Ginzburg-Landau theory and symmetry

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August 30, 1999
Appeared in *Nonlinear Instability, Chaos and Turbulence, Vol 2*

Abstract

We consider mathematical issues concerning Ginzburg-Landau theory, by which we mean the validity, universality and structure of reduced equations near criticality in spatially extended systems. The extraction of Ginzburg-Landau equations (variously known as amplitude, modulation and envelope equations) is part of this theory. We pay particular attention to the Euclidean symmetries present in such systems.

1 Introduction

Patterns that appear in physical, chemical and biological systems are both striking and reproducible. One approach to studying patterns is through bifurcation and symmetry. This approach leads to a description of those aspects of pattern formation that are *universal* — depending only on the symmetries of the underlying problem and the bifurcation that is taking place.

Universal theories of this type go back to Landau [1] in the context of second order phase transitions in crystals. Here, the symmetries are taken to be given and form a finite group Γ. For each finite group Γ, Landau theory leads to low-dimensional universal equations known as *Landau equations* that govern the transitions near criticality. The Landau equations are equivariant with respect to the group Γ, and the resulting structure enables the explicit calculation of many branches of solutions bifurcating from an underlying fully symmetric solution. Typically, the bifurcating solutions are not fully symmetric; this phenomenon is known as *spontaneous symmetry breaking* [2].

Landau theory is not restricted to transitions in crystals, and applies more generally to problems with finite symmetry group Γ. For example, transitions of barium titanate crystals (which have a cubic structure) are governed by the same Landau equations as hydrodynamic instabilities in a cubic domain. We note that the details of the transitions may be different because there are certain Taylor coefficients in the Landau equations that depend on the details of the physics. However, the structure of the equations is identical and there are only a finite number of possibilities for the details to distinguish between.

Landau theory also applies to problems where there are infinitely many symmetries, provided these symmetries form a compact Lie group. This is typically the case for problems in bounded domains. For example, astrophysical and geophysical problems in spherical domains can be studied using Landau theory.

For problems with compact symmetry group, Landau theory provides a fairly complete description of universal phenomena. Moreover, *equivariant bifurcation theory* [3, 4, 5, 6] places Landau theory in a mathematically rigorous setting. See also [7].

In contrast, transitions in spatially extended systems, with unbounded domains, are relatively poorly understood. Such systems arise when the domains are so large that boundaries are deemed unimportant for certain purposes. The modeling equations are then posed on infinite domains and may have a noncompact symmetry group (typically Euclidean symmetry). Examples of spatially extended systems include hydrodynamic systems (Rayleigh-Bénard convection modeled by the Boussinesq equations), chemical systems (modeled by reaction-diffusion equations), and biological systems (spots and stripes on animals). The counterpart of Landau equations are the *Ginzburg-Landau equations* [8, 9]. For an overview, see Cross & Hohenberg [10], or Newell [11]. These are *lower*-dimensional universal equations, but they are still infinite-dimensional and hard to study. Moreover, their validity is problematic and their universality has only recently been understood [12].

In this article, we discuss various aspects of bifurcations in spatially extended systems, including the validity and universality of reduced equations and the extraction where possible of Ginzburg-Landau equations.

We now describe some of our main results. Consider systems of Euclidean equivariant PDEs with $n \geq 1$ unbounded domain variables (and any number of bounded domain variables). For ease of exposition, we restrict ourselves to the so-called 'type I_s' transition, where a fully-symmetric trivial solution undergoes a steady-state bifurcation with nonzero critical wavenumber. (The analogous results for the remaining cases [Hopf bifurcation and/or zero critical wavenumber] can be found in sections 4 and 5.)

- If $n = 1$, then the bifurcation is governed by a single universality class of reduced equations. In particular, the standard Ginzburg-Landau equation on the line is valid and universal.

- If the system of PDEs is a reaction-diffusion equation (any $n \geq 1$), then generically it is possible to reduce from a system of PDEs to a single equation, and for each n this equation is universal.

- If $n = 2$, it is possible generically to reduce from a system of PDEs to a single equation. However, there are precisely two universality classes, *scalar* and *pseudoscalar*, and these lead to quite different phenomena.

- If $n = 3$, then there are infinitely many (but only countably many) universality classes. It is possible generically to reduce the system of PDEs to a smaller system of equations of *minimal* size. However, this minimal size depends on the universality class and is arbitrarily large.

- The situation for each $n \geq 4$ is analogous to the case $n = 3$.

- For all $n \geq 1$, the reduced equations have no bounded domain variables, and incorporate only the unbounded domain variables of the original system of PDEs.

We emphasize that these results have precise mathematical statements and proofs [12]. The first two results listed above might be construed as 'obvious' or 'well-known', but the next three results demonstrate the need for caution. The final result recovers a result of Mielke [13].

The results described so far do not say much directly about the Ginzburg-Landau equations themselves. There are a number of reasons for focusing attention on reduced equations as an intermediate stage: (i) The validity of the reduced equations is rigorous with no approximations, applies to all nondegenerate bifurcations, and is universal. (ii) The derivation of Ginzburg-Landau equations is problematic even formally, and it is not always clear what form the correct equations should take, or even if useful equations exist.

One instance where there is little argument about the form of the Ginzburg-Landau equations is steady-state bifurcation with nonzero wavenumber when $n = 1$. Here, we have the standard Ginzburg-Landau equation on the line

$$\partial A_t = A + D^2 A \pm |A|^2 A, \tag{1}$$

where $A : \mathbb{R} \to \mathbb{C}$ is a complex amplitude. There at least two approaches to justifying this equation rigorously. One approach due to Schneider [14], van Harten [15], Mielke & Schneider [16], shows that solutions to eqn (1) approximate solutions to the underlying system of PDEs over finite but arbitrarily long time-scales near criticality. A second approach due to Melbourne [17] shows that there is a reduction with no approximations to a complex amplitude equation in $A : \mathbb{R} \to \mathbb{C}$ whose lowest order truncation is eqn (1). Moreover, 'nondegenerate' solutions to eqn (1) correspond (for all time) to branches of solutions to the underlying systems of PDEs. This justification of the one-dimensional Ginzburg-Landau equation is identical in spirit to the justification in equivariant bifurcation theory of Landau equations, see section 2. See also Iooss et al. [18] who give a similar justification of the steady Ginzburg-Landau equation (with time-derivatives set to zero).

Even though the Ginzburg-Landau equation (1) is well established, we wish to draw attention to some anomalous properties of this equation. Eqn (1) is constant coefficient signifying translation symmetry. However, the derivation of eqn (1) revolves around the substitution $u(x) = A(x)e^{ix} + \overline{A}(x)e^{-ix}$. Hence, translation symmetry accounts for constant coefficients only in the underlying PDE satisfied by u, and does not explain the constant coefficient structure in eqn (1). This anomaly was pointed out by Pomeau [19] and is discussed further in section 7.

The remainder of this paper is organized as follows. In section 2, we review some of the ideas from Landau theory and equivariant bifurcation theory, treating first the case when the symmetry group Γ is finite, and second the case when Γ is a compact Lie group. Also, we introduce the Swift-Hohenberg equation to illustrate the difficulties that arise when Γ is noncompact. In section 3, we concentrate on the Euclidean group $\Gamma = \mathbf{E}(n)$. We describe the actions of $\mathbf{E}(n)$ that arise in applications, paying special attention to the scalar and pseudoscalar actions mentioned above. In particular, we demonstrate that at least two universality classes are required when considering steady-state bifurcation with nonzero critical wavenumber with $n = 2$. In sections 4 and 5, we sketch the formal arguments that lead to the above results on universality and reduction respectively. (The mathematical details can be found in [12].) In section 6, we consider the problem of extracting Ginzburg-Landau equations, and the structure of these equations is described in section 7.

