, the probability of any box being good converges to 1 as $N \to \infty$. (2)

We will say that a set of boxes is bad if all its boxes are bad.

Our definition of good boxes is slightly different than the standard one in that it asks for all boxes $B(x, y)$ to contain a crossing cluster. The reason for imposing this additional property is because now

every $N\mathbb{L}^d_\boxtimes$-component B of good boxes contains a unique percolation cluster C such that some box of B is C-substantial (and in fact all boxes of B are C-substantial). (3)

This follows easily once we notice that this holds for pairs of neighbouring boxes.

Observe that the boxes in $\partial C(N)$ are never good. Indeed, if some box $B \in \partial C(N)$ is good, then C connects all the $(d-1)$-dimensional faces of B, hence all $N\mathbb{L}^d$-neighbouring boxes of B contain a connected subgraph of C of diameter at least $N/5$, and so they lie in $C(N)$. This contradicts the fact that B belongs to $\partial C(N)$.

Having introduced the above definitions, our aim now is to find a suitable expression for $1-\theta$ in terms of good and bad boxes surrounding o.

With the above definitions we have that, conditioning on the event that C_o is finite and has diameter at least $N/5$, there is a non-empty $N\mathbb{L}^d_\boxtimes$-connected subgraph of bad boxes that separates o from infinity, namely $T := \partial C_o(N)$. However, the event $\{|C_o| < \infty\}$ is not necessarily measurable with respect to the configuration inside T. In other words, we cannot express $1-\theta$ in terms of just the configuration inside T, and instead we have to explore the configuration inside the finite components surrounded by T. To this end, we will expand $\partial C_o(N)$ into a larger object.

3.2 Separating components

A *separating component* is an $N\mathbb{L}^d_\boxtimes$-connected set S of boxes, such that o lies either inside S or in a finite component of $N\mathbb{L}^d_\boxtimes \setminus S$. We will write $\partial_\boxtimes S$ for its vertex boundary —defined in Section 2.1— when viewed as a subgraph of $N\mathbb{L}^d_\boxtimes$. We say that S *occurs* in a configuration ω if all the following hold:

(i) all boxes in S are bad;

(ii) all boxes in $\partial_\boxtimes S$ are good, and

(iii) there is a configuration ω' which coincides with ω in $S \cup \partial_\boxtimes S$, such that $C_o(\omega')$ is finite, and S contains $\partial C_o(\omega')(N)$.

We will say that ω' is a *witness* for the occurrence of S if (i)–(iii) all hold. Note that (iii) implies that

$$\partial^V C_o(\omega') \text{ (and } C_o(\omega')\text{) does not meet the infinite component of } \mathbb{L}^d \setminus S. \quad (4)$$

3.3 Expressing $\theta(p)$ in terms of the probability of the occurrence of a separating component

In this section we show that C_o is finite exactly when some separating component occurs, unless $diam(C_o) < N/5$ which is a case that is easy to deal with. This will allow us to express $\theta(p)$ in terms of the probability of the occurrence of a separating component (see (5)). In the following section we will expand the latter as a sum (with inclusion-exclusion) over all possible separating components. The summands of this sum are well-behaved polynomials, that will allow us to apply Theorem 2.1 to deduce the analyticity of $\theta(p)$.

Lemma 3.2. *For every $p > p_c$ there is $N \in \mathbb{N}$ and an interval (a,b) containing p such that the following holds for every $q \in (a,b) \cap (p_c, 1]$. Conditioning on C_o being finite, and $diam(C_o) \geq N/5$, at least one separating component occurs almost surely.*

Proof. Let S be the maximal connected subgraph of $N\mathbb{L}^d_\boxtimes$ that contains $\partial C_o(N)$ and consists of bad boxes only. This S exists whenever C_o is finite and $diam(C_o) \geq N/5$ because $\partial C_o(N)$ is connected by (1).

We claim that there is some N and an interval (a, b) containing p such that S is $\mathbb{P}_q$-almost surely finite for every $q \in (a, b) \cap (p_c, 1]$. For this, it suffices to show that for some large enough N, the probability $\mathbb{P}_q$(S has size at least n) converges to 0 as n tends to infinity for each such q. The latter follows by combining the union bound with Lemma 3.6 below, which states that

$$\sum_{T \text{ is a separating component of size } n} \mathbb{P}_q(\text{T is bad}) \leq e^{-tn}$$

for some constant $t = t(p) > 0$, for some N, and every q in an interval $(a, b) \cap (p_c, 1]$.

Note that conditions (i) and (ii) are automatically satisfied by the choice of S. The configuration $\omega' := \omega$ satisfies condition (iii), since $C_o(\omega)$ is finite, and S contains $\partial C_o(\omega)(N)$ by definition. Thus S occurs in ω, as desired. □

Note that the proof of Lemma 3.2 finds a concrete occurring separating component whenever C_o is finite and $diam(C_o) \geq N/5$; we denote this separating component by $\mathcal{S}_o$ in this case.

The next two lemmas provide a converse to Lemma 3.2, namely that C_o is finite whenever some separating component occurs.

Whenever ω' is a witness for the occurrence of S, we let $R_o(\omega')$ denote the set of vertices of the infinite component of $\mathbb{L}^d \setminus C_o(\omega')$ lying in S.

