A note on the shadowing lemma and symmetric periodic points *

Michael Dellnitz † Ian Melbourne ‡
Department of Mathematics
University of Houston
Houston, Texas 77204-3476
USA

August 28, 1995

Abstract

A standard use of shadowing lemmas in dynamical systems theory is to prove density of periodic points. When there is symmetry present it is natural to investigate the density of periodic points of a given symmetry. We obtain several results in this direction. As an application, we prove a result about compact group extensions of hyperbolic and nonuniformly hyperbolic dynamical systems.

1 Introduction

Suppose that A is a topologically transitive set for a discrete dynamical system $f : \mathbb{R}^n \to \mathbb{R}^n$. If A possesses a (strong) shadowing property, then it is well known (see for example Shub [20] or Pollicott [19]) that periodic points

*Appeared: *Nonlinearity* **8** (1995) 1067-1075

†Research supported in part by the Deutsche Forschungsgemeinschaft and ONR Grant N00014-94-1-0317. Permanent address: Institut für Angewandte Mathematik, Universität Hamburg, D-20146 Hamburg, Germany.

‡Supported in part by NSF Grant DMS-9403624, by ONR Grant N00014-94-1-0317 and by the CNRS

are dense in A. Suitable versions of the shadowing lemma hold for hyperbolic sets [1, 20] and more generally for nonuniformly hyperbolic sets [10, 19].

When there is symmetry present, we can ask whether periodic orbits with a given amount of symmetry are dense. More precisely, suppose that Γ is a finite group acting on $\mathbb{R}^n$ and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant homeomorphism with $A \subset \mathbb{R}^n$ a topologically transitive set. Following [6, 14] we define the *symmetry group* of A to be the subgroup

$$\Sigma_A = \{\gamma \in \Gamma,\ \gamma A = A\}.$$

Thus elements of Σ_A fix A as a set. We say that A is Σ-*symmetric* if $\Sigma_A = \Sigma$ and A contains at least one point of trivial isotropy. (It follows that the points in $\mathbb{R}^n$ with trivial isotropy form an open dense subset [2] and hence that the set of points in A with trivial isotropy is open and dense in A.)

If a periodic orbit is Σ-symmetric, then it is easily shown that Σ is a cyclic subgroup of Γ (see, for example, [14, Corollary 4.4]). From now on we assume (without loss of generality) that A is Γ-symmetric. We are interested in determining those cyclic subgroups $\Sigma \subset \Gamma$ for which Σ-symmetric periodic orbits are dense in A.

In Section 2, we review the definition(s) of shadowing. We prove that if A is a topologically transitive set satisfying the (strong) shadowing property then periodic orbits with *maximal* cyclic symmetry are dense. For example, if A is $\mathbb{Z}_k$-symmetric, we deduce that $\mathbb{Z}_k$-symmetric periodic orbits are dense in A. However we cannot, in this generality, make a statement about periodic orbits in A that have less (or no) symmetry. In Section 3, we show that if in addition A is topologically mixing then Σ-symmetric periodic orbits are indeed dense in A for all cyclic subgroups $\Sigma \subset \Gamma$. In Section 4, we make the nontrivial extension to sets that are topologically mixing up to a cycle. In particular, our results are valid for topologically transitive sets that are hyperbolic [1, 20] or nonuniformly hyperbolic [18]. Our results generalize to the case when there are continuous symmetries present, see Section 5.

An important consequence of our work is that we obtain sufficient conditions for density of asymmetric periodic orbits (that is, periodic orbits with no symmetry). This is of significance for problems involving compact group extensions. We discuss these issues in Section 6.

2 Shadowing properties and density of periodic orbits

In this section, we state the shadowing lemma as satisfied by hyperbolic sets and derive a result on density of periodic orbits with maximal cyclic symmetry as a consequence. Also we indicate the main differences for nonuniformly hyperbolic sets.