2 Landau theory

Landau theory originated in the context of second order phase transitions in crystals, but applies generally to transitions in physical problems with a given (compact Lie group) of symmetries Γ. We illustrate the theory in the cases of octahedral symmetry and spherical symmetry.

2.1 Finite groups: Octahedral symmetry

The same symmetry group may occur in several quite diverse contexts, such as in phase transitions of crystals, fluid dynamics, and problems in chemical engineering and biology. It turns out that these seemingly unrelated situations may exhibit similar phenomena. Such 'universal' phenomena that depend on the symmetry rather than the physical details are studied in Landau theory.

For example, the finite (48 element) group $\Gamma = \mathbb{O} \oplus \mathbb{Z}_2^c$, generated by rotations and reflections of the cube, occurs in phase transitions of crystals with cubic symmetry, such as barium titanate crystals, see Devonshire [20], and also in fluid problems in cubic domains. Note that there may be no underlying equations to work with (as is the case in the barium titanate example) or that the model equations may be of high or infinite-dimension (as is the case in fluid problems). Nevertheless, whatever form the governing laws of nature may take, these laws should respect the symmetries of the problem.

Suppose that Γ is a finite group and that a fully symmetric 'trivial solution' loses stability as a bifurcation parameter is varied. In variational problems such as the barium titanate example, such a transition will be a steady-state bifurcation where eigenvalues of the linearized equations pass through zero. More generally, as in fluid problems, there is also the possibility of Hopf bifurcation where complex conjugate eigenvalues pass through the imaginary axis. In this section, we concentrate on steady-state bifurcations.

The main idea in Landau theory is that near criticality, the underlying laws should reduce to the eigenspace of the critical eigenvalues and that this eigenspace E is typically as small as possible. It can then be argued that the action of Γ on E is *absolutely irreducible*. That is, the only linear maps $L : E \to E$ that commute with the action of Γ on E are the real scalar multiples of the identity $L = cI$ where $c \in \mathbb{R}$.

The next step in Landau theory is to write down the simplest polynomial equations $f : E \to E$ that are *equivariant* with respect to the action of the symmetry group Γ. So $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$, $x \in E$. Such equations are known as *Landau equations*.

For example, the standard action of $\mathbb{O} \oplus \mathbb{Z}_2^c$ on $\mathbb{R}^3$, given by rotations and reflections of the cube, is easily seen to be absolutely irreducible. The general third order equivariant polynomial is

$$f(x,y,z) = \big(\lambda + a(x^2+y^2+z^2)\big)\begin{pmatrix} x \\ y \\ z \end{pmatrix} + b\begin{pmatrix} x^3 \\ y^3 \\ z^3 \end{pmatrix}, \tag{2}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter and $a, b \in \mathbb{R}$ are constants.

The Landau equation, eqn (2), is supposed to be *universal* for problems with octahedral symmetry. The word universal has a precise meaning here. It is easier to explain where universality breaks down. First, there are several (but finitely many) different absolutely irreducible representations of $\mathbb{O} \oplus \mathbb{Z}_2^c$. We have considered only one here. Second, the constants $a, b \in \mathbb{R}$ are unknown. If the underlying equations are given, then a and b can be computed via asymptotic expansions. In any case, it turns out that there are only finitely many regions in (a, b)-space that give qualitatively different dynamics for eqn (2). So the correct interpretation of the universality of Landau theory is that, given a finite symmetry group Γ, the totality of possible steady-state bifurcations with symmetry Γ can be reduced to finitely many universality classes enumerated by the absolutely irreducible representations of Γ. Moreover, within each universality class, the dynamics is governed by a vector field with finitely many arbitrary Taylor coefficients. The choice of universality class and the values of the Taylor coefficients are determined by the details of the physical problem.

A precise mathematical version of these ideas is formulated in equivariant bifurcation theory [3]. Suppose that there is an underlying model in the form of a Γ-equivariant partial differential equation (PDE). Suppose that there is a steady-state bifurcation from a fully symmetric equilibrium. Then

(i) Generically the critical eigenspace E of the linearized PDE is an absolutely irreducible representation of Γ.

(ii) Center manifold reduction leads to an ordinary differential equation (ODE) $\dot{x} = f(x)$ where $f : E \to E$ is a smooth Γ-equivariant vector field.

(iii) The reduced vector field f is not a polynomial, but the lowest order nonlinear truncation of f is precisely the Landau equation corresponding to the absolutely irreducible representation E.

(iv) Any 'nondegenerate' solution to the Landau equation corresponds to a branch of solutions to the underlying PDE. In the case of an equilibrium x_0 for f, nondegenerate means that $(df)_{x_0}$ is nonsingular.

2.2 Compact groups: Spherical symmetry

Landau theory and equivariant bifurcation theory apply also to problems whose symmetries form a compact Lie group. For example, the three-dimensional compact Lie group $\Gamma = \mathbf{O}(3)$ consisting of 3×3 orthogonal matrices occurs in physical problems with spherical symmetry, such as in geophysics, and PDE models inherit this symmetry.

In some instances, it is possible to reduce a system of $\mathbf{O}(3)$-equivariant PDEs (near a steady-state bifurcation) to a system of ODEs on $\mathbb{R}^3$ that is equivariant under the standard action of $\mathbf{O}(3)$ by rotations and reflections on $\mathbb{R}^3$. The resulting equations are fully justified by center manifold reduction and, when truncated, are precisely the Landau equations. In general, however, it is necessary to consider all absolutely irreducible representations of $\mathbf{O}(3)$. Thus reduction leads to a $(2\ell+1)$-dimensional ODE on the space of spherical harmonics of order ℓ where ℓ is any nonnegative integer. (The action of $\mathbf{O}(3)$ that we considered on $\mathbb{R}^3$ is just the case $\ell = 1$.) Hence, there is a countable infinity of universality classes: namely these $(2\ell + 1)$-dimensional systems of ODEs. Again, it follows from [3] that PDEs with $\mathbf{O}(3)$-symmetry undergoing steady-state bifurcation generically reduce to one of these universality classes. As before, the Landau equations are universal only up to the value of the integer ℓ and the various real Taylor coefficients.

In general, the Landau equation (which is the reduced equation truncated at leading order) is insufficient to determine the local dynamics associated with the bifurcation. A specific example of this is the $\ell = 5$ representation of $\Gamma = \mathbf{O}(3)$ where the reduced equation is an 11-dimensional system of ODEs. The Landau equation is third order, whereas generically at least fifth order terms are required to determine the local bifurcation; see Chossat et al. [21]. In this regard, it is worth noting a result of Field [22] which guarantees that, for any compact Lie group Γ, there is a finite order at which the reduced equations may be truncated such that the truncated equations determine many important features of the local bifurcation (including the branching and stability of equilibria).