Lemma 3.3. *Consider a separating component S, and assume that S occurs in ω. Let ω' be a witness of the occurrence of S. Then no vertex of $R_o(\omega')$ lies in $C_o(\omega)$.*

Proof. Assume that some vertex u of $R_o(\omega')$ lies in $C_o(\omega)$; we will obtain a contradiction.

Since $C_o(\omega)$ contains u, there must exist a path P in ω connecting o to u. This path cannot lie entirely in $S \cup \partial_{\boxtimes} S$ because ω and ω' coincide in that set of boxes and $u \notin C_o(\omega')$. Hence $N\mathbb{L}^d_{\boxtimes} \setminus (S \cup \partial_{\boxtimes} S)$ must have some finite component. Let E denote the minimal edge cut of $C_o(\omega')$. Clearly, P must meet E, since u lies in the infinite component of $\mathbb{L}^d \setminus C_o(\omega')$. Let e be an edge of E that P contains. Notice that no common edge of P and E lies in $S \cup \partial_{\boxtimes} S$, because the edges of E are closed in ω', the edges of P are open in ω, and the two configurations coincide in $S \cup \partial_{\boxtimes} S$. Hence e must lie in one of the finite components $\mathcal{B}_{in}$ of $N\mathbb{L}^d_{\boxtimes} \setminus (S \cup \partial_{\boxtimes} S)$. Write $\mathcal{B}$ for the set of those boxes in $\partial_{\boxtimes} S$ that have a $N\mathbb{L}^d_{\boxtimes}$-neighbour in $\mathcal{B}_{in}$. (Thus $\mathcal{B}$ is the vertex boundary of $\mathcal{B}_{in}$.) See Figure 2.

It is not hard to see that some box B of $\mathcal{B}$ is $C_o(\omega')$-substantial, which then implies that all boxes of $\mathcal{B}$ are $C_o(\omega')$-substantial because they are all good. Indeed, notice that one of the two endvertices of e lies in $C_o(\omega')$ by the definition of the set E. As S contains a $C_o(\omega')$-substantial box, some box B of $\mathcal{B}$ must be $C_o(\omega')$-substantial, as claimed, because $\mathcal{B}$ is the vertex boundary of $\mathcal{B}_{in}$.

Our aim now is to show that we can connect u to the subgraph of $C_o(\omega')$ inside $\mathcal{B}$ with a path in ω' lying entirely in $S \cup \partial_{\boxtimes} S$. This will imply that u belongs to $C_o(\omega')$, contradicting that $u \in R_o(\omega')$.

For this, consider the subpath Q of P that starts at u and ends at the last vertex of the intersection of $\mathcal{B}_{in}$ and $\mathcal{B}$ (notice that although $\mathcal{B}_{in}$ and $\mathcal{B}$ are

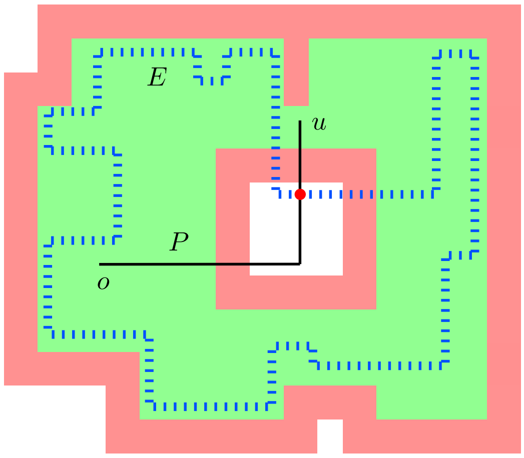

Figure 2: The situation in the proof of Lemma 3.3. The separating component S is depicted in green and its boundary $\partial_{\boxtimes} S$ in red (if colour is shown). When two boxes of S and $\partial_{\boxtimes} S$ overlap, their intersection is depicted also in geen.

disjoint sets of boxes, the subgraphs of $\mathbb{L}^d$ inside them overlap). If Q is not contained in $S \cup \partial_{\boxtimes} S$, then we can modify it to ensure that it does lie entirely in $S \cup \partial_{\boxtimes} S$. Indeed, notice that each $N\mathbb{L}^d_{\boxtimes}$-component F of $\partial_{\boxtimes} S$ contains a unique ω-cluster C such that some box of F is C-substantial by (3), because all its boxes are good. Moreover, each time Q exits $S \cup \partial_{\boxtimes} S$, it has to first visit the unique such percolation cluster of some $N\mathbb{L}^d_{\boxtimes}$-component F of $\partial_{\boxtimes} S$, and then eventually revisit the same percolation cluster of F. We can thus replace the subpaths of Q that lie outside of $S \cup \partial_{\boxtimes} S$ by open paths lying entirely in $\partial_{\boxtimes} S$ that share the same endvertices. Thus we may assume that Q is contained in $S \cup \partial_{\boxtimes} S$ as claimed.

Now notice that Q contains a subpath of diameter greater than $N/5$ lying entirely in some box B of $\mathcal{B}$. This box is $C_o(\omega')$-substantial, hence $C_o(\omega')$ and Q must meet. Then following the edges of Q, which are all open in ω', we arrive at u, and thus u belongs $C_o(\omega')$, as desired. □

We now use this to prove

Lemma 3.4. *Whenever some separating component occurs in a configuration ω, the cluster $C_o(\omega)$ is finite.*

Proof. We will prove the following slightly stronger statement: whenever a separating component S occurs in a configuration ω, a minimal (finite) edge cut of closed edges occurs in ω which separates o from infinity and lies in $S \cup \partial_{\boxtimes} S$.

