Statistical Limit Theorems for Suspension Flows

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Abstract

In dynamical systems theory, a standard method for passing from discrete time to continuous time is to construct the suspension flow under a roof function. In this paper, we give conditions under which statistical laws, such as the central limit theorem and almost sure invariance principle, for the underlying discrete time system are inherited by the suspension flow.

As a consequence, we give a simpler proof of the results of Ratner (1973) and recover the results of Denker and Philipp (1984) for Axiom A flows. Moreover, we obtain several new results for nonuniformly and partially hyperbolic flows, including frame flows on negatively curved manifolds satisfying a pinching condition.

1 Introduction

Let (X, μ) be a probability space. Suppose that $S : X \to X$ is a measure-preserving transformation and that $\Phi : X \to \mathbb{R}$ is a measurable observation. Consider the sequence of partial sums $\Phi_N = \sum_{j=0}^{N-1} \Phi \circ S^j$. Provided S is ergodic and $\Phi \in L^1(X)$, Birkhoff's ergodic theorem (or the strong law of large numbers) states that

$$\textbf{(SLLN)} \qquad \lim_{N\to\infty} \frac{\Phi_N}{N} = \int_X \Phi \, d\mu \ \text{ a.e.}$$

Under certain hypotheses, the central limit theorem (CLT) states that

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(CLT) $$\lim_{N\to\infty} \mu\Big\{\frac{\Phi_N - N\int_X \Phi\, d\mu}{\sqrt{N}} < c\Big\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{c} e^{-x^2/2\sigma^2}\, dx.$$

The CLT has been established for large classes of dynamical systems. Usually the dynamical system X is assumed to have some hyperbolicity properties (Axiom A or nonuniformly/partially hyperbolic, see for example [25, 27, 30, 31, 7, 11, 22]) or expansivity properties (for example [28, 19]), and the observation Φ is assumed to have certain regularity properties such as Hölder continuity or bounded variation.

Suspension flows There is a standard method for constructing continuous time dynamical systems (or flows) from a discrete time dynamical system $S : X \to X$.

A measurable function $r : X \to \mathbb{R}$ is called a *roof function* if r is positive almost everywhere and $r \in L^1$. Define the *suspension*

$$X_r = \{(x, u) \in X \times \mathbb{R} : 0 \le u \le r(x)\}/\sim,$$

where $(x, r(x))$ is identified with $(Sx, 0)$. The *suspension flow* $S_t : X_r \to X_r$ is given by $S_t(x, u) = (x, u + t)$ computed subject to the identifications. (If $S : X \to X$ is not invertible, then S_t is a semi-flow, but we shall still speak of suspension flows.)

Let $\overline{r} = \int_X r\, d\mu$. Then $\mu_r = \mu \times \ell/\overline{r}$ is an invariant probability measure for the suspension flow, where ℓ stands for the Lebesgue measure on the real line. Given an L^2 observation $\phi : X_r \to \mathbb{R}$, we consider the family of integrals $\phi_T = \int_0^T \phi \circ S_t\, dt$.

Central limit theorems for suspension flows (convergence in distribution of $\frac{1}{\sqrt{T}}(\phi_T - T\int_{X_r} \phi\, d\mu_r)$) were obtained by Ratner [25] for Hölder observations in the Anosov context. It follows from Bowen [3] that Ratner's results hold for general Axiom A flows. Denker and Philipp [9] gave a relatively elementary proof of Ratner's theorem, using the fact that the hyperbolic diffeomorphism $S : X \to X$ satisfies strong statistical properties, in particular the almost sure invariance principle (ASIP).

In this paper, we give mild conditions under which the CLT for a general suspension flow $S_t : X_r \to X_r$ follows from the corresponding result for the discrete dynamical system $S : X \to X$. Our proof is not constrained to the hyperbolic context and is more general, and simpler, than the original proof of [25]. In addition our proof does not rely on the ASIP. A major advantage of our approach is that the roof function r need not be bounded. Hence, we are able to prove the CLT in a number of situations where [9, 25] do not apply, see Section 5.

We assume throughout that $\phi : X_r \to \mathbb{R}$ is an observation of mean zero, and $r \in L^a(X)$, $\phi \in L^b(X_r)$, where a, b are moderately large. We define

$$\Phi(x) = \int_0^{r(x)} \phi(x, u)\, du.$$

Then $\Phi : X \to \mathbb{R}$ is a mean zero observation on X, and the aim is to deduce statistical properties of ϕ on the suspension flow from the corresponding statistical properties of Φ on the base dynamics.

Central Limit Theorem The first version of our main theorem is as follows. In this result, we do not require invertibility of $S : X \to X$.

Theorem 1.1 *Let $S : X \to X$ be an ergodic transformation. Suppose that $(1 - \frac{1}{a})(1 - \frac{1}{b}) \geq \frac{1}{2}$. Suppose that Φ and r each satisfy the CLT. Then ϕ satisfies the CLT. Moreover, if the CLT for Φ has variance $\sigma_1^2 \geq 0$, then the CLT for ϕ has variance $\sigma^2 = \sigma_1^2/\overline{r}$.*

Remark 1.2 (a) The regularity hypothesis $(1 - \frac{1}{a})(1 - \frac{1}{b}) \geq \frac{1}{2}$ is optimal in the sense that otherwise Φ need not lie in L^2 (see Remark 2.4).

(b) The proof of Ratner [25] relies heavily on the additional assumptions that (Φ, r) jointly satisfy a 2-dimensional CLT, and that $\Phi \in L^\infty$.

(c) We do not require that the CLTs for Φ and r are nondegenerate (they may have zero variance). Moreover, the result remains true if $\frac{1}{\sqrt{N}}(r_N - N\overline{r})$ converges in distribution, regardless of the limit distribution. ∎

We have the following generalization of Theorem 1.1.