2.3 Noncompact groups: Euclidean symmetry

There are many difficulties that arise in the attempt to generalize Landau theory from compact groups to noncompact groups. Consider, for example, the generalized n-dimensional Swift-Hohenberg equation [23]

$$\partial_t u = -(\Delta + 1)^2 u + \lambda u + bu^2 + cu^3, \tag{3}$$

where $u : \mathbb{R}^n \to \mathbb{R}$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian. Here, $\lambda \in \mathbb{R}$ is the bifurcation parameter and $b, c \in \mathbb{R}$ are constants. Since this equation is constant coefficient, equivariance with respect to translations $u(x) \mapsto u(x + a)$, $a \in \mathbb{R}^n$, is immediate. Equivariance with respect to rotations and reflections, $u(x) \mapsto u(Ax)$, $A \in \mathbf{O}(n)$, follows from standard properties of the Laplacian. Hence, eqn (3) is an example of an $\mathbf{E}(n)$-equivariant PDE.

The Swift-Hohenberg equation has a fully-symmetric trivial solution $u(x) \equiv 0$. The Fourier ansatz $u(x) = e^{ik\cdot x}$, $k \in \mathbb{R}^n$, leads to eigenvalues $-(|k|^2 - 1)^2 + \lambda$ for the linearized PDE. It follows from the principle of linear stability that the trivial solution is asymptotically stable for $\lambda < 0$ and unstable for $\lambda > 0$ so that there is a bifurcation at $\lambda = 0$. Moreover, the critical eigenvalues pass through zero so that this is a steady-state bifurcation. The critical eigenfunctions have wavenumbers k with $|k|$ close to the *critical wavenumber* $k_c = 1$. In contrast to the situation in the previous subsections, the eigenvalue passing through zero is not isolated in the spectrum of the linearized operator and a continuum of wavenumbers is excited for $\lambda > 0$ small. This is the 'continuous spectrum' difficulty. In addition, when $n \geq 2$, there is the 'rotational degeneracy' whereby the critical eigenfunctions with $|k| = k_c = 1$ span an infinite-dimensional space.

This example serves to illustrate two difficulties that are well-recognized in the fluid dynamics literature:

(a) The presence of continuous spectrum obstructs reduction to the critical eigenspace — noncritical eigenvalues that are close to critical must be retained in the reduced equations.

(b) The critical eigenspaces themselves may be of infinite-dimension (as occurs for $n \geq 2$ above due to the rotational degeneracy).

Hence, in general the reduced equations will be PDEs instead of ODEs. The continuous spectrum also means that center manifold reduction is highly problematic, if not impossible. There are many other difficulties. For example:

(c) The representation theory of noncompact Lie groups is not completely understood. Moreover, noncompact Lie groups such as $\mathbf{E}(n)$ have uncountably many distinct irreducible representations whereas compact Lie groups have at most countably many distinct irreducible representations.

(d) The determinacy result of Field [22] relies on compactness of the symmetry group and does not extend to the case of Euclidean symmetry.

The methods in Melbourne [17, 12] provide a solution to difficulties (a) and (c), as described in this article. The approximation results of [14, 15, 16] provide an alternative solution to (a). However, difficulties (b) and (d) remain unresolved.

3 Actions of $\mathbf{E}(n)$; scalar and pseudoscalar PDEs

The simplest classes of $\mathbf{E}(n)$-equivariant PDEs are the scalar and pseudoscalar PDEs mentioned in the introduction. In particular, the Swift-Hohenberg equation, eqn (3), is an example of a scalar PDE. Scalar and pseudoscalar PDEs are defined in subsection 3.1, where we also contrast certain aspects of the local bifurcations, following [24].

In subsection 3.2, we consider systems and describe a general class of $\mathbf{E}(n)$-equivariant systems of PDEs to which the results in this paper apply.

3.1 Scalar and pseudoscalar PDEs

The Euclidean group $\mathbf{E}(n)$ consists of isometries of $\mathbb{R}^n$. These include orthogonal transformations $A \in \mathbf{O}(n)$ (such as rotations and reflections) and translations $a \in \mathbb{R}^n$. Every element $\gamma \in \mathbf{E}(n)$ can be written uniquely as the combination of an orthogonal transformation A and a translation a. The standard (affine) action of $\gamma = (A, a) \in \mathbf{E}(n)$ on $\mathbb{R}^n$ is given by $\gamma x = Ax + a$.

The standard action of $\mathbf{E}(n)$ on $\mathbb{R}^n$ induces a linear action on functions $u : \mathbb{R}^n \to \mathbb{R}$ given by $u(x) \mapsto u(\gamma^{-1}x)$. (We have $\gamma^{-1}x$ instead of γx for purely technical reasons.) For example, the Swift-Hohenberg equation (3), is equivariant under this action of $\mathbf{E}(n)$. More generally, we denote bounded domain variables in the problem by $z \in \Omega$ and consider domains of the form $\mathbb{R}^n \times \Omega$. Then the *scalar action* of $\mathbf{E}(n)$ on functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}$ is given by

$$u(x, z) \mapsto u(\gamma^{-1}x, z). \tag{4}$$

Note that the action transforms the unbounded domain variables $x \in \mathbb{R}^n$ of the function u in the standard way, leaving the bounded domain variables $z \in \Omega$ untouched. Also, there is no action on the range $\mathbb{R}$ of u.

There is a second action of $\mathbf{E}(n)$ on functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}$, called the *pseudoscalar action*, see Bosch Vivancos et al. [24],

$$u(x, z) \mapsto (\det A)u(\gamma^{-1}x, z). \tag{5}$$

The only difference from the scalar action is that the action of reflections in the domain is coupled with the range transformation $u \mapsto -u$.

When $n = 2$, an example of a pseudoscalar equation is the PDE

$$\partial_t u = -(\Delta + 1)^2 u + \lambda u + bQ(u) + cu^3, \tag{6}$$

where

$$Q(u) = \partial_{x_1}(\Delta u \partial_{x_2} u) - \partial_{x_2}(\Delta u \partial_{x_1} u) = \mathrm{curl}[\Delta u \nabla u]. \tag{7}$$

The linear and cubic order terms are unchanged from eqn (3) and (being odd) are equivariant with respect to both the scalar and pseudoscalar actions. Evidently, the new quadratic term is equivariant with respect to translations. The reflection $(x_1, x_2) \mapsto (-x_1, x_2)$ leads to a minus sign (since there is an odd number of x_1 partial derivatives throughout $Q(u)$). Equivariance with respect to rotations is less obvious, but this is an elementary calculation.

The trivial solution for eqn (6) undergoes a steady-state bifurcation at $\lambda = 0$ with critical wavenumber $k_c = 1$ identical at linear order to the bifurcation for eqn (3). However, the nonlinear PDEs exhibit quite different dynamics, as is shown in Bosch Vivancos et al. [24]. For example, even the simplest solutions such as equilibrium rolls in eqn (3) are replaced by antirolls in eqn (6). It is well-known that rolls arise in planar convection problems for a three-dimensional incompressible fluid. However, transitions in two-dimensional incompressible fluids (Kolmogorov flow in the Navier-Stokes equations on $\mathbb{R}^2$) lead to antirolls. Both of these fluid problems are $\mathbf{E}(2)$-equivariant, but the solutions have different symmetries, as shown in Figure 1. Note that rolls and antirolls have identical translation symmetry: all translations in one horizontal direction and discrete translations in the other (that is, the solutions are homogeneous in one direction and spatially periodic in the other). Also, both solution are preserved by 180° rotation in the plane. However, reflections in the horizontal coordinate axes preserve rolls but reverse the orientation of antirolls. Instead, antirolls have glide-reflection symmetry (reflection combined with a half-period translation parallel to the axis of reflection).