First, we give the definition of the shadowing property that is satisfied by hyperbolic topologically transitive sets. For reasons explained in Remark 2.8 below, we call this the *strong* shadowing property.

Definition 2.1 Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, that $A \subset \mathbb{R}^n$ and that $\delta > 0$. A sequence $\mathbf{y} = \{y_m\}_{m=-\infty}^{+\infty}$ of points $y_m \in A$ is a *δ-pseudoorbit in A* if $|f(y_m) - y_{m+1}| < \delta$ for all m.

Given $\epsilon > 0$, $x \in A$ is an *ϵ-tracing point in A* for a pseudoorbit $\mathbf{y}$ in A if $|f^m(x) - y_m| < \epsilon$ for all m.

Definition 2.2 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. A subset $A \subset \mathbb{R}^n$ has the *strong shadowing property* if

(i) Given $\epsilon > 0$ there exists $\delta > 0$ such that every δ-pseudoorbit in A has an ϵ-tracing point in A, and

(ii) There is an $\epsilon_0 > 0$ such that the ϵ-tracing points in (i) are unique provided $\epsilon < \epsilon_0$.

Proposition 2.3 *Suppose that A is a topologically transitive set.*

(a) If A is hyperbolic, then A satisfies the strong shadowing property.

(b) If A satisfies the strong shadowing property, then periodic points are dense in A.

Proof Part (a) can be found, for example, in [20, Chapter 8] (combination of Propositions 8.7, 8.11 and 8.20). Part (b) is also well-known, and is a special case of Lemma 2.5. ∎

Remark 2.4 We can drop the transitivity assumption in Proposition 2.3(a) provided we insist that A has a local product structure. This is a consequence of Smale's spectral decomposition theorem [20, 21] which states that a hyperbolic set with local product structure is a disjoint union of finitely many transitive sets each of which is topologically mixing up to a cycle. The corresponding result for nonuniformly hyperbolic sets is due to Pesin [18] (there may now be a countable infinity of transitive sets in the decomposition).

A consequence of the spectral decomposition theorems is that all of the results stated in this paper are valid more generally for (nonuniformly) hyperbolic sets with local product structure

Proposition 2.3(b) has the following analogue in the equivariant context.

Lemma 2.5 *Let Γ be a finite group acting orthogonally on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Γ-equivariant homeomorphism with a Γ-symmetric topologically transitive set A. Suppose that A satisfies the strong shadowing property. If Σ is a maximal cyclic subgroup of Γ, then Σ-symmetric periodic orbits are dense in A.*

Proof Let $\Sigma \subset \Gamma$ be a maximal cyclic subgroup. Since points of trivial isotropy are dense in A, it is sufficient to prove that periodic orbits P with $\Sigma_P = \Sigma$ are dense in A. Choose σ to be a generator of Σ and let U be a nonempty open subset of A. We show that there is a point $x \in U$ such that $f^q(x) = \sigma x$ for some integer q. Then x is a periodic point and $\sigma \in \Sigma_P$ where P is the corresponding periodic orbit. It follows that $\Sigma \subset \Sigma_P$. Since Σ_P is cyclic and Σ is maximal, we have $\Sigma_P = \Sigma$.

It remains to construct the periodic point x. Choose $z \in U$ and let $\epsilon, \delta > 0$ be such that every δ-pseudoorbit has a unique ϵ-tracing point and such that $B_{\delta+\epsilon}(z) \in U$. Let $V = B_\delta(z)$. By topological transitivity, $f^q(V) \cap \sigma V \neq \emptyset$ for some integer q.