Theorem 1.3 *Let $S : X \to X$ be an invertible ergodic transformation. Suppose that $(1 - \frac{1}{a})(1 - \frac{1}{b}) \geq \frac{1}{2}$ and that $\frac{1}{\sqrt{N}}\Phi_N \to_d G$. Suppose further that there exist $\alpha \in (0, 1]$, $\beta \in (0, 1)$ with $\alpha\beta \leq \frac{1}{2}$ such that*

(i) $\frac{1}{|N|^\alpha}\Phi_N \to 0$ a.e. *as $N \to \pm\infty$ (where $\Phi_N = -\sum_{j=1}^{|N|} \Phi \circ S^{-j}$ for $N < 0$), and*

(ii) $\frac{1}{N^\beta}(r_N - N\overline{r})$ *converges in distribution as $N \to \infty$.*

Then $\frac{1}{\sqrt{T}}\phi_T \to_d G/\sqrt{\overline{r}}$ as $T \to \infty$.

Remark 1.4 (a) For invertible dynamical systems, the SLLN and the CLT hold in backwards time ($N, T \to -\infty$) if and only if they hold in forwards time. Hence in Theorem 1.1 it is not necessary to assume that $S : X \to X$ is invertible, since we can pass to the natural extension. In particular, Theorem 1.1 is a special case of Theorem 1.3 with $\alpha = 1$ and $\beta = \frac{1}{2}$. Theorem 1.3 shows that the CLT condition on r can be weakened when the SLLN condition on Φ is strengthened.

(b) In many situations, including the hyperbolic setting, hypotheses (i) and (ii) of Theorem 1.3 are valid for all $\alpha, \beta > \frac{1}{2}$, so certainly $\alpha\beta \leq \frac{1}{2}$.

(c) For ease of exposition, we have focused on convergence in distribution for which the normalizing factors are $\sqrt{N}$ and $\sqrt{T}$. It is easy to generalize Theorem 1.3 to allow for different normalizing factors. Let $c > 0$. Assume that $a, b \geq 1$ satisfy $(1 - \frac{1}{a})(1 - \frac{1}{b}) \geq (1 - \frac{1}{c})$ and that $\alpha \in (0, 1]$, $\beta \in (0, 1)$ satisfy $\alpha\beta \leq c$. Let $r \in L^a(X)$, $\Phi \in L^b(X_r)$ satisfy conditions (i) and (ii) of Theorem 1.3 for the revised values of α and β. If $\frac{1}{N^c}\Phi_N \to_d G$, then $\frac{1}{T^c}\phi_T \to_d G/\overline{r}^{\,c}$. ∎

Conjecture 1.5 The conclusion of Theorem 1.3 remains valid if, instead of conditions (i) and (ii), there exists $\beta \in (0,1)$ such that

$$\frac{1}{|N|^\beta}(r_N - N\overline{r}) \to 0 \text{ a.e. as } N \to \pm\infty.$$

Other limit laws Our results on the CLT extend in a straightforward way to the functional central limit theorem (or weak invariance principle). For example, suppose that $S : X \to X$ is ergodic, and let $r \in L^\infty(X)$ and $\phi \in L^2(X_r)$. Suppose that r satisfies the CLT. If Φ satisfies the functional CLT, then so does ϕ.

We also consider almost sure results such as the law of the iterated logarithm (LIL) and the ASIP, generalising and simplifying results of Wong [29] and Denker and Philipp [9] respectively. The arguments are easier than for the CLT.

We note that the corresponding results for time-one maps of hyperbolic flows are more delicate than those for the flow itself. See [11, 22].

The remainder of the paper is organized as follows. Section 2 contains an approximation result. Our main results on the CLT are proved in Section 3. In Section 4, we prove the LIL and ASIP for suspension flows. In Section 5, we apply our results to nonuniformly hyperbolic and partially hyperbolic flows, including frame flows on negatively curved manifolds satisfying a pinching condition.

2 An approximation result

Throughout this section, we assume that $r : X \to \mathbb{R}$ is a roof function, so r is positive almost everywhere, and $r \in L^1(X)$ with mean $\overline{r} = \int_X r\,d\mu$. We assume that $S : X \to X$ is measure-preserving and that r satisfies the SLLN, so

$$r_N = N\overline{r} + o(N) \text{ as } N \to \infty, \tag{2.1}$$

almost everywhere, where $r_N(x) = r(x) + r(Sx) + \cdots + r(S^{N-1}x)$.

It follows from (2.1) that $\lim_{N\to\infty} r_N = \infty$ almost everywhere. Given $T > 0$, we define $n[x,T]$ to be the largest integer n such that $r_n(x) \le T$. That is

$$r_{n[x,T]}(x) \le T < r_{n[x,T]+1}(x). \tag{2.2}$$

For almost every $x \in X$, $n[x,T]$ is defined for all $T > 0$ and $\lim_{T\to\infty} n[x,T] = \infty$ a.e. By the ergodic theorem, it follows easily (cf. [8, Lemma 11.2.1]) that

$$\lim_{T\to\infty} \frac{T}{n[x,T]} = \overline{r} \text{ a.e.} \tag{2.3}$$

Given $\phi : X_r \to \mathbb{R}$, we define $\Phi(x) = \int_0^{r(x)} \phi(x,u)du$ as usual, and $\Psi(x) = \int_0^{r(x)} |\phi(x,u)|du$. We have the following basic approximation result:

Lemma 2.1 *Suppose that $1 \le a,b,p \le \infty$, and $(1-\frac{1}{a})(1-\frac{1}{b}) \ge (1-\frac{1}{p})$. Let $r \in L^a(X)$, $\phi \in L^b(X_r)$. Then*

(a) *$\phi_T(x,0) = \Phi_{n[x,T]}(x) + o(T^{1/p})$ as $T \to \infty$, for almost all $x \in X$, and*

(b) *$\frac{1}{T^{1/p}}\big(\phi_T(x,u) - \Phi_{n[x,T]}(x)\big) \to 0$ as $T \to \infty$, in probability (on X_r).*

Remark 2.2 If $a = b = \infty$ in Lemma 2.1, then $\phi_T(x,u)$, $\phi_T(x,0)$ and $\Phi_{n[x,T]}(x)$ differ by amounts that are $O(1)$ almost everywhere. ∎

The remainder of this section is concerned with the proof of Lemma 2.1.