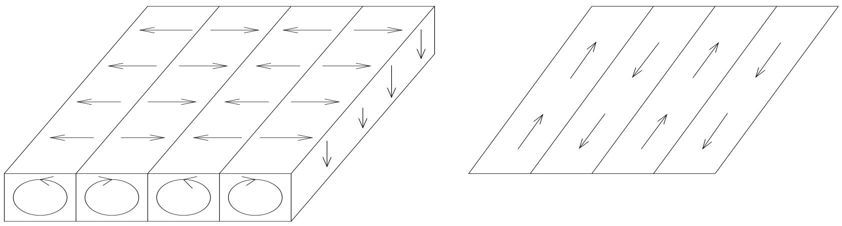

Figure 1: Rolls and antirolls for a three-dimensional and a two-dimensional incompressible fluid in $\mathbf{E}(2)$-equivariant fluid problems. The planar rotation and translation symmetries are identical. Rolls are invariant also under reflections in the horizontal coordinate axes, whereas antirolls are invariant under glide-reflections

Similarly, hexagons in eqn (3) are replaced by oriented hexagons in eqn (6), the latter having no reflection (or even glide-reflection) symmetry. Moreover, whereas hexagons typically bifurcate transcritically (with amplitude proportional to λ) in eqn (3), oriented hexagons bifurcate sub- or supercritically (with amplitude proportional to $\sqrt{\lambda}$) in eqn (6). More comprehensive details on the bifurcation of doubly spatially periodic solutions can be found for scalar equations in Dionne & Golubitsky [25] and for pseudoscalar equations in Bosch Vivancos et al. [24].

This discussion indicates the need for at least two universality classes for steady-state bifurcation with $\mathbf{E}(2)$ symmetry. In fact, if we restrict ourselves to the case of nonzero critical wavenumber, then it turns out that there are precisely *two* universality classes, namely the scalar class and the pseudoscalar class! In particular, taken together, the classifications of doubly spatially periodic solutions in [25, 24] are universal, in the sense that precisely one of the classifications is valid in a given bifurcation.

3.2 $\mathbf{E}(n)$-equivariant systems of PDEs

Next, we consider systems of PDEs involving vector-valued functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$. Here, $s \geq 1$ represents the size of the system of PDEs. The most common actions of the Euclidean group that arise in practice are the *reaction-diffusion action* and the *vector field action*. As the name suggests, reaction-diffusion equations transform under the reaction diffusion action which is given by

$$u(x, z) \mapsto u(\gamma^{-1}x, z). \tag{8}$$

This is the obvious generalization to systems of the scalar action. Recall that reaction-diffusion equations are of the form

$$\partial_t u = D\Delta u + f(u), \tag{9}$$

where $u = (u_1, \cdots, u_s) : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$, D is a constant $s \times s$ matrix and $f : \mathbb{R}^s \to \mathbb{R}^s$ is a nonlinear function. It is not difficult to verify that eqn (9) transforms under the action (8).

On the other hand, fluid equations such as the Navier-Stokes equations transform under the vector field action

$$u(x, z) \mapsto Au(\gamma^{-1}x, z). \tag{10}$$

Here, the domain action is the standard one, but every orthogonal transformation in the domain is coupled with the identical transformation in the range variables.

The Boussinesq equations are the result of coupling the Navier-

Stokes equation and the heat equation, and are given by

$$\begin{aligned}
\partial V/\partial t &= -(V\cdot\nabla)V - \nabla p + \Delta V + \sqrt{R}\theta k,\\
\partial\theta/\partial t &= -(V\cdot\nabla)\theta + \mathrm{Pr}^{-1}(\Delta\theta + \sqrt{R}V\cdot k),\\
\operatorname{div} V &= 0,
\end{aligned} \tag{11}$$

where $V = (V_1, V_2, V_3)$ is the velocity field of the fluid, θ is temperature, and p is pressure. Here, R and Pr are parameters and $k = (0,0,1)$. We suppose that the equations are posed on $\mathbb{R}^2 \times [0,1]$ with suitable boundary conditions. Eqns (11) model convection in a planar layer, and transform under a combination of the vector field and scalar actions; the velocity field of the fluid transforms under the vector field action of $\mathbf{E}(2)$ (the action of $\mathbf{E}(3)$ restricted to the planar layer) and the temperature and pressure transform under the scalar actions. Write $u = (V_1, V_2, V_3, \theta, p) : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^5$. Then, we have the action

$$u(x,z) \mapsto \rho_A u(\gamma^{-1}x, z), \tag{12}$$

with $\rho_A = \begin{pmatrix} A & 0 \\ 0 & I_3 \end{pmatrix}$, where A is a 2×2 orthogonal matrix (rotation or reflection) and I_3 is the 3×3 identity matrix.

In this article, we consider systems of PDEs equivariant with respect to the above actions of $\mathbf{E}(n)$. More generally, we consider all actions of $\mathbf{E}(n)$ on functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$ of the form

$$u(x,z) \mapsto \rho_A u(\gamma^{-1}x, z), \tag{13}$$

where ρ_A is any action of $\mathbf{O}(n)$ on the range $\mathbb{R}^s$. This class of actions appears to include those actions that occur in applications. Note that elements of $\mathbf{E}(n)$ are required to act in the standard way on the unbounded domain variables $\mathbb{R}^n$ and that translations act trivially on the range $\mathbb{R}^s$.

4 Universality

In this section, we give a complete description of universality for bifurcations with Euclidean symmetry, following [12]. We consider systems of PDEs that are equivariant with respect to an action of $\mathbf{E}(n)$ of the form (13). Suppose that such a system of PDEs undergoes steady-state bifurcation from a fully symmetric equilibrium, so that real eigenvalues pass through zero. Then it is shown in Melbourne [12] that generically the kernel of the linearized PDE is an absolutely irreducible representation of $\mathbf{E}(n)$.

Although the technical details of this result are beyond the scope of this article, we now give a concrete description of the structure of the critical eigenspaces that arise in $\mathbf{E}(n)$-equivariant steady-state bifurcation.

Suppose that u_0 is a critical eigenfunction in the form of a single Fourier mode so that $u_0(x) = be^{ik\cdot x}$ where $k \in \mathbb{R}^n$ and b is a (complexified) vector in $\mathbb{C}^s$. By symmetry, γu_0 is a critical eigenfunction for each $\gamma = (A, a) \in \mathbf{E}(n)$. We compute that

$$\begin{aligned}(\gamma u_0)(x) &= \rho_A u_0(\gamma^{-1}x) = \rho_A u_0(A^{-1}(x-a)) \\ &= \rho_A b\, e^{ik\cdot A^{-1}(x-a)} = \rho_A b\, e^{iAk\cdot(x-a)} \\ &= e^{-iAk\cdot a}\rho_A b\, e^{iAk\cdot x}. \end{aligned} \tag{14}$$

Hence, the net effect is that the wave vector k is rotated/reflected onto Ak while $b \mapsto e^{-iAk\cdot a}\rho_A b$. The group orbit of the critical eigenfunction u_0 leads to a space spanned by eigenfunctions with constant wavenumber $|k| = k_c$. Moreover, every such $k \in \mathbb{R}^n$ occurs.

If $k_c = 0$, then the action of $\mathbf{E}(n)$ on u_0 reduces to $b \mapsto \rho_A b$, and we obtain a representation of $\mathbf{O}(n)$. It follows that the irreducible actions of $\mathbf{E}(n)$ that arise in steady-state bifurcations with zero critical wavenumber are precisely the irreducible actions of $\mathbf{O}(n)$.