For this, consider a witness ω' of the occurrence of S, and let ω'' be the configuration which coincides with ω (and ω') on every edge lying in $S \cup \partial_{\boxtimes} S$, and every other edge of ω'' is open. Note that S occurs in ω'' since it occurs in ω. Thus $C_o(\omega'')$ contains no vertex of $R_o(\omega')$ by Lemma 3.3. This implies that $C_o(\omega'')$ contains no vertex in the infinite component X of $N\mathbb{L}^d_{\boxtimes} \setminus S$, because any path P in $\mathbb{L}$ connecting o to X has to first visit $R_o(\omega')$. To see that the latter statement is true, consider the last vertex u of $\partial^V C_o(\omega')$ that P contains.

Notice that the subpath of P after u, which is denoted Q, visits only vertices of the infinite component of $\mathbb{L}^d \setminus C_o(\omega')$, and furthermore that u lies either in S or in a finite component of $\mathbb{L}^d \setminus S$ by (4). In the first case, u lies in $R_o(\omega')$. In the second case, Q has to visit S, hence $R_o(\omega')$.

We have just proved that $C_o(\omega'')$ can only contain vertices in S and the finite components of $N\mathbb{L}^d_{\boxtimes} \setminus S$. Since S is a finite set of boxes, $C_o(\omega'')$ is finite as well. Hence a minimal edge cut of closed edges separating o from infinity occurs in ω''. This minimal edge cut must lie entirely in $S \cup \partial_{\boxtimes} S$, because all edges not in $S \cup \partial_{\boxtimes} S$ are open. This is the desired minimal edge cut since it occurs in ω as well. We will denote it by $\partial^b \mathcal{S}_o$. □

Lemmas 3.2 and 3.4 combined allow us to express the event that C_o is finite in terms of the event that some separating component occurs. To do so, let us write D_N to denote the event $\{diam(C_o) < N/5\}$. Thus we have proved that

$$\begin{aligned} 1 - \theta(p) =& \mathbb{P}_p(C_o \text{ is finite}) \\ =& \mathbb{P}_p(D_N) + \mathbb{P}_p(|C_o| < \infty, D_N^{\mathsf{c}}) \\ =& \mathbb{P}_p(D_N) + \mathbb{P}_p(\text{some separating component occurs}, D_N^{\mathsf{c}}). \end{aligned} \tag{5}$$

Here and below, the notation $X, Y, \ldots$ denotes the intersection of the events $X, Y, \ldots$.

3.4 Expanding θ as an infinite sum of polynomials

Notice that $\mathbb{P}_p(D_N)$ is a polynomial in p, since the event D_N depends only on the state of finitely many edges.

Following our technique from [9] we will now use the inclusion-exclusion principle to expand the right-hand side $\mathbb{P}_p(\text{some separating component occurs}, D_N^{\mathsf{c}})$ of (5) as an infinite sum of polynomials, corresponding to all possible separating components that could occur.

Notice that any two occurring separating components are disjoint because they are connected, their boxes are bad, and they are surrounded by good boxes by definition.

Lemma 3.5. *For every $p > p_c$ there is some integer $N = N(p) > 0$ and an interval (a, b) containing p such that the expansion*

$$\mathbb{P}_q(\textit{some } S \textit{ occurs}, D_N^{\mathsf{c}}) = \sum_{S \in MS^N} (-1)^{c(S)+1} \mathbb{P}_q(S \textit{ occurs}, D_N^{\mathsf{c}}) \tag{6}$$

holds for every $q \in (a, b) \cap (p_c, 1]$, where MS^N denotes the set of all finite collections of pairwise disjoint separating components S, and $c(S)$ denotes the number of separating components of S.

Lemma 3.5 will follow easily from the next lemma. We will use the notation MS_n^N to denote the set of those finite collections of pairwise disjoint separating components $\{S_1, S_2, \ldots, S_k\}$ such that $|S_1| + |S_2| + \ldots + |S_k| = n$. The superscript reminds us of the dependence of the boxes on N.

Lemma 3.6. *For every $p > p_c$, there are $N = N(p) > 0$, $t = t(p) > 0$ and an interval (a, b) containing p such that*

$$\sum_{S \in MS_n^N} \mathbb{P}_q(S \textit{ is bad}) \leq e^{-tn} \tag{7}$$

for every $n \geq 1$ and every $q \in (a, b) \cap (p_c, 1]$.

Proof. To prove the desired exponential decay we will use a standard renormalization technique with a few modifications. We will first prove the exponential decay when $q = p$, and then we will use a continuity argument to obtain the desired assertion.

We will first show that there exists a constant $k > 0$ depending only on d such that for every $S \in MS_n^N$ we have

$$\mathbb{P}_p(\text{S is bad}) \leq c^{n/k},$$

where $c := \mathbb{P}_p(B(o) \text{ is bad})$. Indeed, it is not hard to see that there is a constant $k = k(d) > 0$ such that for every $S \in MS_n^N$ there is a subset Y of S of size at least n/k, all boxes of which are pairwise disjoint. As each box of Y is bad whenever S occurs, we have

$$\mathbb{P}_p(\text{S is bad}) \leq \mathbb{P}_p(\text{Y is bad}).$$

By independence $\mathbb{P}_p(\text{Y is bad}) = c^{n/k}$ and the assertion follows.