Proposition 2.3 *Suppose that $1 \le a,b,p \le \infty$, and $(1-\frac{1}{a})(1-\frac{1}{b}) \ge (1-\frac{1}{p})$. Let $r \in L^a(X)$, $\phi \in L^b(X_r)$. Then $\Phi, \Psi \in L^p(X)$ and $\|\Phi\|_p \le \|\Psi\|_p \le \|r\|_a \|\phi\|_b$.*

Proof Assume $1 < a,b,p < \infty$, hence $b < p$ (the other cases are easier).

By Hölder's inequality with $1/b' + 1/b = 1$, we obtain

$$\Psi(x) \le \Big(\int_0^{r(x)} 1\,dt\Big)^{1/b'} \Big(\int_0^{r(x)} |\phi(x,t)|^b\,dt\Big)^{1/b} = r(x)^{(b-1)/b}\Big(\int_0^{r(x)} |\phi(x,t)|^b\,dt\Big)^{1/b}.$$

We apply Hölder's inequality once more with $1/c + 1/d = 1$ where $d = b/p$, hence $c = b/(b-p)$, to obtain

$$\begin{aligned}\int_X \Psi(x)^p\,d\mu &\le \int_X r(x)^{p(b-1)/b}\Big(\int_0^{r(x)} |\phi(x,t)|^b\,dt\Big)^{p/b}\,d\mu \\ &= \int_X r(x)^{p(b-1)/b}\Big(\int_0^{r(x)} |\phi(x,t)|^b\,dt\Big)^{1/d}\,d\mu \\ &\le \Big(\int_X r(x)^{cp(b-1)/b}\,d\mu\Big)^{1/c}\Big(\int_X\int_0^{r(x)} |\phi(x,t)|^b\,dt\,d\mu\Big)^{1/d}\end{aligned}$$

which is finite provided that $r \in L^{p(b-1)/(b-p)}(X)$ and $\phi \in L^b(X_r)$.

Setting $p(b-1)/(b-p) = a$ leads to the required condition. ∎

Remark 2.4 The value of p is optimal, as can be seen from the following examples. If $a < \infty$, given $r \in L^a(X)$, take $\phi(x,t) = r(x)^{(a-1)/b}$; then $\phi \in L^b(X_r)$ and $\|\Phi\|_p = \|\Psi\|_p = \|r\|_a^{a/p}$, where $(1-\frac{1}{a})(1-\frac{1}{b}) = (1-\frac{1}{p})$. If $a = \infty$ take $\phi(x,t) = c(x)$ with $c \in L^b(X)$ and $\mathrm{supp}(c) \subset \{r \ge \|r\|_\infty/2\}$. ∎

For fixed $N \ge 1$, it is easily proved by induction that

$$\phi_{r_N(x)}(x,0) = \Phi_N(x). \tag{2.4}$$

Hence $|\phi_T(x,0) - \Phi_{n[x,T]}(x)| \le A_T(x)$, where $A_T(x) = \int_{r_{n[x,T]}(x)}^T |\phi \circ S_t(x,0)|\,dt$.

Proposition 2.5 *Suppose that r satisfies the SLLN (2.1) and $\Psi \in L^p(X)$ for some $p \geq 1$. Then $A_T = o(T^{1/p})$ as $T \to \infty$ a.e.*

Proof For $N = n[x,T]$ compute that

$$\begin{aligned} A_T(x) &\leq \int_{r_N(x)}^{r_{N+1}(x)} |\phi \circ S_t(x,0)|\, dt = \int_0^{r(S^N x)} |\phi \circ S_t \circ S_{r_N(x)}(x,0)|\, dt \\ &= \int_0^{r(S^N x)} |\phi \circ S_t(S^N x, 0)|\, dt = \Psi(S^N x). \end{aligned}$$

Since $\Psi \in L^p(X)$, the ergodic theorem applies to $\Psi^p \in L^1(X)$ so that $\Psi^p(S^n x) = o(n)$ almost everywhere. Hence, $\Psi(S^n x) = o(n^{1/p})$ almost everywhere. It follows that $A_T = o(n[x,T]^{1/p})$ almost everywhere. By (2.3), $A_T(x) = o(T^{1/p})$ almost everywhere. ∎

Part (a) of Lemma 2.1 follows immediately from Propositions 2.3 and 2.5. To prove part (b), compute that

$$\phi_T(x,u) = \textstyle\int_0^T \phi \circ S_t(x,u)dt = \int_0^T \phi \circ S_{t+u}(x,0)dt = \int_u^{T+u} \phi \circ S_t(x,0)dt,$$

and so by (2.4),

$$\begin{aligned} |\phi_T(x,u) - \Phi_{n[x,T]}(x)| &= |\phi_T(x,u) - \phi_{r_{n[x,T]}(x)}(x,0)| \\ &= \big|\textstyle\int_u^{T+u} \phi \circ S_t(x,0)dt - \int_0^{r_{n[x,T]}(x)} \phi \circ S_t(x,0)dt\big| \\ &\leq A_T(x) + B_T(x,u), \end{aligned} \tag{2.5}$$

where $B_T(x,u) = \int_T^{T+u} |\phi \circ S_t(x,0)|\, dt + \int_0^u |\phi \circ S_t(x,0)|\, dt$ for $u \in [0, r(x)]$.