If $k_c \neq 0$, then there is a copy of $\mathbf{O}(n-1)$ that fixes k while $b \mapsto e^{-iAk\cdot a}\rho_A b$ for $A \in \mathbf{O}(n-1)$. The remainder of $\mathbf{O}(n)$ smears the wavevectors around the sphere $|k| = k_c$. When $n \geq 2$, the subspace spanned by this group orbit is infinite-dimensional. Actions of $\mathbf{E}(n)$ that are obtained in this way are irreducible if and only if the vector $b \in \mathbb{C}^s$ is restricted to an irreducible representation of $\mathbf{O}(n-1)$. Moreover, we obtain a distinct irreducible representation of $\mathbf{E}(n)$ for each irreducible representation of $\mathbf{O}(n-1)$ and each choice of $k_c > 0$.

Since the exact value of $k_c > 0$ is unimportant, we see that the irreducible actions of $\mathbf{E}(n)$ that arise in steady-state bifurcations with nonzero critical wavenumber are in one-to-one correspondence with the irreducible actions of $\mathbf{O}(n-1)$.

Theorem 4.1 (Melbourne [12]) *Suppose that an* $\mathbf{E}(n)$*-equivariant system of PDEs undergoes steady-state bifurcation with critical wave number* $k_c \geq 0$*. Generically, the critical eigenspace is an absolutely irreducible representation of* $\mathbf{E}(n)$*. If* $k_c = 0$*, then the universality classes are enumerated by the irreducible representations of* $\mathbf{O}(n)$*. If* $k_c > 0$*, then the universality classes are enumerated by the irreducible representations of* $\mathbf{O}(n-1)$*.*

Remark 4.2 There is a classification of the irreducible representations of $\mathbf{E}(n)$ (in a somewhat different context) due to Ito [26] and Mackey [27]. The above description of the action of $\mathbf{E}(n)$ on the critical eigenfunctions provides a concrete realization of their classification.

Specializing to the case of nonzero wavenumbers, we see that the results on universality listed in the introduction follow immediately from theorem 4.1. For instance, if $n = 1$, then $\mathbf{O}(n-1)$ collapses to the trivial group and has a single irreducible representation, whereas if $n = 2$, then $\mathbf{O}(n-1)$ is the two-element group $\{\pm 1\}$ which has two one-dimensional irreducible representations (the nontrivial element -1 acting trivially or nontrivially).

The representation ρ_A of $\mathbf{O}(n-1)$ on the critical eigenfunctions is a restriction of the original action ρ_A of $\mathbf{O}(n-1)$ in eqn (13). If we begin with the reaction-diffusion action (8), then the action of $\mathbf{O}(n-1)$ on the critical eigenfunctions can only be the trivial action. Hence there is a unique universality class for reaction-diffusion equations, for each $n \geq 1$.

In this section, we have focused on steady-state bifurcation, but the results for Hopf bifurcation are entirely analogous and can be proved using the same methods.

Theorem 4.3 *Suppose that an* $\mathbf{E}(n)$*-equivariant system of PDEs undergoes Hopf bifurcation with critical wavenumber* $k_c \geq 0$*. Generi-*

cally, the critical eigenspace is the direct sum of two absolutely irreducible representations of $\mathbf{E}(n)$. *If* $k_c = 0$, *then the universality classes are enumerated by the irreducible representations of* $\mathbf{O}(n)$. *If* $k_c > 0$, *then the universality classes are enumerated by the irreducible representations of* $\mathbf{O}(n-1)$.

As is the case for compact Lie groups [3], the critical eigenspace at Hopf bifurcation is generically an irreducible representation of $\mathbf{E}(n) \times S^1$ where the copy of S^1 arises from phase-shift symmetry.

5 Reduced systems of minimal size

In section 4, we investigated the universality properties of steady-state bifurcations in systems with Euclidean symmetry and we enumerated the universality classes in terms of the irreducible representations of $\mathbf{E}(n)$. In this section, we analyze the structure of reduced universal equations corresponding to these bifurcations.

For the moment, we are not concerned with extracting Ginzburg-Landau equations. Rather, we are concerned with deriving minimal systems of equations involving functions $v : \mathbb{R}^n \to \mathbb{R}^{s'}$, reducing from the original system of PDEs involving functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$. The aim of the reduction is threefold:

1. To factor out the bounded domain variables Ω.
2. To reduce the size s of the system to a *minimal* size s'.
3. To determine the action of $\mathbf{E}(n)$ on the reduced equations.

It is not always possible to reduce to a single equation, with $s' = 1$. The reason for this is purely algebraic: s' must be large enough to permit an action (13) of $\mathbf{E}(n)$ on the reduced system that restricts to the required action (14) on the critical eigenfunctions. For example, consider $\mathbf{E}(3)$-equivariant steady-state bifurcation with nonzero wavenumber. Then the action of $\mathbf{E}(3)$ on the critical eigenfunctions corresponds to an action of $\mathbf{O}(n-1) = \mathbf{O}(2)$. Suppose that this is the standard two-dimensional action of $\mathbf{O}(2)$. The smallest action of $\mathbf{O}(n) = \mathbf{O}(3)$ that contains this action of $\mathbf{O}(2)$ is the standard action on $\mathbb{R}^3$. Hence, even formally, the smallest possible value of s' is $s' = 3$.

In the remainder of this section, we recall the results in [12] for steady-state bifurcation with nonzero wavenumber, and then we state the analogous results for the other bifurcations.

5.1 Reduction for steady-state bifurcation, nonzero wavenumber

Suppose that an $\mathbf{E}(n)$-equivariant system of PDEs undergoes steady-state bifurcation with nonzero wavenumber. As shown in section 4, the action (14) of $\mathbf{E}(n)$ on the critical eigenfunctions is determined by a representation of $\mathbf{O}(n-1)$. Choose s' and an action ρ'_A of $\mathbf{O}(n)$ on $\mathbb{R}^{s'}$ such that the action ρ'_A restricted to the subgroup $\mathbf{O}(n-1)$ contains the aforementioned representation of $\mathbf{O}(n-1)$. Moreover, choose s' as small as possible. Then s' and ρ'_A are said to be *minimal* with respect to the representation of $\mathbf{O}(n-1)$.

Theorem 5.1 (Melbourne [12]) *Suppose that an $\mathbf{E}(n)$-equivariant system of PDEs involving functions $u : \mathbb{R}^n \times \Omega \to \mathbb{R}^s$ undergoes steady-state bifurcation with nonzero critical wavenumber $k_c > 0$. Choose $s' \geq 1$ and an action ρ'_A of $\mathbf{O}(n)$ on $\mathbb{R}^{s'}$ such that s' and ρ'_A are minimal with respect to the action of $\mathbf{O}(n-1)$ on the critical eigenfunctions. Then generically there is a reduction to a system of equations involving functions $v : \mathbb{R}^n \to \mathbb{R}^{s'}$ that is equivariant with respect to the $\mathbf{E}(n)$ action $v(x) \mapsto \rho'_A v(\gamma^{-1}x)$.*

The reduction preserves essential solutions near criticality (essential solutions are those that remain bounded and small in space and time). Naturally, $s' \leq s$.