Let k denote the order of σ (so $\sigma^k = 1$). Choose $y_0 \in V$ so that $f^q(y_0) \in \sigma V$ and define $y_0, y_1, \ldots, y_{kq-1}$ to be the sequence

$$\begin{array}{cccc} y_0, & f(y_0), & \ldots, & f^{q-1}(y_0); \\ \sigma y_0, & \sigma f(y_0), & \ldots, & \sigma f^{q-1}(y_0); \\ \vdots & \vdots & & \vdots \\ \sigma^{k-1} y_0, & \sigma^{k-1} f(y_0), & \ldots, & \sigma^{k-1} f^{q-1}(y_0). \end{array}$$

Extend to a kq-periodic bi-infinite sequence $\mathbf{y}$. Since V and hence γV has diameter δ for all $\gamma \in \Gamma$, $\mathbf{y}$ is a δ-pseudoorbit with ϵ-tracing point x say. In addition $y_0 \in B_\delta(z)$, and hence $x \in B_\epsilon(y_0) \subset B_{\delta+\epsilon}(z) \subset U$. Moreover, $\sigma^{-1}f^q(x)$ is also an ϵ-tracing point so by uniqueness $\sigma^{-1}f^q(x) = x$. ∎

Remark 2.6 The shadowing property as stated in Definition 2.2 holds for hyperbolic transitive sets [20] but only in a slightly more complicated form for nonuniformly hyperbolic transitive sets [19]. The most important difference is that tracing points are no longer guaranteed to lie in A. However, periodic tracing points can be shown to lie in A [9] (see also [19, p. 96]) and Lemma 2.5 remains valid for this more complicated notion of shadowing.

Corollary 2.7 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Γ-equivariant diffeomorphism. Suppose that A is a Γ-symmetric (nonuniformly) hyperbolic topologically transitive set and that Σ is a maximal cyclic subgroup of Γ. Then Σ-symmetric periodic orbits are dense in A.*

We improve substantially upon this result in later sections.

Remark 2.8 As pointed out to us by the referee, the 'standard' notion of shadowing does not require that the tracing point lies in A and, in general, even tracing points that are periodic need not lie in A. Indeed, there are are examples such as the Feigenbaum limit set which satisfy the 'standard' shadowing property but nonuniform hyperbolicity fails and A contains *no* periodic points. In this case, A is contained in the closure of the periodic points.

We emphasize that the results in this paper are valid only when the strong shadowing property is satisfied (that is, periodic tracing points lie in A) and that this property is only known to be valid under the assumption of hyperbolicity or nonuniformly hyperbolicity. However, many of our results, in particular, Lemma 2.5, Theorem 3.2 and Theorem 4.3, have an obvious generalization to the case when the 'standard' shadowing property is in force. For example, the conclusion of Lemma 2.5 becomes: 'If Σ is a maximal cyclic subgroup of Γ, then A is contained in the closure of the Σ-symmetric periodic points'.

In the remainder of this paper, we shall suppress the adjectives 'strong/standard'. All of our results are valid in the strong case and the results in Sections 3 and 4 are valid in the standard case with the slightly weakened conclusion described in the previous paragraph. However, the results in Section 5 depend crucially on the density of periodic orbits in A and hence are valid solely in the strong case.

3 Topological mixing and the shadowing property

We continue to assume that Γ is a finite group acting orthogonally on $\mathbb{R}^n$ and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant homeomorphism.

Definition 3.1 An invariant set A is *topologically mixing* under a homeomorphism f if for any nonempty open sets $U, V \subset A$ there is an integer q_0 such that $f^q(U) \cap V \neq \emptyset$ for all $q \geq q_0$.

Theorem 3.2 *Suppose that A is a Γ-symmetric topologically mixing set satisfying the shadowing property. Then Σ-symmetric periodic orbits are dense in A for all cyclic subgroups $\Sigma \subset \Gamma$.*

Proof Fix a cyclic subgroup $\Sigma \subset \Gamma$ and a generator $\sigma \in \Sigma$. Let U be a nonempty open subset of A and choose $V \subset U$ as in the proof of Lemma 2.5. Since A is topologically mixing, there is an integer q_0 such that $f^q(V) \cap \sigma V \neq \emptyset$ for all $q \geq q_0$. Choose $q \geq q_0$ so that $q > |\Gamma|$ is prime. As in the proof of Lemma 2.5 we have the existence of a periodic point $x \in U$ such that the corresponding periodic orbit P satisfies $\Sigma \subset \Sigma_P$. Moreover $f^{q|\Sigma|}(x) = x$. In particular,

(i) $|\Sigma|$ divides $|\Sigma_P|$, and

(ii) $|\Sigma_P|$ divides $q|\Sigma|$.