We will now find an exponential upper bound for the number of elements of $S \in MS_n^N$. Since $N\mathbb{L}^d_\boxtimes$ is isomorphic to $\mathbb{L}^d_\boxtimes$, there is a constant $\mu > 0$ depending only on d and not on N, such that the number of connected subgraphs of $N\mathbb{L}^d_\boxtimes$ with n vertices containing a given vertex is at most μ^n. However, an element of MS_n^N might contain multiple separating components, and there are in general several possibilities for the reference vertices that each of them contains. To remedy this, consider one of the d axis $X = (-x_1, x_0 = B(o), x_1)$ of $N\mathbb{L}^d_\boxtimes$ that contain the box $B(o)$, and let X^+, X^- be its two infinite subpaths starting from $B(o)$. We will first show that any separating component of size n contains one of the first n elements of X^+. Indeed, consider an occurring separating component S of size n, and notice that S has to contain some vertex x^+ of X^+, and some vertex x^- of X^-. The graph distance between x^+ and x^- is at most n, as there is a path in S connecting them. This implies that x^+ is one of the first n elements of X^+, as desired.

Consider now a constant $M > 0$ such that $m\mu^m \leq M^m$ for every integer $m \geq 1$. Consider also a partition $\{m_1, m_2, \ldots, m_k\}$ of n. It follows that the number of collections $\{S_1, S_2, \ldots, S_k\}$ with $|S_i| = m_i$ is at most $m_1 m_2 \ldots m_k \mu^n \leq M^n$, since we have at most $m_i \mu^{m_i}$ choices for each S_i. A well known result of Hardy & Ramanujan [16] implies that the number of partitions of n is at most $r^{\sqrt{n}}$ for some constant $r > 0$. We can now deduce that the size of MS_n^N is at most $r^{\sqrt{n}} M^n$, implying that

$$\sum_{S \in MS_n^N} \mathbb{P}_p(\text{S is bad}) \leq r^{\sqrt{n}} M^n c^{n/k}.$$

Notice that in the right hand side of the above inequality only c depends on N. It is a standard result that c converges to 0 as N tends to infinity [15, Theorem 7.61]. Choosing N large enough so that $Mc^{1/k} < 1$, we obtain the desired exponential decay.

Now notice that $c(q) = \mathbb{P}_q(B(o) \text{ is bad})$ is a polynomial in q, hence a continuous function, since it depends only on the state of the edges inside $B(o)$. This implies that we can choose an interval (a,b) containing p such that $Mc(q)^{1/k} < 1$ for every $q \in (a,b) \cap (p_c, 1]$. This completes the proof. $\square$

Lemma 3.5 follows now easily:

Proof of Lemma 3.5. Lemma 3.6 shows that $\sum_{S \in MS^N} \mathbb{P}_q(\text{S occurs}, D_N^{\mathsf{c}})$ is finite, hence only finitely many separating components occur in almost any percolation configuration ω by the Borel-Cantelli lemma. Now the standard inclusion-exclusion principle implies that

$$\mathbb{1}_{\{\text{S occurs}, D_N^{\mathsf{c}}\}} = \sum_{S \in MS^N} (-1)^{c(S)+1} \mathbb{1}_{\{\text{S occurs}, D_N^{\mathsf{c}}\}}.$$

Taking expectations we obtain

$$\mathbb{P}_q(\text{S occurs}, D_N^{\mathsf{c}}) = \mathbb{E}_q\Big(\sum_{S \in MS^N} (-1)^{c(S)+1} \mathbb{1}_{\{\text{S occurs}, D_N^{\mathsf{c}}\}}\Big).$$

Since

$$\mathbb{E}_q\Big(\sum_{S \in MS^N} \mathbb{1}_{\{\text{S occurs}, D_N^{\mathsf{c}}\}}\Big) = \sum_{S \in MS^N} \mathbb{P}_q(\text{S occurs}, D_N^{\mathsf{c}})$$

and the latter sum is finite, Fubini's theorem implies that

$$\mathbb{E}_q\Big(\sum_{S \in MS^N} (-1)^{c(S)+1} \mathbb{1}_{\{\text{S occurs}, D_N^{\mathsf{c}}\}}\Big) = \sum_{S \in MS^N} (-1)^{c(S)+1} \mathbb{P}_q(\text{S occurs}, D_N^{\mathsf{c}}).$$

The proof is complete. $\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Consider some $p \in (p_c, 1]$. Let $N, t > 0$, and the interval (a,b) containing p, be as in Lemma 3.6. Then the expression

$$1 - \theta(q) = \mathbb{P}_q(D_N) + \sum_{n=1}^{\infty} \sum_{S \in MS_n^N} (-1)^{c(S)+1} \mathbb{P}_q(\text{S occurs}, D_N^{\mathsf{c}})$$

holds for every $q \in (a,b) \cap (p_c, 1]$, and furthermore

$$\Big| \sum_{S \in MS_n^N} (-1)^{c(S)+1} \mathbb{P}_q(\text{S occurs}, D_N^{\mathsf{c}}) \Big| \leq e^{-tn}$$

for every $q \in (a,b) \cap (p_c, 1]$. The probability $\mathbb{P}_q(D_N)$ is a polynomial in q, hence analytic, because it depends on finitely many edges. Moreover, the event $\{\text{S occurs}, D_N^{\mathsf{c}}\}$ depends only on the state of the edges lying in $S \cup \partial_{\boxtimes} S$ and the box $B(o,N)$. The number of edges of each box is $O(N^d)$, hence the event $\{\text{S occurs}, D_N^{\mathsf{c}}\}$ depends only on $O(N^d n)$ edges. The desired assertion follows now from Theorem 2.1. $\square$