In certain cases, we can obtain $s' = 1$. Again, a heuristic argument is possible. If we start off with the reaction-diffusion action (8), then the action of $\mathbf{O}(n-1)$ on critical eigenfunctions is necessarily trivial and can be realized by the scalar action. Also, if $n = 1$, then $\mathbf{O}(n-1)$ is trivial; so the action is trivial and can be realized by the scalar action. Hence, we may reduce to a scalar equation for reaction-diffusion equations and when $n = 1$. If $n \geq 2$ and we do not have the reaction-diffusion action, then there is no guarantee that reduction to a scalar equation is possible, only when the action of $\mathbf{O}(n-1)$ at the critical eigenfunction level happens to be trivial.

When $n = 2$, $\mathbf{O}(n-1) = \{\pm 1\}$ may act trivially or nontrivially. If $\mathbf{O}(n-1)$ acts trivially, we choose the scalar action. If $\mathbf{O}(n-1)$ acts nontrivially, we choose the pseudoscalar action. Thus, generically we achieve $s' = 1$ when $n = 2$, but there are two distinct universality classes.

Many important examples, such as the Boussinesq equations, are $\mathbf{E}(2)$-equivariant and reduce to a scalar equation. We would like to stress that this behavior is not universal — there are the two possibilities, scalar and pseudoscalar, that are equally likely from the mathematical point of view. Moreover, there is no obvious physical reason for distinguishing between the two, and it is necessary just as in Landau theory to see how the critical eigenfunctions transform under the action of $\mathbf{E}(2)$. This point appears to have been first observed by Sattinger [28] and is still often overlooked.

5.2 Reduction for the other bifurcations

We now generalize the results of the previous subsection to steady-state bifurcation with zero wavenumber and to Hopf bifurcation.

In the case of steady-state bifurcation with zero wavenumber, the critical eigenspace is generically an irreducible representation of $\mathbf{O}(n)$. Let s' be the dimension of this representation, and let ρ'_A be the representation. Generically, there is a reduction to a system of equations involving functions $v : \mathbb{R}^n \to \mathbb{R}^{s'}$ that is equivariant with respect to the $\mathbf{E}(n)$ action $v(x) \mapsto \rho'_A v(\gamma^{-1}x)$.

The corresponding results for Hopf bifurcation are exactly as would be expected from theorem 4.3 on universality. In the case of zero wavenumber, the critical eigenspace is generically the direct sum of two irreducible representations ρ'_A of $\mathbf{O}(n)$ of dimension s'. Generically, there is a reduction to a system of equations involving functions $v : \mathbb{R}^n \to \mathbb{R}^{2s'}$ that is equivariant with respect to the $\mathbf{E}(n)$

action $v(x) \mapsto (\rho'_A \oplus \rho'_A)v(\gamma^{-1}x)$.

Finally, the critical eigenspace for Hopf bifurcation with nonzero wavenumber generically transforms according to the direct sum of two isomorphic irreducible representations of $\mathbf{O}(n-1)$. Choose s' and a representation ρ'_A of $\mathbf{O}(n)$ of dimension s' as small as possible that restricts to the desired irreducible representation of $\mathbf{O}(n-1)$. Generically, there is a reduction to a system of equations involving functions $v : \mathbb{R}^n \to \mathbb{R}^{2s'}$ that is equivariant with respect to the $\mathbf{E}(n)$ action $v(x) \mapsto (\rho'_A \oplus \rho'_A)v(\gamma^{-1}x)$.

6 Ginzburg-Landau equations

In this section, we consider the derivation of Ginzburg-Landau equations near criticality for bifurcations with Euclidean symmetry. We consider the four cases corresponding to steady-state or Hopf bifurcation with zero or nonzero wavenumber. In the physics nomenclature [10], steady-state and Hopf are denoted by subscripts s and o respectively (o for oscillatory) and zero or nonzero wavenumber is denoted by III or I. (There is also type II which corresponds to zero wavenumber in a conservative system [10], but we do not consider such systems here.)

The main obstruction to deriving useful Ginzburg-Landau equations is the rotational degeneracy in the nonzero wavenumber cases (steady-state and Hopf) for $n \geq 2$.

6.1 Steady-state bifurcation, zero wavenumber (type III_s)

This case is often ignored (see for example [10]) but is instructive in the matter of universality.

First, we consider the case of the reaction-diffusion action (8). By the results in section 5, we may perform a preliminary reduction to a scalar equation involving functions $u : \mathbb{R}^n \to \mathbb{R}$. The critical Fourier modes have the form $b_k e^{ik\cdot x}$, where $k \sim 0$. The linear terms of the reduced equation have the form

$$\partial_t b_k = (\lambda - k^2 q(k^2)) b_k, \tag{15}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter, $q(k)$ is real and smooth, and generically $q(0) > 0$. Write

$$\lambda = \epsilon, \quad u(x) = \epsilon A(X), \quad X = \sqrt{\epsilon} x, \quad T = \epsilon t. \tag{16}$$

Note that the amplitude function $A : \mathbb{R}^n \to \mathbb{R}$ is real valued. At linear level, A satisfies $\partial A_T = A + q(0)\Delta A + O(\epsilon)$. Generically, there is a quadratic term cA^2, $c \in \mathbb{R}$. Rescaling the coefficients $q(0)$ and c yields the Ginzburg-Landau equation

$$\partial_T A = A + \Delta A \pm A^2. \tag{17}$$

There are technical issues which we have passed over here concerning the mathematical treatment of higher derivatives as higher order terms. For details, see [17, 12]. It follows from the implicit function theorem that nondegenerate solutions to this equation correspond to transcritical branches (amplitude proportional to λ) for the original system of PDEs.

Now, we consider the complications that set in when we do not have the reaction-diffusion action. The case $n = 1$ is straightforward. There are two universality classes leading to the following two possibilities for the Ginzburg-Landau equation:

$$\partial_T A = A + D^2 A \pm A^2, \qquad \text{or} \qquad \partial_T A = A + D^2 A \pm A^3. \tag{18}$$

These two possibilities are distinguished by the action of reflections on the critical eigenfunctions (either $u(x) \mapsto u(-x)$ or $u(x) \mapsto -u(-x)$).

The case $n = 2$ is not straightforward! If rotations in $\mathbf{E}(2)$ happen to act trivially on the critical eigenfunctions, then we obtain the two possibilities (18) with D^2 replaced by Δ. In addition, there are the (countably many) two-dimensional representations of $\mathbf{O}(2)$. The minimal reduced equations involve functions $u : \mathbb{R}^2 \to \mathbb{R}^2$ and lead to a system of two coupled Ginzburg-Landau equations. For the standard two-dimensional representation, a tedious calculation shows that the linear terms have the form

$$\partial_T A_1 = A_1 + c\frac{\partial^2 A_1}{\partial x_1^2} + d\frac{\partial^2 A_1}{\partial x_2^2} + (c-d)\frac{\partial^2 A_2}{\partial x_1 \partial x_2},$$
$$\partial_T A_2 = A_2 + (c-d)\frac{\partial^2 A_1}{\partial x_1 \partial x_2} + d\frac{\partial^2 A_2}{\partial x_1^2} + c\frac{\partial^2 A_2}{\partial x_2^2}, \qquad (19)$$

where $c, d \in \mathbb{R}$. The remaining two-dimensional representations force $c = d$, but the terms involving higher order derivatives are complicated.