Proposition 2.6 *If $r \in L^1(X)$, $\phi \in L^1(X_r)$, and $q > 0$, then $\frac{1}{T^q} B_T$ converges to zero in probability.*

Proof It suffices to estimate the first term in the definition of B_T, so we write $B_T(x,u) = \int_T^{T+u} |\phi \circ S_t(x,0)|\, dt$. Define $V_T(x) = B_T(x, r(x)) = \int_0^{r(x)} |\phi \circ S_T(x,t)|\, dt$. Since S_T is measure-preserving, $\|V_T\|_1 = \overline{r}\|\phi\|_1$ and so $V_T/T^q \to 0$ in L^1. In particular, $V_T/T^q \to 0$ in probability. Fix $\varepsilon > 0$ and define

$$E_T = \{x \in X : V_T(x)/T^q \geq \varepsilon\}, \qquad F_T = \{(x,u) \in X_r : B_T(x,u)/T^q \geq \varepsilon\}.$$

Then $\mu(E_T) \to 0$. Since $B_T(x,u) \leq V_T(x)$ and $r \in L^1$, it follows that $\mu_r(F_T) \leq (1/\overline{r}) \int_{E_T} r\, d\mu \to 0$. Hence $B_T/T^q \to 0$ in probability as required. ∎

Part (b) of Lemma 2.1 is immediate from equation (2.5) and Propositions 2.3, 2.5 and 2.6 (with $q = 1/p$.)

3 Central Limit Theorem

In this section, we prove Theorem 1.3 extending the central limit theorem (CLT) from observations on X to observations on the suspension flow X_r.

Since two different measures μ and μ_r are involved, the following device for passing between X and X_r is required.

Lemma 3.1 *Let $S : X \to X$ be ergodic and let X_r be the suspension with respect to the L^1 roof function r. Let $\Phi : X \to \mathbb{R}$ be an L^2 observation and set $\Phi_N = \sum_{j=0}^{N-1} \Phi \circ S^j$.*

Define $\widehat{\Phi}_N : X_r \to \mathbb{R}$ by $\widehat{\Phi}_N(x,u) = \Phi_N(x)$. If $\Phi_N/\sqrt{N} \to_d G$ on X, then $\widehat{\Phi}_N/\sqrt{N} \to_d G$ on X_r.

Proof It follows from Eagleson [12, Section 4.2a] (see also [1, Section 3.6]) that $\Phi_N/\sqrt{N}$ converges in distribution to G relative to any probability measure ν that is absolutely continuous with respect to the underlying ergodic measure μ. Taking ν to be the measure with density $r/\overline{r}$, we have $\int_X e^{it\Phi_N/\sqrt{N}} r/\overline{r} d\mu \to E(e^{itG})$ for all $t \in \mathbb{R}$. But

$$\int_{X_r} e^{it\widehat{\Phi}_N/\sqrt{N}} d\mu_r = (1/\overline{r}) \int_X \int_0^r e^{it\Phi_N/\sqrt{N}} d(\mu \times \ell) = \int_X e^{it\Phi_N/\sqrt{N}} r/\overline{r} d\mu.$$

Hence $\int_{X_r} e^{it\widehat{\Phi}_N/\sqrt{N}} d\mu_r \to E(e^{itG})$ and so $\widehat{\Phi}_N/\sqrt{N} \to_d G$. ∎

Remark 3.2 The result of Eagleson [12] used in the above argument states that central limit theorems for ergodic sequences are automatically "mixing in the sense of Rényi" (a concept introduced in [26]). ∎

Now let $\phi : X_r \to \mathbb{R}$ be a mean zero observation. We assume that $r \in L^a(X)$, $\phi \in L^b(X_r)$ where $(1 - \frac{1}{a})(1 - \frac{1}{b}) \geq \frac{1}{2}$. Recall that $\Phi(x) = \int_0^{r(x)} \phi(x,u) du$ and define $\widehat{\Phi}(x,u) = \Phi(x)$ as in Lemma 3.1. Let $S : X \to X$ and Φ_N, $N \in \mathbb{Z}$, be as described in Theorem 1.3. Define

$$m[x,T] = n[x,T] - [T/\overline{r}] \qquad \text{and} \qquad W_T(x) = \frac{1}{\sqrt{T}} \Phi_{m[x,T]}(S^{[T/\overline{r}]}x).$$

Lemma 3.3 *Suppose that $r \in L^1(X)$ satisfies the SLLN* (2.1) *and that*

$$W_T \to_d 0 \;\text{ on } X \quad \text{and} \quad \frac{1}{\sqrt{N}} \widehat{\Phi}_N \to_d G \;\text{ on } X_r \text{ as } T, N \to \infty.$$

Then

$$\frac{1}{\sqrt{T}} \widehat{\Phi}_{n[x,T]} \to_d G/\sqrt{\overline{r}} \;\text{ on } X_r.$$

Proof By (2.3), it is equivalent to show that $\frac{1}{\sqrt{n[x,T]}}\widehat{\Phi}_{n[x,T]}(x,u) \to_d G$ on X_r. We compute that

$$\frac{1}{\sqrt{n[x,T]}}\widehat{\Phi}_{n[x,T]}(x,u) - \frac{1}{\sqrt{[T/\overline{r}]}}\widehat{\Phi}_{[T/\overline{r}]}(x,u) = \frac{1}{\sqrt{n[x,T]}}\Big(\widehat{\Phi}_{n[x,T]}(x,u) - \widehat{\Phi}_{[T/\overline{r}]}(x,u)\Big)$$
$$+ \Big(\frac{1}{\sqrt{n[x,T]}} - \frac{1}{\sqrt{[T/\overline{r}]}}\Big)\widehat{\Phi}_{[T/\overline{r}]}(x,u)$$
$$= \mathrm{I}+\mathrm{II}.$$

By definition of $\widehat{\Phi}_N$, term I is independent of u and

$$\mathrm{I} = \frac{1}{\sqrt{n[x,T]}}\Big(\Phi_{[T/\overline{r}]+m[x,T]}(x) - \Phi_{[T/\overline{r}]}(x)\Big) = \frac{\sqrt{T}}{\sqrt{n[x,T]}}\frac{1}{\sqrt{T}}\Phi_{m[x,T]}(S^{[T/\overline{r}]}x)$$
$$= \frac{\sqrt{T}}{\sqrt{n[x,T]}}W_T(x) \to_d 0 \text{ on } X$$

by (2.3) and the hypothesis on W_T. Since $r \in L^1(X)$, it follows that $\mathrm{I} \to_d 0$ on X_r.