Since q is prime and $|\Sigma_P| < q$ it follows from (ii) that $|\Sigma_P|$ divides $|\Sigma|$. We deduce from (i) that $|\Sigma_P| = |\Sigma|$ and so $\Sigma_P = \Sigma$ as required. ∎

We say that a periodic orbit P is *asymmetric* if $\Sigma_P = 1$.

Corollary 3.3 *Under the hypotheses of Theorem 3.2, asymmetric periodic orbits are dense.*

4 Topologically mixing up to a cycle

In this section, we relax the assumption that A is topologically mixing.

Definition 4.1 A topologically transitive set A is *topologically mixing up to a cycle* if A can be written as the disjoint union of closed sets $A_0, \ldots, A_{r-1}$ each of which is invariant and topologically mixing under f^r.

It follows that if $j \in \{0, \ldots, r-1\}$ and $U \subset A_0$, $V \subset A_j$ are nonempty open subsets, then there exists an integer q_0 such that $f^{qr+j}(U) \cap V \neq \emptyset$ for all $q \geq q_0$.

Remark 4.2 If A is a hyperbolic topologically transitive set then A is topologically mixing up to a cycle [1, 20, 21]. This property holds also for nonuniformly hyperbolic transitive sets, see Pesin [18]. As mentioned in Remark 2.4, these notions generalize via the spectral decomposition to (nonuniformly) hyperbolic sets with local product structure.

Theorem 4.3 *Let Γ be a finite group acting orthogonally on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Γ-equivariant homeomorphism. Suppose that a Γ-symmetric set A satisfies the following conditions:*

(i) A possesses the shadowing property,

(ii) A is topologically mixing up to a cycle, and

(iii) A is not a periodic orbit.

Then Σ-symmetric periodic orbits are dense in A for all cyclic subgroups $\Sigma \subset \Gamma$. In particular, asymmetric periodic orbits are dense in A.

Proof Let $A_0, \ldots, A_{r-1}$ be the components of A as in Definition 4.1. Suppose that $\Sigma \subset \Gamma$ is a cyclic subgroup and let Δ be the subgroup of Σ that fixes A_0. Then $\Sigma/\Delta \cong \mathbb{Z}_k$, where k divides r, and $\mathbb{Z}_k$ acts fixed-point freely on the components of A. Since f is Γ-equivariant, it follows easily that the group orbit $\mathbb{Z}_k(A_0)$ is given by $\mathbb{Z}_k(A_0) = A_0 \cup A_{r/k} \cup \cdots \cup A_{(k-1)r/k}$. In particular, we may choose a generator $\sigma \in \Sigma$ such that $\sigma A_0 = A_{r/k}$.

Set $\ell = r/k$ and let $a_1 > a_2 > \cdots > a_s \geq 2$ be the distinct primes appearing in the prime factorization of ℓ. We shall prove the following result.