3.5 Exponential tail of $\partial^b \mathcal{S}_o$

Lemma 3.6 easily implies that the size of $\partial^b \mathcal{S}_o$, as defined in the proof of Lemma 3.4, has an exponential tail:

Theorem 3.7. *For every $p > p_c$, there are constants $N = N(p) > 0$ and $t = t(p) > 0$ such that*

$$\mathbb{P}_p(|\partial^b \mathcal{S}_o| \geq n) \leq e^{-tn}$$

for every $n \geq 1$.

Proof. Assume that $|\partial^b \mathcal{S}_o| \geq n$, and consider the separating component S associated to C_o. Then the boxes of $S \cup \partial_{\boxtimes} S$ must contain at least n edges. Hence the number of boxes of $S \cup \partial_{\boxtimes} S$ is at least cn/N^d for some constant $c > 0$. Moreover, we have $|\partial_{\boxtimes} S| \leq (3^d - 1)|S|$, because each box of $\partial_{\boxtimes} S$ has at least one neighbour in S, and each box in S has at most $3^d - 1$ neighbours. Therefore, S contains at least $cn/(3N)^d$ boxes. The desired assertion follows from Lemma 3.6. $\square$

We recall that for every $p \in (p_c, 1 - p_c]$, the probability $\mathbb{P}_p(|\partial C_o| \geq n)$ does not decay exponentially in n [22, 10]. This implies that for those values of p, $\partial^b \mathcal{S}_o$ has typically smaller order of magnitude than the standard minimal edge cut of C_o.

As a corollary, we re-obtain a result of Pete [28] which states that when C_o is finite, the number of touching edges between C_o and the unique infinite cluster, which we denote C_∞, has an exponential tail. A *touching edge* is an edge in $\partial^E C_o \cap \partial^E C_\infty$. We denote the number of (closed) touching edges joining C_o to the infinite component C_∞ by $\phi(C_o, C_\infty)$.

Corollary 3.8. *For every $p > p_c$, there is some $c = c(p, d) > 0$ such that*

$$\mathbb{P}_p(|C_o| < \infty, \phi(C_o, C_\infty) \geq t) \leq e^{-ct}$$

for every $t \geq 1$.

Proof. The result follows from Theorem 3.7 by observing that C_∞ has to lie in the unbounded component of $\mathbb{L}^d \setminus \partial^b \mathcal{S}_o$, hence all relevant edges belong to $\partial^b \mathcal{S}_o$. $\square$

4 Analyticity of τ

In the previous section we proved that θ is analytic above p_c for every $d \geq 3$. Some further challenges arise when one tries to prove that other functions describing the macroscopic behaviour of our model are analytic functions of p. The main obstacle is that events of the form $\{x \text{ is connected to } y\}$ are not fully determined, in general, by the configuration inside $S \cup \partial_{\boxtimes} S$. In this section we show how one can remedy this issue, and we will prove that the k-point function τ and its truncated version τ^f are analytic functions above p_c for every $d \geq 3$. We will then deduce that the truncated susceptibility $\mathbb{E}(|C_o|; |C_o| < \infty)$ and the free energy $\mathbb{E}(|C_o|^{-1})$ are analytic functions as well.

Given a k-tuple $\boldsymbol{x} = \{x_1, \ldots, x_k\}, k \geq 2$ of vertices of $\mathbb{Z}^d$, the function $\tau_{\boldsymbol{x}}(p)$ denotes the probability that $\boldsymbol{x}$ is contained in a cluster of Bernoulli percolation on $\mathbb{Z}^d$ with parameter p. Similarly, $\tau^f_{\boldsymbol{x}}(p)$ denotes the probability that $\boldsymbol{x}$ is contained in a *finite* cluster. We will write $MS^N(\boldsymbol{x})$ for the set of all finite collections of separating components surrounding some vertex of $\boldsymbol{x}$, and $MS^N_n(\boldsymbol{x})$ for the set of those elements of $MS^N(\boldsymbol{x})$ that have size n.

Arguing as in the proof of Lemma 3.6 we obtain the following:

Lemma 4.1. *For every $p > p_c$, there are $N = N(p) > 0$, $t = t(p) > 0$, and an interval (a, b) containing p, such that*

$$\sum_{S \in MS^N_n} \mathbb{P}_q(S \text{ occurs}) \leq e^{-tn} \tag{8}$$

for every $n \geq 1$ and every $q \in (a, b) \cap (p_c, 1]$.

We are now ready to prove that τ and τ^f are analytic.

Theorem 4.2. *For every $d \geq 3$ and every finite set $\boldsymbol{x}$ of vertices of $\mathbb{Z}^d$, the functions $\tau_{\boldsymbol{x}}(p)$ and $\tau^f_{\boldsymbol{x}}(p)$ admit analytic extensions to a domain of $\mathbb{C}$ that contains the interval $(p_c, 1]$.*

Moreover, for every $p \in (p_c, 1]$ and every finite set $\boldsymbol{x}$ such that $diam(\boldsymbol{x}) \geq N/5$, there is a closed disk $D(p, \delta), \delta > 0$ and positive constants $c_1 = c_1(p, \delta), c_2 = c_2(p, \delta)$ such that

$$|\tau^f_{\boldsymbol{x}}(z)| \leq c_1 e^{-c_2 diam(\boldsymbol{x})}$$

for every $z \in D(p, \delta)$ for such an analytic extension $\tau^f_{\boldsymbol{x}}(z)$ of $\tau^f_{\boldsymbol{x}}(p)$.