Next, working on X_r,

$$\mathrm{II} = \Big(\frac{\sqrt{[T/\overline{r}]}}{\sqrt{n[x,T]}} - 1\Big)\Big(\frac{1}{\sqrt{[T/\overline{r}]}}\widehat{\Phi}_{[T/\overline{r}]}(x,u)\Big).$$

The second factor converges in distribution. The first factor converges (by (2.3)) to 0 a.e. Hence $\mathrm{II} \to_d 0$ on X_r. ∎

The key part of the argument is to verify that $W_T \to_d 0$. This can be carried out on X, with no mention of the suspension X_r.

Lemma 3.4 *Let $r, \Phi \in L^1(X)$, $r > 0$. Suppose that there exist $\alpha \in (0,1]$, $\beta \in (0,1)$ with $\alpha\beta \le \frac{1}{2}$ such that*

(i) $\frac{1}{|N|^\alpha}\Phi_N \to 0$ a.e. *as $N \to \pm\infty$, and*

(ii) $\frac{1}{N^\beta}(r_N - N\overline{r})$ *converges in distribution as $N \to \infty$.*

Then $W_T \to_d 0$ as $T \to \infty$.

Proof Fix $a > 0$. We show that $\mu(|W_T| \ge a) \to 0$ as $T \to \infty$.

Let $\varepsilon > 0$. By hypothesis (i), we can choose a set $\widetilde{X} \subset X$ with $\mu(\widetilde{X}) > 1 - \varepsilon$ and N_0 such that $|\frac{1}{M^\alpha}\Phi_M| < \varepsilon$ on $\widetilde{X}$ for all $|M| \ge N_0$. For each value of T, we define

$$X'_T = \{x \in X : |m[x,T]| \le N_0\}, \qquad X''_T = \{x \in X : |m[x,T]| > N_0\}.$$

First, we estimate the contribution from X'_T. For $x \in X'_T$, we have

$$|W_T(x)| \le \frac{1}{\sqrt{T}} \sum_{j=-N_0}^{N_0-1} |\Phi \circ S^j(S^{[T/\overline{r}]}(x))| = \frac{1}{\sqrt{T}} \Psi(S^{[T/\overline{r}]}(x)),$$

where $\Psi = \sum_{j=-N_0}^{N_0-1} |\Phi \circ S^j|$. Since Ψ lies in L^1, there is a constant C and a subset $Y \subset X$ with $\mu(Y) > 1 - \varepsilon$ such that $\Psi(y) \le C$ for all $y \in Y$. Let $Y_T = X'_T \cap (S^{[T/\overline{r}]})^{-1}(Y)$. Then $\mu(Y_T) > \mu(X'_T) - \varepsilon$ (since S is measure-preserving), and $\Psi(S^{[T/\overline{r}]}(x)) < C$ for all $x \in Y_T$. Hence

$$\begin{aligned} \mu(x \in X'_T : |W_T(x)| \ge a) &< \mu(x \in Y_T : |W_T(x)| \ge a) + \varepsilon \\ &\le \mu(x \in Y_T : \tfrac{1}{\sqrt{T}} C \ge a) + \varepsilon = \varepsilon, \end{aligned}$$

for T sufficiently large. Therefore, we can choose T_0 so that

$$\mu(|W_T| \ge a) \le \mu(x \in X''_T : |W_T(x)| \ge a) + \varepsilon, \text{ for all } T \ge T_0.$$

Next, we estimate the contribution from X''_T. Let $\widetilde{X}''_T = X''_T \cap (S^{[T/\overline{r}]})^{-1}(\widetilde{X})$ and note that $\mu(\widetilde{X}''_T) \ge \mu(X''_T) - \varepsilon$ (since S is measure-preserving). Hence,

$$\mu(|W_T| \ge a) \le \mu(x \in \widetilde{X}''_T : |W_T(x)| \ge a) + 2\varepsilon.$$

Write

$$W_T(x) = \frac{|m[x,T]|^\alpha}{\sqrt{T}} \, \frac{1}{|m[x,T]|^\alpha} \Phi_{m[x,T]}(S^{[T/\overline{r}]}(x)).$$

On X''_T, we have $|m[x,T]| > N_0$ and so $\left|\frac{1}{|m[x,T]|^\alpha} \Phi_{m[x,T]}(y)\right| < \varepsilon$ for all $x \in X''_T$ and $y \in \widetilde{X}$. It follows from the definitions that $\left|\frac{1}{|m[x,T]|^\alpha} \Phi_{m[x,T]}(S^{[T/\overline{r}]}(x))\right| < \varepsilon$ on $\widetilde{X}''_T$. Hence $|W_T(x)| < \varepsilon \frac{|m[x,T]|^\alpha}{\sqrt{T}}$ for $x \in \widetilde{X}''_T$, so that

$$\begin{aligned} \mu(|W_T| \ge a) &\le \mu\Big(x \in \widetilde{X}''_T : \frac{|m[x,T]|^\alpha}{\sqrt{T}} \ge a/\varepsilon\Big) + 2\varepsilon \\ &\le \mu\Big(\frac{|m[x,T]|}{T^{1/(2\alpha)}} \ge (a/\varepsilon)^{1/\alpha}\Big) + 2\varepsilon \le \mu\Big(\frac{|m[x,T]|}{T^\beta} \ge (a/\varepsilon)^{1/\alpha}\Big) + 2\varepsilon, \end{aligned}$$