Lemma 4.4 *Given any nonempty open set $U_0 \subset A_0$ there is a point $x \in U_0$ and a positive integer b such that*

(a) $f^{b\ell}(x) = \sigma x$,

(b) $f^{b\ell/a_i}(x) \notin \Gamma \cdot x$ *for each* $i = 1, \ldots, s$, *and*

(c) $b > |\Gamma|$ *is prime.*

It follows from (a) that x is a periodic point and that $\Sigma \subset \Sigma_P$ where P is the corresponding periodic orbit. Moreover, the prime period of P is given by $|P| = b\ell|\Sigma|$. Let $d = |\Sigma_P/\Sigma|$. Then $|\Sigma_P| = d|\Sigma|$ divides $|P|$ so that d divides $b\ell$. It follows from (c) that d divides ℓ. Hence we can write $d = a_1^{m_1} \cdots a_s^{m_s}$ where $m_i \geq 0$. If $d = 1$ then we are done so we may assume that $m_{i_0} \geq 1$ for some i_0. Since $|P| = (b\ell/d)|\Sigma_P|$ we have that $f^{jb\ell/d}(x) \in \Sigma_P \cdot x$ for any positive integer j. Taking $j = d/a_{i_0}$ we obtain a contradiction to condition (b). Hence the theorem follows from the lemma. ∎

Proof of the lemma Set $U_j = f^j(U_0) \subset A_j$ and let $U = \Gamma(U_0 \cup \cdots \cup U_{r-1})$. Shrinking U_0 if necessary we may assume that $A_0 - U$ has nonempty interior. (It is here that we make use of hypothesis (iii) in Theorem 4.3.) Let $z \in U_0$ and choose z' in the interior of $A_0 - U$. Then we can choose $\epsilon, \delta > 0$ so that every δ-pseudoorbit has a unique ϵ-tracing point and so that

$$B_{\delta+\epsilon}(f^j(z)) \subset U_j, \quad B_{\delta+\epsilon}(f^j(z')) \subset A_j - U_j, \quad j = 0, \ldots, r-1.$$

Let $V_j = B_\delta(f^j(z))$, $W_j = B_\delta(f^j(z'))$, $j = 0, \ldots, r-1$, and set $V = V_0 \cup \cdots \cup V_{r-1}$, $W = W_0 \cup \cdots \cup W_{r-1}$.

Suppose that $f^m(A_0) \subset \sigma A_0$ for m a positive integer and let $m_i = m/a_i$. Note that $0 < m_1 < \cdots < m_s < m$. Since A is mixing up to a cycle, it follows that there is an integer m_0 such that if $m \geq m_0$ then we can satisfy the following conditions simultaneously:

$$f^{m_1}(V_0) \cap W \neq \emptyset, \quad f^{m_{i+1}-m_i}(W_j) \cap W \neq \emptyset, \; i = 1, \ldots, s-1,$$

$$f^{m-m_s}(W_j) \cap V \neq \emptyset \text{ for each } j.$$

Specifically, we have

$$f^{m_1}(V_0) \cap W_{\hat{m}_1} \neq \emptyset, \quad f^{m_{i+1}-m_i}(W_{\hat{m}_i}) \cap W_{\hat{m}_{i+1}} \neq \emptyset, \; i = 1, \ldots, s-1,$$

$$f^{m-m_s}(W_{\hat{m}_s}) \cap \sigma V_0 \neq \emptyset,$$

where $\hat{m}_i \equiv m_i \bmod r$. Since each W_j has diameter δ, we can construct a finite δ-pseudoorbit $\mathbf{y} = \{y_\alpha\}_{\alpha=0}^m$ such that $y_0 \in V_0$, $y_m \in \sigma V_0$ and $y_{m_i} \in W$ for $i = 1, \ldots, s$. Moreover V_0 has diameter δ and we can extend using equivariance and periodicity to a bi-infinite δ-pseudoorbit $\mathbf{y}$. Let x be an ϵ-tracing point. Then $x \in U_0$ and $f^{m_i}(x) \in A - \Gamma(U_0)$ for $i = 1 \ldots, s$. It follows from uniqueness of the tracing point that $f^m(x) = \sigma x$. This establishes parts (a) and (b).