6.2 Steady-state bifurcation, nonzero wavenumber (type I_s)

Suppose that an $\mathbf{E}(1)$-equivariant system of PDEs undergoes steady-state bifurcation with nonzero wavenumber k_c. For definiteness, we suppose that $k_c = 1$. By the results of section 5, we may perform a preliminary reduction to a scalar equation involving functions $u : \mathbb{R} \to \mathbb{R}$. We then make the ansatz (change of coordinates)

$$\lambda = \epsilon^2, \quad u(x) = \epsilon(A(X)e^{ix} + \overline{A}(X)e^{-ix}), \quad X = \epsilon x, \quad T = \epsilon^2 t, \qquad (20)$$

where $A : \mathbb{R} \to \mathbb{C}$ is a slowly varying complex amplitude function. (Note that A has too many degrees of freedom since u is real-valued. Again, we refer to [17] for a mathematical justification that this is a well-defined change of coordinates.) This change of coordinates leads to the standard Ginzburg-Landau equation on the line; see eqn (1). As mentioned in the introduction, it follows from [12] that eqn (1) is universal and that 'nondegenerate' solutions correspond to branches of solutions to the underlying equations. There are also the approximate determinacy results of [14, 15, 16].

When $n = 2$, we can reduce to a single equation in $u : \mathbb{R}^2 \to \mathbb{R}$ transforming under either the scalar action or pseudoscalar action. The next step of extracting amplitude equations is highly nontrivial. There are various amplitude equations in the literature, none of which is completely satisfactory; see the survey [10]. (This is in contrast to the zero wavenumber case, where satisfactory, though sometimes complicated, amplitude equations can be found for all n.) The problem here is the rotational degeneracy mentioned in section 2.3. Approaches to this problem can be found in [29, 30, 31, 32]. We are presently working on extending the methods in [17, 12] to this situation.

6.3 Hopf bifurcation, zero wavenumber (type III_o)

This is similar to steady-state bifurcation with zero wavenumber; see also Schneider [33]. Universal Ginzburg-Landau equations can be written down for all n and are determined by the irreducible representations of $\mathbf{O}(n)$. For $n = 1$, and for reaction-diffusion equations (all n) we can apply the results of section 5 and reduce first to a scalar equation involving functions $u : \mathbb{R}^n \to \mathbb{R}$. Suppose that the critical eigenvalues are given by $\pm i\omega$ where $\omega > 0$. We make the standard substitution

$$\lambda = \epsilon^2, \quad u(x) = \epsilon(A(X)e^{i\omega t} + \overline{A}(X)e^{-i\omega t}), \quad X = \epsilon x, \quad T = \epsilon^2 t, \tag{21}$$

where $A : \mathbb{R}^n \to \mathbb{C}$ is a complex amplitude function. This ansatz leads to the complex Ginzburg-Landau equation

$$\partial_T A = A + c\Delta A + d|A|^2 A, \tag{22}$$

where $c, d \in \mathbb{C}$.

When the action of $\mathbf{E}(n)$ on the critical eigenfunctions corresponds to a higher-dimensional representation of $\mathbf{O}(n)$, we may obtain systems of coupled Ginzburg-Landau equations but the couplings are highly nontrivial as in eqn (19).

6.4 Hopf bifurcation, nonzero wavenumber (type I_o)

As in the case of steady-state bifurcation with nonzero wavenumber, there is a rotational degeneracy that sets in when $n \geq 2$. However, a new difficulty arises even when $n = 1$. Knobloch and De Luca [34] argue in favor of nonlocal *mean field Ginzburg-Landau equations* and these equations have been justified (approximately over large time intervals) by Pierce and Wayne [35] and Schneider [36]. So far, we have not attempted to apply the methods in [17, 12] to this situation.

7 Normal form symmetry of amplitude equations

In this section, we investigate the structure, particularly the symmetry, of Ginzburg-Landau equations such as the Ginzburg-Landau equation on the line (1) which we restate for convenience:

$$\partial A_t = A + D^2 A \pm |A|^2 A. \tag{23}$$

Recall, from section 6.2, that this equation is derived via an ansatz $u(x) = A(x)e^{ix} + \overline{A}(x)e^{-ix}$ from a scalar equation in u. (The ϵ's do not affect our analysis of symmetries and are suppressed in this section.) The scalar action of $\mathbf{E}(1)$ on u induces on A the action

$$A(x) \mapsto A(x+a)e^{ia}, \tag{24}$$

$$A(x) \mapsto \overline{A}(-x). \tag{25}$$

However, inspection of eqn (23) reveals the additional nonphysical 'symmetries'

$$A \mapsto Ae^{i\theta}, \tag{26}$$

$$A \mapsto \overline{A}. \tag{27}$$

It turns out that the symmetry (27) is a consequence of the low order truncation — higher order terms include $i|A|^2DA$, and $iA^2D\overline{A}$ which break the symmetry (27). It is commonplace for physicists to incorporate such terms.

We are more interested in the circle symmetry (26) which is retained by higher order truncations. A calculation shows that terms of the form

$$A^p\overline{A}^q e^{i(p-q-1)x} \tag{28}$$

break the circle symmetry (26) if $p - q \neq 1$ whilst preserving the physical symmetries (24, 25). Melbourne [17] proves that such terms can be removed to arbitrarily high order. That is, exact amplitude equations include such terms, but such terms can be neglected in any finite order truncation. Thus, the circle symmetry (26) is a *normal form symmetry* that can be justified to arbitrarily high order.

7.1 Implications for solutions

From the point of view of dynamical systems and bifurcation theory, it is evident that nonconstant coefficient terms such as (28) are unavoidable in the tail (this is particularly easy to see if the underlying scalar PDE is not odd in u — an easy argument [17] shows that the amplitude equation is not odd in A, whereas the constant coefficient terms are odd) and have significant consequences for solutions of the Ginzburg-Landau equation.

Normal form symmetry occurs in steady-state/Hopf mode interaction in systems without symmetry. The equations in normal form can be solved fairly completely [37], but the full equations have delicate chaotic dynamics. Although the normal form equations control many of the details of the bifurcation, this is only a first step to understanding the full equations. Dynamical systems theorists do not advocate ignoring the effects of the tail altogether; neither should Ginzburg-Landau theorists.

Coullet et al. [38] used these ideas from dynamical systems to obtain time-independent spatially chaotic solutions in the Ginzburg-Landau equation. They did this by adding an external 'periodic forcing' term to the standard truncation of the Ginzburg-Landau equation, so as to break the translation invariance of the underlying problem. It follows from our results that such terms already occur *internally* and it is not necessary to break the underlying translation invariance. (This example demonstrates that the effects that we are talking about are of interest to physicists.)

There is one class of solutions that is particularly sensitive to terms that break the normal form symmetry. Suppose that the non-truncated amplitude equation in A has the normal form symmetry to all orders and hence is constant coefficient. It follows from standard implicit function theorem arguments that for each $\omega > 0$, there is a branch of spatially periodic equilibria with period $2\pi/\omega$ bifurcating from the trivial solution $A = 0$ at $\lambda = \lambda_\omega > 0$. Moreover, $\lambda_\omega \to 0$ as $\omega \to 0$. Provided ω is small, these spatially periodic solutions correspond to branches of solutions for the underlying PDE. But if ω is irrational, we have obtained branches of spatially quasiperiodic solutions with independent frequencies 1 and ω. This is absurd, since we have somehow bypassed the problem of small divisors. Of course, this argument breaks down precisely because of the presence of terms in the tail of the amplitude equation that are not constant coefficient. Iooss & Los [39] show that those quasiperiodic solutions with ω Diophantine exist for the underlying PDE and therefore survive the terms in the tail.