since $\alpha\beta \le \frac{1}{2}$. A direct calculation starting from hypothesis (ii) shows that the sequence $\frac{|m[x,T]|}{T^\beta}$ converges in distribution (cf. [25, p. 188]). Denote the limit distribution by ν (seen as a probability measure on the real line). Then

$$\mu\Big(\frac{|m[x,T]|}{T^\beta} \ge (a/\varepsilon)^{1/\alpha}\Big) \to \nu\big(|t| \ge (a/\varepsilon)^{1/\alpha}\big),$$

and hence there exists $T_\varepsilon \ge T_0$ such that

$$\mu(|W_T| \ge a) \le \nu\big(|t| \ge (a/\varepsilon)^{1/\alpha}\big) + 3\varepsilon$$

for $T \ge T_\varepsilon$. Since $\nu(|t| \ge A) \to 0$ as $A \to \infty$, this completes the proof. ∎

Proof of Theorem 1.3 Hypotheses (i) and (ii) of the theorem translate immediately into the corresponding hypotheses for Lemma 3.4. Hence $W_T \to_d 0$ on X. By Lemmas 3.1 and 3.3, $\frac{1}{\sqrt{T}}\widehat{\Phi}_{n[x,T]} \to_d G/\sqrt{\bar{r}}$ on X_r. Finally, by Lemma 2.1(b) with $p = 2$, $\frac{1}{\sqrt{T}}\phi_T \to_d G/\sqrt{\bar{r}}$ on X_r. ∎

4 LIL and ASIP for suspension flows

In this section, we show how to extend the LIL and ASIP from observations on X to observations on the suspension flow X_r. As usual, $\phi : X_r \to \mathbb{R}$ has mean zero and $\Phi(x) = \int_0^{r(x)} \phi(x,u)du$.

We say that ϕ satisfies the LIL if

(LIL) $$\limsup_{T\to\infty} \frac{\phi_T}{\sqrt{2T\log\log T}} = \sigma \;\text{ a.e.}$$

and similarly for Φ.

Theorem 4.1 *Suppose that $S : X \to X$ is a measure-preserving transformation and that $S_t : X_r \to X_r$ is the suspension flow corresponding to a roof function $r : X \to \mathbb{R}$ satisfying the SLLN (2.1). Suppose that $r \in L^a(X)$ and $\phi \in L^b(X_r)$ where $(1-\frac{1}{a})(1-\frac{1}{b}) \ge \frac{1}{2}$.*

If Φ satisfies the LIL with variance σ_1^2, then ϕ satisfies the LIL with variance $\sigma^2 = \sigma_1^2/\bar{r}$.

Proof Define $q : (3,\infty) \to (0,\infty)$ by $q(T) = (2T\log\log T)^{-1/2}$. Noting that $T \mapsto n[x,T]$ is a monotone surjection, we deduce that $\limsup_{T\to\infty} q(n[x,T])\Phi_{n[x,T]}(x) = \sigma_1$ a.e. By (2.3), $\limsup_{T\to\infty} q(T)\Phi_{n[x,T]}(x) = \sigma$ a.e. It follows from Lemma 2.1(a) that

$$\limsup_{T\to\infty} q(T)\phi_T(x,0) = \sigma \;\text{ a.e.} \tag{4.1}$$

Choose $X' \subset X$ with $\mu(X') = 1$ such that (4.1) holds for all $x \in X'$. Then

$$\limsup_{T\to\infty} q(T)\phi_T(x,u) = \limsup_{T\to\infty} q(T)\phi_{T+u}(x,0) = \limsup_{T\to\infty} q(T)\phi_T(x,0) = \sigma$$

for all $x \in X'$ and $u \in [0, r(x)]$. ∎

We say that ϕ satisfies the ASIP if (after redefining $\{\phi_T \mid T \geq 0\}$ on a richer probability space without changing the joint distributions of ϕ_T) there exists $\varepsilon > 0$ and a Brownian motion W with variance σ^2 such that $\phi_T = W(T) + o(T^{1/2-\varepsilon})$ as $T \to \infty$ almost everywhere. Similarly for Φ. Many statistical laws including the CLT and LIL are implied by the ASIP, see Philipp and Stout [24].

Theorem 4.2 *Let $S_t : X_r \to X_r$ be the suspension flow corresponding to a roof function $r : X \to \mathbb{R}$. Suppose that $r \in L^a(X)$ and $\phi \in L^b(X_r)$ where $(1-\frac{1}{a})(1-\frac{1}{b}) > \frac{1}{2}$. Suppose further that*

$$r_N = N\overline{r} + o(N^{1-\delta}) \ \text{ as } N \to \infty \tag{4.2}$$

almost everywhere, for some $\delta > 0$.

If Φ satisfies the ASIP with variance σ_1^2, then ϕ satisfies the ASIP with variance $\sigma^2 = \sigma_1^2/\overline{r}$.

Proof By assumption (shrinking δ if necessary), there is a sequence of random variables $\{S_N,\ N \geq 1\}$ equal in distribution to the sequence $\{\Phi_N,\ N \geq 1\}$ such that

$$S_N = W_1(N) + o(N^{1/2-\delta}) \ \text{ as } N \to \infty \tag{4.3}$$

almost everywhere, where W_1 is a Brownian motion with variance σ_1^2.

Thanks to condition (4.2), we can strengthen (2.3) to conclude that

$$n[x,T] = T/\overline{r} + o(T^{1-2\delta}) \ \text{ as } T \to \infty, \tag{4.4}$$

almost everywhere, where for convenience we have replaced δ by 2δ.