It remains to check that m can be chosen so that the condition $b > |\Gamma|$ prime in part (c) is valid. Observe that $f^m(A_0) \subset \sigma A_0 = A_{r/k}$ if and only if $m = qr + r/k$ for some integer q. We compute that $b = kq+1$. By Dirichlet's Theorem, the arithmetic progression $kq + 1$, $q \geq 0$ contains infinitely many primes. Now choose $q > |\Gamma|$ such that $m \geq m_0$ and $kq + 1$ is prime. ∎

Corollary 4.5 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant diffeomorphism. If A is a Γ-symmetric (nonuniformly) hyperbolic topologically transitive set and A is not a periodic orbit, then Σ-symmetric periodic orbits are dense in A for all cyclic subgroups $\Sigma \subset \Gamma$.*

5 Continuous symmetry groups

Suppose now that $\Gamma \subset \mathbf{O}(n)$ is a compact Lie group acting on $\mathbb{R}^n$. Our results for Γ finite in the previous sections extend to the case Γ compact in a reasonably obvious way.

Let Γ^0 denote the connected component of Γ that contains the identity. We have the quotient $H = \Gamma/\Gamma^0$ and the canonical projection $\pi : \Gamma \to H$. Factoring out the action of Γ^0 we obtain the orbit space $X = \mathbb{R}^n/\Gamma^0$. Observe that H acts on X and that a Γ-equivariant homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ projects down to an H-equivariant homeomorphism $\tilde{f} : X \to X$.

Recall that an ω-limit set $P \subset \mathbb{R}^n$ is a relative periodic orbit if the projection $\widetilde{P} \subset X$ is a periodic orbit for $\tilde{f}$. Equivalently, P is contained in the union of finitely many Γ-orbits in $\mathbb{R}^n$. Now suppose that $A \subset \mathbb{R}^n$ is an invariant set for f and let $\widetilde{A}$ denote the corresponding invariant set for $\tilde{f}$. We say that A has the *relative shadowing property* if $\widetilde{A}$ has the shadowing property. Similarly, A is *relatively topologically mixing (up to a cycle)* if $\widetilde{A}$ is topologically mixing (up to a cycle). We have the following result.

Proposition 5.1 *Suppose that A contains some points of trivial isotropy and satisfies the relative shadowing property.*

(a) If A is topologically transitive and K is a maximal cyclic subgroup of $\pi(\Sigma_A)$, then relative periodic orbits P with $\pi(\Sigma_P) = K$ are dense in A.

(b) If A is relatively topologically mixing up to a cycle and A is not a relative periodic orbit, then relative periodic orbits P with $\pi(\Sigma_P) = K$ are dense in A for each cyclic subgroup K of $\pi(\Sigma_A)$.

Proof Passing to the orbit space, we have a $\pi(\Sigma_A)$-symmetric topologically transitive set satisfying the shadowing property (and topologically mixing up to a cycle in part (b)). Now apply Lemma 2.5 and Theorem 4.3. ∎

We end this section by indicating the validity of the hypotheses in Proposition 5.1 under suitable hyperbolicity hypotheses. Recall that a set A is Γ-hyperbolic if it is 'hyperbolic transverse to the continuous group action'. In other words, at each point $x \in A$, there is a uniform splitting of $\mathbb{R}^n$ into directions that are stable, unstable and tangent to the group orbit $\Gamma \cdot x$. The definition of Γ-nonuniform hyperbolicity is also the obvious one.

It seems plausible that Γ-(nonuniformly) hyperbolic transitive sets possess the relative shadowing property and are relatively topologically mixing up to a cycle. This is certainly the case if the set consists of points of trivial isotropy. More generally, we say that a point $x \in \mathbb{R}^n$ has *discrete isotropy* if $\Sigma_x \cap \Gamma^0 = 1$.