Proof. We start by showing that $\tau^f_{\boldsymbol{x}}(p)$ is analytic. Suppose $\boldsymbol{x} = \{x_1, \ldots, x_k\}$, and let A be the event that $diam(C_{x_i}) \geq N/5$ for every $i \leq k$. We will write $\{\boldsymbol{x} \text{ is connected}\}$ to denote the event that all vertices of $\boldsymbol{x}$ belong to the same cluster, which we denote $C_{\boldsymbol{x}}$. When $C_{\boldsymbol{x}}$ is finite and both events $\{\boldsymbol{x} \text{ is connected}\}$ and A occur, we will write $\mathcal{S}_{\boldsymbol{x}}$ for the separating component of the latter cluster, namely the $N\mathbb{L}^d_{\boxtimes}$-component of $\partial C_{\boldsymbol{x}}(N)$. The event $\{\text{S occurs}\}$ is defined as in the previous section except that now C_o is replaced by $C_{\boldsymbol{x}}$, i.e. the event $\{\boldsymbol{x} \text{ is connected}\}$ occurs in a witness ω', and S contains $\partial C_{\boldsymbol{x}}(\omega')(N)$. With the above definitions we have

$$\tau^f_{\boldsymbol{x}}(p) = \mathbb{P}_p(A^c, \boldsymbol{x} \text{ is connected}) + \sum_S \mathbb{P}_p(A, \boldsymbol{x} \text{ is connected}, \mathcal{S}_{\boldsymbol{x}} = S),$$

where the sum ranges over all possible separating components separating all of $\boldsymbol{x}$ from infinity.

Our aim is to further decompose the events of the above expansion into simpler ones that we have better control of, and then use the inclusion-exclusion principle. We will first introduce some notation. Given a separating component S as above, we first decompose $\boldsymbol{x}$ into two sets $\boldsymbol{x}_{out}$ and $\boldsymbol{x}_{in}$, where $\boldsymbol{x}_{out}$ denotes the set of those vertices of $\boldsymbol{x}$ lying in some finite component of $N\mathbb{L}^d_{\boxtimes} \setminus (S \cup \partial_{\boxtimes} S)$, and $\boldsymbol{x}_{in} := \boldsymbol{x} \setminus \boldsymbol{x}_{out}$ its complement. We write $\{\boldsymbol{x} \to S\}$ for the event that no separating component separating some $x_i \in \boldsymbol{x}$ from S occurs; to be more precise, the event $\{\boldsymbol{x} \to S\}$ means that for each $x_i \in \boldsymbol{x}_{out}$, no separating component that surrounds x_i and lies entirely in some of the finite components of $N\mathbb{L}^d_{\boxtimes} \setminus (S \cup \partial_{\boxtimes} S)$ occurs.

Consider now some vertex x in $\boldsymbol{x}_{out}$, and let F be the component of $\partial_{\boxtimes} S$ that separates x from S. We claim that when S and the events A, $\{\boldsymbol{x} \to S\}$

all occur, then x is connected to the unique large cluster of F. In particular, if another vertex of $\boldsymbol{x}$ lies in the same finite component of $N\mathbb{L}^d_{\boxtimes} \setminus (S \cup \partial_{\boxtimes} S)$ as x does, then both vertices are connected to the unique large cluster of F, hence they are connected to each other. To see that the claim holds, notice that C_x has to be finite, because $S \cup \partial_{\boxtimes} S$ contains a minimal edge cut of closed edges that surrounds all vertices of $\boldsymbol{x}$, hence x. Now $\partial C_x(N)$ has to intersect S, because it cannot lie entirely in $N\mathbb{L}^d_{\boxtimes} \setminus (S \cup \partial_{\boxtimes} S)$ by our assumption. This implies that x is connected to some vertex inside S, hence it must first visit the unique large cluster of F, as desired.

We now define $\mathcal{C}$ to be the event that

- all vertices of $\boldsymbol{x}_{in}$ are connected to each other with open paths lying in $S \cup \partial_{\boxtimes} S$,
- the unique large percolation clusters of the components F of $\partial_{\boxtimes} S$ that separate some $x_i \in \boldsymbol{x}_{out}$ from S are connected to each other with open paths lying $S \cup \partial_{\boxtimes} S$,
- all vertices of $\boldsymbol{x}_{in}$ are connected to all such percolation clusters with open paths lying in $S \cup \partial_{\boxtimes} S$.