Shrinking δ once again as necessary so that $(1-\frac{1}{a})(1-\frac{1}{b}) > \frac{1}{2} + \delta$, Lemma 2.1(a) guarantees that

$$\phi_T(x,0) = \Phi_{n[x,T]}(x) + o(T^{1/2-\delta}) \ \text{ as } T \to \infty \text{ a.e.}$$

Moreover, r is finite almost everywhere, so for μ-a.e. x and each $u \in [0, r(x)]$,

$$\phi_T(x,u) = \phi_{T+u}(x,0) = \Phi_{n[x,T+u]}(x) + o(T^{1/2-\delta}) \ \text{ as } T \to \infty. \tag{4.5}$$

As in [24, p. 23], without loss we may identify Φ_N with S_N and we may suppose that they are defined on the same probability space as ϕ_T and W_1. Hence combining (4.3) and (4.5) yields

$$\phi_T(x,u) = W_1(n[x, T+u]) + o(T^{1/2-\delta}) \ \text{ as } T \to \infty \text{ a.e.}$$

By equation (4.4), $n[x,T+u] = T/\overline{r} + o(T^{1-2\delta})$ and so it follows as in [24, p. 10 and p. 24] that for any $\delta' < \delta$,

$$W_1(n[x,T+u]) = W_1(T/\overline{r}) + o(T^{1/2-\delta'}) = W(T) + o(T^{1/2-\delta'}),$$

where W is a Brownian motion with variance $\sigma^2 = \sigma_1^2/\overline{r}$. Hence

$$\phi_T(x,u) = W(T) + o(T^{1/2-\delta'}) \text{ as } T \to \infty,$$

almost everywhere, as required. ∎

Remark 4.3 (a) Condition (4.2) is satisfied if $r \in L^2(X)$ and r is cohomologous in L^2 to a martingale [13, p. 238]. A result of Gordin [16] guarantees that r is cohomologous to a martingale in a wide class of examples.

(b) Improved error terms in the ASIP for Φ and in (4.2) can be used to obtain an improved error term in the ASIP for ϕ. So if Φ satisfies the ASIP with error $o(N^{1/p})$ for some $p > 2$, then ϕ satisfies the ASIP with error $o(T^{1/p'})$, for any $p' < p$, provided that $r_N = N\overline{r} + o(N^{2/p})$ and $(1 - \frac{1}{a})(1 - \frac{1}{b}) \geq (1 - \frac{1}{p})$.

(c) When X is Axiom A, Hölder observations satisfy the ASIP with error term $o(N^{1/4+\alpha})$ for any $\alpha > 0$, see [14]. Moreover, Hölder roof functions satisfy the condition in part (a) (cf. [9, 14]). Hence, when r and ϕ are Hölder, we obtain the ASIP with error term $o(T^{1/4+\alpha})$ improving slightly upon the result of [9]. ∎

5 Applications

Hyperbolic flows As discussed in the introduction, Ratner [25] established the CLT for Hölder observations on a hyperbolic basic set X, and also (by a quite complicated argument) the CLT for Hölder observations on the suspension X_r by a Hölder roof function r. The CLT for general hyperbolic flows then follows from Bowen [3]. In this paper (see Theorem 1.1), we have given an elementary and general argument that yields the CLT for hyperbolic flows, based on the CLT for hyperbolic basic sets. (See Remark 1.2 for a comparison of the two proofs.)

Nondegeneracy The statistical limit theorems in this paper are said to be *degenerate* if $\sigma^2 = 0$ (equivalently $\sigma_1^2 = 0$). In our abstract setting, it is not possible to formulate useful criteria for nondegeneracy.

However, if Φ is a Hölder continuous observation on a hyperbolic basic set X equipped with a Gibbs measure μ, then it is well-known that $\sigma_1^2 = 0$ if and only if there is a constant K_1 such that $|\Phi_N|_\infty \leq K_1$ for all N, which is equivalent to the fact that Φ is a Hölder coboundary (see [14, §5] for an explicit statement). Provided r and ϕ are Hölder, so in particular $\phi_T(x,u) = \Phi_{n[x,T]}(x) + O(1)$, we conclude that $\sigma^2 = 0$ if and only if there is a constant K such that $|\phi_T|_\infty \le K$ for all T.

This condition for degeneracy has significant advantages over the L^2 condition in [25]. For continuous observations, it means that $\phi_p(x,u) = 0$ whenever $S_p(x,u) = (x,u)$ (equivalently, $\Phi_P(x) = 0$ whenever $S^P x = x$) so we can guarantee nondegeneracy by perturbing the observation along a single periodic trajectory.

Nonuniformly hyperbolic flows The CLT has been established for certain classes of nonuniformly hyperbolic and partially hyperbolic diffeomorphisms. The CLT for the suspension flow follows from Theorem 1.1.

For example, Young [30, 31] proves the CLT for smooth enough observations of a class of nonuniformly hyperbolic diffeomorphisms. By Theorem 1.1, we have

Corollary 5.1 *Provided r and ϕ are smooth enough, the CLT (and its functional version) hold for suspension flows over the nonuniformly hyperbolic transformations considered in [30, 31].* ∎

Nonuniformly hyperbolic diffeomorphisms The methods in this paper apply also to nonuniformly hyperbolic diffeomorphisms (and more generally to induced transformations). The tower construction in Young [30, 31] realises such diffeomorphisms as discrete suspensions over a uniformly hyperbolic base, where the roof function r takes only integer values. Provided r lies in L^2, the results in this paper immediately yield the CLT. The condition that r is L^2 is more natural and general than the condition in [31] (though the emphasis there is on establishing rates of decay of correlations which is a much harder problem).

The application of the ideas in this paper to nonuniformly hyperbolic diffeomorphisms has been carried out recently by Gouëzel [18] (our main results are summarized in [18, Appendix A]). This provides a significantly simplified proof of results in Gouëzel [17].