Proposition 5.2 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant diffeomorphism and that A is a Γ-(nonuniformly) hyperbolic topologically transitive set consisting entirely of points of discrete isotropy. Then A has the relative shadowing property and is relatively topologically mixing up to a cycle.*

Proof Let U denote the points in $\mathbb{R}^n$ of discrete isotropy. Then U is open and invariant under the diffeomorphism f. Moreover, $A \subset U$ and Γ^0 acts fixed-point freely on U. Hence, the orbit space $\widetilde{U} = U/\Gamma^0$ is a manifold containing $\widetilde{A}$. It is immediate that $\widetilde{A}$ is (nonuniformly) hyperbolic for the diffeomorphism $\tilde{f} : \widetilde{U} \to \widetilde{U}$. Hence $\widetilde{A}$ satisfies the shadowing property and is topologically mixing up to a cycle. It follows from the definitions that A has the required properties. ∎

6 Compact group extensions of nonuniformly hyperbolic dynamical systems

An outstanding problem in ergodic theory is the study of smooth compact group extensions of dynamical systems and the generic lifting of properties such as ergodicity, mixing and so on. There are a great number of results in the measurable and continuous contexts, see for example [12, 22]. However, the smooth context has been largely overlooked until recently. Exceptions to this can be found in [3, 4, 16].

In this section, we are concerned with lifting transitivity. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map commuting with the action of a compact Lie group $\Gamma \subset \mathbf{O}(n)$. Suppose that A is a topologically transitive set for f. Our aim is to prove that generically $\Gamma^0 \subset \Sigma_A$ under reasonable hypotheses on the dynamics in A. Genericity is with respect to smooth perturbations along the continuous group orbits, the dynamics at the orbit space level being preserved.

The case when Γ is abelian is completely understood — it is shown in [15] that both generically and prevalently, $\Gamma^0 \subset \Sigma_A$ regardless of the underlying dynamics. The situation is much less clear when Γ is not abelian though there are some partial results when Γ^0 is abelian [15]. One problem is that it is necessary to exclude certain regular dynamics. Recently, there has been progress when A is assumed to be hyperbolic [8, 17]. The aim of this section is to point out a very simple approach when Γ^0 is abelian and A is hyperbolic or even nonuniformly hyperbolic. The main strength of the results of [8, 17] lies in the case when Γ is compact and Γ^0 is nonabelian.

The idea behind our approach, following [13], is that provided certain relative periodic orbits P are dense in A, generically Σ_A contains a maximal torus in Γ^0. The previous sections were concerned precisely with deriving such a density condition.

First, we recall some results of Krupa [11] and Field [7]. Suppose that P is a relative periodic orbit consisting of points of trivial isotropy. Then Σ_P is an abelian subgroup of Γ of the form $\Sigma_P \cong \mathbf{T}^p \times \mathbb{Z}_q$. Moreover, generically p (but not q) is maximal with respect to containment. In more sophisticated language, Σ_P is topologically cyclic and is generically a Cartan subgroup [5].

It is not always the case that generically $p = \dim \Gamma$ even when Γ^0 is abelian. (For example, if $\Gamma = \mathbf{O}(2)$ and Σ_P contains a reflection, then Σ_P must be isomorphic to $\mathbb{Z}_2$.) However, provided $\pi(\Sigma_P) = 1$ it is generically the case that Σ_P is a maximal torus in Γ^0. Of course, the condition that $\pi(\Sigma_P) = 1$ is equivalent to asking that the corresponding periodic orbit $\widetilde{P} \subset X$ is asymmetric. Hence we deduce the following.

Corollary 6.1 *Suppose that A is not a relative periodic orbit and contains some points with trivial isotropy. Suppose further that A possesses the relative shadowing property and is relatively topologically mixing up to a cycle. Then generically Σ_A contains a maximal torus in Γ^0. In particular, when Γ_0 is abelian, generically $\Gamma_0 \subset \Sigma_A$.*

Combining Corollary 6.1 with Proposition 5.2 we obtain the following result.