(It is possible that either $\boldsymbol{x}_{in}$ or $\boldsymbol{x}_{out}$ is the empty set, in which case the third item and one of the first two are empty statements.) We claim that when S and the events A, $\{\boldsymbol{x} \to S\}$ and $\{\boldsymbol{x} \text{ is connected}\}$ all occur, then the event $\mathcal{C}$ occurs as well. Indeed, consider a vertex $x \in \boldsymbol{x}_{out}$, and let F be the component of $\partial_{\boxtimes} S$ that separates x from S, as above. Any open path connecting x to some vertex of $\boldsymbol{x}_{out}$ which does not lie in the same finite component of $N\mathbb{L}^d_{\boxtimes}$ that x does, has to first visit the unique large percolation cluster of F. Hence it suffices to prove that when two vertices x_i and x_j of $\boldsymbol{x}_{in}$ lie in the same cluster, there is always an open path connecting them lying entirely in $S \cup \partial_{\boxtimes} S$. To this end, assume that there is a path P in ω connecting x_i to x_j, which does not lie entirely in $S \cup \partial_{\boxtimes} S$. Arguing as in the proof of Lemma 3.3, we can modify P to obtain an open path P' connecting x_i to x_j which lies entirely in $S \cup \partial_{\boxtimes} S$. The desired claim follows now easily.

Combining the above claims, we conclude that the events $\{A, \boldsymbol{x} \text{ is connected}, \mathcal{S}_{\boldsymbol{x}} = S\}$ and $\{A, \mathcal{C}, \boldsymbol{x} \to S, S \text{ occurs}\}$ coincide, and thus

$$\mathbb{P}_p(A, \boldsymbol{x} \text{ is connected}, \mathcal{S}_{\boldsymbol{x}} = S) = \mathbb{P}_p(A, \mathcal{C}, \boldsymbol{x} \to S, S \text{ occurs}).$$

Using the inclusion-exclusion principle we obtain that

$$\begin{aligned}\mathbb{P}_p(A, \mathcal{C}, \boldsymbol{x} \to S, S \text{ occurs}) = \mathbb{P}_p(A, \mathcal{C}, S \text{ occurs}) + \\ \sum_T (-1)^{c(T)} \mathbb{P}_p(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs}),\end{aligned} \tag{9}$$

where the latter sum ranges over all finite collections T of separating components separating $\boldsymbol{x}$ from S. Collecting now all the terms we obtain that

$$\begin{aligned}\tau^f_{\boldsymbol{x}}(p) = \mathbb{P}_p(A^{\mathsf{c}}, \boldsymbol{x} \text{ is connected}) + \\ \sum_S \Big(\mathbb{P}_p(A, \mathcal{C}, S \text{ occurs}) + \sum_T (-1)^{c(T)} \mathbb{P}_p(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs}) \Big).\end{aligned} \tag{10}$$

Notice that by combining S and T we obtain an element of $MS^N(\boldsymbol{x})$, hence we can use Lemma 4.1, and then argue as in the proof of Theorem 3.1 to obtain that $\tau_{\boldsymbol{x}}^f$ is analytic above p_c.

We will now prove the analyticity of $\tau_{\boldsymbol{x}}$. Since $\tau_{\boldsymbol{x}}^f$ is analytic, it suffices to prove that $\tau_{\boldsymbol{x}} - \tau_{\boldsymbol{x}}^f$ is analytic. It is well-known that the infinite cluster is unique in our setup [7], and this implies that $\tau_{\boldsymbol{x}} - \tau_{\boldsymbol{x}}^f = \mathbb{P}(|C_{x_1}| = \infty, \ldots, |C_{x_k}| = \infty)$. The latter probability is complementary to $\mathbb{P}(\cup_{i=1}^k \{|C_{x_i}| < \infty\})$, which is in turn equal to

$$\mathbb{P}(\cup_{i=1}^k \{|C_{x_i}| < \infty\}) = \mathbb{P}(A^{\mathsf{c}}) + \mathbb{P}(\big(\cup_{i=1}^k \{|C_{x_i}| < \infty\}\big) \cap A).$$

Define the event $\{\text{S occurs for some } x_i \in \boldsymbol{x}\}$ as in the previous section expect that now we require the existence of a witness ω' such that S contains $\partial C_{x_i}(\omega')(N)$ for some $x_i \in \boldsymbol{x}$. We can expand the latter term as an infinite sum using the inclusion-exclusion principle to obtain

$$\mathbb{P}(\big(\cup_{i=1}^k \{|C_{x_i}| < \infty\}\big) \cap A) = \sum (-1)^{c(S)+1} \mathbb{P}(\text{S occurs for some } x_i \in \boldsymbol{x}, A),$$

where now we require our separating components to surround some $x_i \in \boldsymbol{x}$. Arguing as in the proof of Theorem 3.1 we obtain that $\tau_{\boldsymbol{x}} - \tau_{\boldsymbol{x}}^f$ is analytic, as desired.

For the second claim of the theorem, notice that when $diam(\boldsymbol{x}) \geq N/5$, the probability $\mathbb{P}(A^{\mathsf{c}}, \boldsymbol{x} \text{ is connected})$ is equal to 0. Hence our expansion for $\tau_{\boldsymbol{x}}^f$ simplifies to

$$\tau_{\boldsymbol{x}}^f(p) = \sum_S \Big(\mathbb{P}(A, \mathcal{C}, S \text{ occurs}) + \sum_T (-1)^{c(T)} \mathbb{P}(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs})\Big).$$

Our goal is to show that for every $p > p_c$ there are some constants $\delta, t > 0$ such that

$$\Big|\sum_{|S|=n} \Big(\mathbb{P}_p(A, \mathcal{C}, S \text{ occurs}) + \sum_T (-1)^{c(T)} \mathbb{P}_p(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs})\Big)\Big| \leq e^{-tn} \tag{11}$$

for every $z \in D(p, \delta)$ for the analytic extensions of the above probabilities. Then the desired claim will follow easily from the observation that any plausible separating component S of $\boldsymbol{x}$ must have size $\Omega(diam(\boldsymbol{x}))$.