A class of partially hyperbolic flows Suppose that Λ is a hyperbolic basic set for a smooth flow S_t with an equilibrium measure μ_Λ corresponding to a Hölder potential, and G is a compact connected Lie group with Haar measure ν. If $h_t : \Lambda \to G$ is a smooth cocycle ($h_{s+t} = h_s\, h_t \circ S_s$), then we form the G-extension flow $S_{t,h}$ on $\Lambda \times G$, $S_{t,h}(y,g) = (S_t(y), gh_t(y))$, with invariant measure $\mu_\Lambda \times \nu$. Such G-extensions are amongst the simplest examples of partially hyperbolic flows (We note that the arguments in [9, 25] do not seem to apply to such flows.)

Theorem 5.2 *Let G be a semisimple compact connected Lie group and suppose that $\Lambda \times G$ is an ergodic G-extension of a hyperbolic basic set Λ for a hyperbolic flow. Then the CLT (and its functional version) hold for sufficiently smooth observations $\phi : \Lambda \times G \to \mathbb{R}$. Moreover, the LIL and ASIP hold for sufficiently smooth observations.*

Proof By Bowen [3], Λ can be modelled as a symbolic flow X_r where X is a subshift of finite type, $r : X \to \mathbb{R}$ is a Hölder roof function, with equilibrium measure μ_r where μ is an equilibrium measure on X. Hence $\Lambda \times G$ is realised as $X_r \times G \cong (X \times G)_r$ with ergodic measure $(\mu \times \nu)_r$. (We regard r as defined on $X \times G$ but independent of the G-component.) Since $\Lambda \times G$ is ergodic, it follows that $\mu \times \nu$ is an ergodic measure on $X \times G$.

Dolgopyat [10, Corollary 4.8] proves rapid mixing for Hölder observations on $X \times G$ that are sufficiently smooth in the G direction and deduces the CLT using [20]. The CLT follows also from [22] and so does the ASIP. By the results in this paper, these results lift to the suspension $(X \times G)_r$ and hence to $\Lambda \times G$. ∎

Remark 5.3 (Application to frame flows) Let V be a negatively-curved orientable n-dimensional manifold, with unit tangent bundle $M = SV$. Let $\widehat{M}$ be the space of positively oriented orthonormal n-frames in TV. The frame flow on $\widehat{M}$ is an $\mathbf{SO}(n-1)$-extension of the geodesic flow on M. Recall that $\mathbf{SO}(n-1)$ is semisimple provided $n \geq 4$. Ergodicity of the frame flow has been proven by [4] when n is odd, $n \neq 7$, and by [5, 6] in the remaining cases under a pinching condition on the curvature. By Theorem 5.2, the ASIP etc. hold for sufficiently smooth observations on such frame flows. ∎

If G is not assumed to be semisimple, then the situation is more complicated. Below, we describe some results for suspensions $(X \times G)_r$ where X is a hyperbolic basic set for a diffeomorphism. Note that the hypotheses are now on $X \times G$ and for this reason the results are considerably weaker than in the semisimple case.

Suppose that X is a hyperbolic basic set with Gibbs measure μ and G is a compact connected Lie group with Haar measure ν. If $S : X \to X$ is the underlying Axiom A dynamics and $h : X \to G$ is a smooth cocycle, the G-extension $S_h : X \times G \to X \times G$ is given by $S_h(x, g) = (Sx, gh(x))$. Field *et al.* [15] showed that the G-extension $X \times G$ is ergodic with respect to $\mu \times \nu$ for an open dense set of smooth extensions $h : X \to G$. Such a G-extension S_h is said to be *stably ergodic*.

For restricted classes of G-extensions (G semisimple covered above, but also X Anosov infranilmanifold), Dolgopyat [10] proved that stable ergodicity implies rapid decay of correlations, and hence the CLT, for sufficiently regular observations $\Phi : X \times G \to \mathbb{R}$.

Corollary 5.4 *Provided r and ϕ are smooth enough, the CLT (and its functional version) hold for suspension flows over the G-extensions considered in [10].* ∎

In the context of equivariant flows, it is natural to consider *equivariant* observations [23, 14], and stronger results are possible.

Theorem 5.5 *Let G be a compact connected Lie group and suppose that $Y \times G$ is an ergodic G-extension of a hyperbolic basic set Y for a hyperbolic flow. Let ρ be a representation of G on the vector space $\mathbb{R}^k$, and consider mean zero observations $\phi : Y \times G \to \mathbb{R}^k$ of the form $\phi(y, g) = \rho_g v(y)$, where $v : Y \to \mathbb{R}^k$ is Hölder. Then ϕ satisfies a k-dimensional CLT, and each one-dimensional projection of ϕ satisfies the ASIP.*

Furthermore, the CLT is degenerate (singular covariance matrix) if and only if some component of ϕ_T is uniformly bounded, and the CLT is nondegenerate for an open and dense set of equivariant Hölder observations ϕ.

Proof As usual, we can reduce to the case where $Y = X_r$ is the suspension of a subshift of finite type X and r is Hölder. Then $Y \times G = X_r \times G = (X \times G)_r$.

Define $\Phi(x, g) = \int_0^{r(x)} \phi(x, g, u)du = \rho_g V(x)$ where $V(x) = \int_0^{r(x)} v(x, u)du$. Then $\Phi : X \times G \to \mathbb{R}^k$ is a Hölder equivariant observation and it follows from [14, 21] that Φ satisfies the CLT. The CLT for ϕ follows from our main results and the Cramer-Wold technique. Similarly, it follows from [14] and Theorem 4.2 that each one-dimensional projection of ϕ satisfies the ASIP.

As in the Axiom A case, the statements about nondegeneracy for ϕ follow from the corresponding statements for Φ which were obtained by Nicol *et al.* [23]. ∎

Remark 5.6 The ideas in [14, 23] were applied by Ashwin *et al.* [2] to provide an explanation of hypermeander of spiral waves in planar excitable media. There, the spiral tip undergoes Brownian-like motion in the plane. Theorem 5.5 bridges the gap between discrete and continuous time [2, Section 5]. ∎

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