Corollary 6.2 *Suppose that $\Gamma \subset \mathbf{O}(n)$ is a compact Lie group acting on $\mathbb{R}^n$ with Γ^0 abelian and that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Γ-equivariant diffeomorphism. Let A be a Γ-(nonuniformly) hyperbolic topologically transitive set, but not a relative periodic orbit, consisting entirely of points of discrete isotropy and containing some points of trivial isotropy. Then generically, $\Gamma^0 \subset \Sigma_A$.*

Acknowledgments We are grateful to Mike Field for helpful discussions and to the referee for useful comments on the definition of shadowing. MD acknowledges the very kind hospitality of the Department of Mathematics of the University of Houston where part of this research has been done. IM acknowledges the hospitality of the Institut Non-Linéaire de Nice where part of this work was carried out.

References

[1] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lecture Notes in Math. **470**, Springer, Berlin, 1975.

[2] G. E. Bredon. *Introduction to Compact Transformation Groups.* Pure and Appl. Math. **46**, Academic Press, New York, 1972.

[3] M. I. Brin. Topological transitivity of one class of dynamic systems and flows of frames on manifolds of negative curvature. *Functional Anal.* **9** (1975) 8-16.

[4] M. I. Brin, J. Feldman and A. Katok. Bernoulli diffeomorphisms and group extensions of dynamical systems with non-zero characteristic exponents. *Annals of Math.* **113** (1981) 159-179.

[5] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups*, Grad. Texts in Math. **98**, Springer, New York 1985.

[6] P. Chossat and M. Golubitsky. Symmetry-increasing bifurcation of chaotic attractors. *Physica D* **32** (1988) 423-436.

[7] M. J. Field. Local structure for equivariant dynamics. In *Singularity Theory and its Applications* Part II, (M. Roberts, I. Stewart, eds.), Lecture Notes in Math. **1463**, Springer, Berlin, 1991, 142-166.

[8] M. J. Field. The structure of transversally hyperbolic basic sets for equivariant diffeomorphisms. In preparation.

[9] A. Katok and L. Mendoza. Smooth ergodic theory. Unpublished notes.

[10] T. Krüger and S. Troubetzkoy. Markov partitions and shadowing for nonuniformly hyperbolic systems with singularities. *Ergod. Th. Dynam. Sys.* **12** (1992) 487-508.

[11] M. Krupa. Bifurcations of relative equilibria. *SIAM J. Appl. Math.* **21** (1990) 1453-1486.

[12] G. W. Mackey. Ergodic theory and virtual groups. *Math. Ann.* **166** (1966) 187-207.

[13] I. Melbourne. Generalizations of a result on symmetry groups of attractors. In *Pattern Formation: Symmetry Methods and Applications* (J. Chadam *et al.* eds.) Fields Institute Communications **5**, AMS, 1996, 281-295.

[14] I. Melbourne, M. Dellnitz and M. Golubitsky. The structure of symmetric attractors. *Arch. Rational Mech. Anal.* **123** (1993) 75-98.

[15] I. Melbourne and I. Stewart. Symmetric ω-limit sets for smooth Γ-equivariant dynamical systems with Γ^0 abelian. *Nonlinearity* **10** (1997) 1551-1567.

[16] M. G. Nerurkar. On the construction of smooth ergodic skew-products. *Ergod. Th. Dynam. Sys.* **8** (1988) 311-326.

[17] W. Parry. Skew products of shifts with a compact Lie group. Warwick University preprint, 1995.

[18] Y. Pesin. Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties. *Ergod. Th. Dynam. Sys.* **12** (1992) 123-151.

[19] M. Pollicott. *Lectures on Ergodic Theory and Pesin Theory on Compact Manifolds*. London Math. Soc. Lecture Notes **180**, Cambridge Univ. Press, Cambridge 1993.

[20] M. Shub. *Global Stability of Dynamical Systems*. Springer, New York 1987.

[21] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967) 747-817.

[22] R. J. Zimmer. Random walks on compact groups and the existence of cocycles. *Israel J. Math.* **26** (1977) 84-90.