Notice that the event A depends only on the edges in the boxes $B(x_i)$, $x_i \in \boldsymbol{x}$. Moreover, the events $\mathcal{C}$ and $\{S \text{ occurs}\}$ depend on $O(|S|)$ edges, while the event $\{T \text{ occurs}\}$ depends on $O(|T|)$ edges. We can now use Lemma 2.2 to conclude that there is a constant $c = c(p, \delta, N) > 1$ (perhaps slightly larger than that of Lemma 2.2) such that

$$|\mathbb{P}_z(A, \mathcal{C}, S \text{ occurs})| \leq c^{|S|} \mathbb{P}_{p'}(A, \mathcal{C}, S \text{ occurs})$$

and

$$|\mathbb{P}_z(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs})| \leq c^{|S|+|T|} \mathbb{P}_{p'}(A, T \text{ occurs}, \mathcal{C}, S \text{ occurs})$$

for every $z \in D(p,\delta)$, where $p' = p + \delta$ if $p < 1$, and $p' = 1 - \delta$ if $p = 1$. Moreover, we can always choose c in such a way that $c \to 1$ as $\delta \to 0$. Hence we have

$$\begin{aligned} &\Big|\sum_{|S|=n}\Big(\mathbb{P}_z(A,\mathcal{C},S \text{ occurs}) + \sum_T (-1)^{c(T)}\mathbb{P}_z(A,T \text{ occurs},\mathcal{C},S \text{ occurs})\Big)\Big| \leq \\ &\qquad c^n \sum_{|S|=n}\Big(\mathbb{P}_{p'}(A,\mathcal{C},S \text{ occurs}) + \sum_T c^{|T|}\mathbb{P}_{p'}(A,T \text{ occurs},\mathcal{C},S \text{ occurs})\Big) \end{aligned} \tag{12}$$

by the triangle inequality. It follows from Lemma 4.1 that the sum

$$\sum_{|S|=n}\Big(\mathbb{P}_{p'}(A,\mathcal{C},S \text{ occurs}) + \sum_T \mathbb{P}_{p'}(A,T \text{ occurs},\mathcal{C},S \text{ occurs})\Big)$$

decays exponentially in n, and by choosing δ small enough we can ensure that

$$c^n \sum_{|S|=n}\Big(\mathbb{P}_{p'}(A,\mathcal{C},S \text{ occurs}) + \sum_T c^{|T|}\mathbb{P}_{p'}(A,T \text{ occurs},\mathcal{C},S \text{ occurs})\Big)$$

decays exponentially in n as well, hence (11) holds. The proof is now complete. $\square$

Using Theorem 4.2 we can now prove the following results.

Theorem 4.3. *For every $k \geq 1$ and every $d \geq 3$, the functions $\chi^f_k(p) := \mathbb{E}_p(|C(o)|^k; |C(o)| < \infty)$ are analytic in p on the interval $(p_c, 1]$.*

Proof. Let us show that $\chi^f(p) := \mathbb{E}(|C(o)|; |C(o)| < \infty)$ is analytic. The case $k \geq 2$ will follow similarly. We observe that, by the definitions,

$$\chi^f(p) = \sum_{x\in\mathbb{Z}^d} \tau^f_{\{o,x\}} = 1 + \sum_{x\in\mathbb{Z}^d\setminus\{o\}} \tau^f_{\{o,x\}}.$$

The probabilities $\tau^f_{\{o,x\}}$ admit analytic extensions by Theorem 4.2, and so it suffices to prove that the sum $\sum_{x\in\mathbb{Z}^d\setminus\{o\}} \tau^f_{\{o,x\}}$ converges uniformly on an open neighbourhood of $(p_c, 1]$. This follows easily from the estimates of the second sentence of Theorem 4.2, and the polynomial growth of $\mathbb{Z}^d$. $\square$

Theorem 4.4. *For every $d \geq 3$, the free energy $\kappa = \mathbb{E}(|C_o|^{-1})$ is analytic in p on the interval $(p_c, 1]$.*

Proof. It is known [3] that κ is differentiable on $(p_c, 1)$ with derivative equal to

$$f(p) := \frac{1}{2(1-p)} \sum_{x\in N(o)} \big(1 - \tau_{\{o,x\}}(p)\big).$$

Since each $\tau_{\{o,x\}}$ is analytic on the interval $(p_c, 1]$, and $\tau_{\{o,x\}}(1) = 1$, f is analytic on $(p_c, 1]$ as well. So far we know that κ coincides with a primitive F of f only on $(p_c, 1)$, which implies that κ is analytic on that interval. In fact, κ coincides with F on the whole interval $(p_c, 1]$. Indeed, we simply need to verify that κ is continuous from the left at 1. To see this notice that $\kappa(1) = 1 - \theta(1) = 0$ and $\kappa(p) \leq 1 - \theta(p)$. Since θ is continuous from the left at 1, which follows e.g. by Theorem 3.1, we have that κ is continuous from the left at 1 as well, hence coincides with F on the whole interval $(p_c, 1]$. It now follows that κ is analytic in p on the interval $(p_c, 1]$, as desired. $\square$

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