We have made the comparison with low-codimension bifurcation theory. In fact, the tail is likely to be of even more importance for the Ginzburg-Landau equations than in bifurcation theory. (i) In the bifurcation theory, the exotic behavior often occurs in thin cuspoidal wedges in parameter space. In the Ginzburg-Landau equation, there is only one parameter so that the thin wedges are everything. (ii) Normal form symmetry leads to group orbits of solutions. Often these group orbits are normally hyperbolic so that breaking the symmetry in the tail picks out some of these solutions. Solutions A on a group orbit are essentially the same, but as pointed out in Pomeau [19] the corresponding solutions $u = Ae^{ix} + \overline{A}e^{-ix}$ need not be physically identical. Hence, with probability one, simulation of the truncated Ginzburg-Landau equation yields spurious nonphysical solutions. The situation is worse if there is nontrivial dynamics in the truncated equation. Breaking the normal form symmetry constitutes 'forced symmetry breaking' which is a poorly understood and highly complicated subject.

7.2 Normal form symmetry in the complex Ginzburg-Landau equation

Analogous questions arise for the complex Ginzburg-Landau equation (22) in $\mathbf{E}(n)$-equivariant Hopf bifurcation with zero wavenumber. We consider mainly the case of reaction-diffusion equations, and so we first reduce to a scalar equation in $u : \mathbb{R}^n \to \mathbb{R}$. Normal form symmetries arise again, and for similar reasons, but take a rather different form. Since the ansatz $u(x) = A(x)e^{i\omega t} + \overline{A}(x)e^{i\omega t}$ is constant coefficient (in x!), the complex amplitude A transforms under the action of $\mathbf{E}(n)$ in the same way as the scalar equation did. In particular, the amplitude equations are constant coefficient to all orders. However, the scalar equation for u has a symmetry that is usually taken for granted — translations in time. (The equations are autonomous.) Once again, we obtain the normal form symmetry $A \mapsto Ae^{i\theta}$ in the complex Ginzburg-Landau equation. Further, the time-dependence in the ansatz means that the amplitude equations are nonautonomous. The autonomous nature of the complex Ginzburg-Landau equation is a normal form symmetry that can be retained in truncations of arbitrarily high order, but which is broken in the tail.

In the case $n = 1$, there are two universality classes and hence two kinds of complex Ginzburg-Landau equation: scalar and pseudoscalar. The normal form symmetry means that the distinction between the scalar and pseudoscalar equations is seen only at arbitrarily high order (since the actions are equivalent for odd terms).

Acknowledgment This work was supported in part by NSF Grant DMS-9704980.

References

[1] Landau, L.D., Second order phase transitions, *Phys. Z. Sowjetunion*, **11**, pp. 545-549, 1938.

[2] Michel, L., Symmetry defects and broken symmetry. Configurations. Hidden symmetry, *Rev. of Mod. Phys.*, **52**, pp. 617-651, 1980.

[3] Golubitsky, M., Stewart, I.N. & Schaeffer, D.G., Singularities and Groups in Bifurcation Theory, Vol. 2. *Appl. Math. Sci.*, **69**, Springer, New York, 1988.

[4] Ruelle, D., Bifurcations in the presence of a symmetry group, *Arch. Rat. Mech. Anal.*, **51**, pp. 136-152, 1973.

[5] Sattinger, D., *Group Theoretic Methods in Bifurcation Theory*, Lecture Notes in Math., **762**, Springer, Berlin, 1979.

[6] Vanderbauwhede, A., Local Bifurcation and Symmetry, *Pitman Research Notes in Math.* **75**, Boston, 1982.

[7] Crawford, J.D. & Knobloch, E., Symmetry and symmetry-breaking bifurcations in fluid dynamics, *Annu. Rev. Fluid Mech.*, **21**, pp. 341-387, 1991.

[8] Landau, L.D. & Lifshitz, E.M., *Statistical Physics*, Part 1, Pergamon, New York, 1958.

[9] Stewartson, K. & Stuart, J.T., A nonlinear instability theory for a wave system in plane Poiseuille flow, *J. Fluid Mech.*, **48**, pp. 529-540, 1971.

[10] Cross, M.C. & Hohenberg, P.C., Pattern formation outside of equilibrium, *Rev. of Mod. Phys.*, **65**, pp. 851-1112, 1993.

[11] Newell, A.C., The dynamics and analysis of patterns. In: *Lectures in the Sciences of Complexity, 1*, Addison-Wesley, New York, pp. 107-173, 1989.

[12] Melbourne, I., Steady-state bifurcation with Euclidean symmetry, *Trans. Amer. Math. Soc.*, **351**, pp. 1575-1603, 1999.

[13] Mielke, A., Reduction of PDEs on domains with several unbounded directions: A first step towards modulation equations, *ZAMP*, **43**, pp. 449-470, 1992.

[14] Schneider, G., Error estimates for the Ginzburg-Landau approximation, *ZAMP*, **45**, pp. 433-457, 1994.

[15] van Harten, A., On the validity of the Ginzburg-Landau equation, *J. Nonlin. Sci.*, **1**, pp. 397-422, 1991.

[16] Mielke, A. & Schneider, G., Attractors for modulation equations on unbounded domains — existence and comparison, *Nonlinearity*, **8**, pp. 743-768, 1995.

[17] Melbourne, I., Derivation of the time-dependent Ginzburg-Landau equation on the line, *J. Nonlin. Sci.*, **8**, pp. 1-15, 1998.

[18] Iooss, G., Mielke, A. & Demay, Y., Theory of steady Ginzburg-Landau equation in hydrodynamic stability problems, *Eur. J. Mech., B/Fluids*, **8**, pp. 229-268, 1989.

[19] Pomeau, Y., Nonadiabatic phenomena in cellular structures. In: *Cellular Structures in Instabilities*, eds J. Wesfried and S. Zalesky, Springer, New York, pp. 207-214, 1984.

[20] Devonshire, A.F., Theory of barium titanate. Part I, *Phil. Mag.*, **40**, pp. 1040-1063, 1949.

[21] Chossat, P., Lauterbach, R. & Melbourne, I., Steady-state bifurcation with $\mathbf{O}(3)$-symmetry, *Arch. Rat. Mech. Anal.*, **113**, pp. 313-376, 1990.

[22] Field, M.J., Symmetry breaking for compact Lie groups, *Mem. Amer. Math. Soc.*, **120**, 1996.

[23] Swift, J.B. & Hohenberg, P.C., Hydrodynamic fluctuations at the convective instability, *Phys. Rev. A*, **15**, pp. 319-328, 1977.

[24] Bosch Vivancos, I., Chossat, P. & Melbourne, I., New planforms in systems of partial differential equations with Euclidean symmetry, *Arch. Rat. Mech. Anal.*, **131**, pp. 199-224, 1995.

[25] Dionne, B. & Golubitsky, M., Planforms in two and three dimensions, *ZAMP*, **43**, pp. 36-62, 1992.

[26] Ito, S., Unitary representations of some linear groups II, *Nagoya Math. J.*, **5**, pp. 79-96, 1953.

[27] Mackey, G.W., Induced representations of locally compact groups I, *Annals of Math.*, **55**, pp. 101-139, 1952.

[28] Sattinger, D., Group representation theory, bifurcation theory and pattern formation, *J. Funct. Anal.*, **28**, pp. 58-101, 1978.

[29] Newell, A.C. & Whitehead, J.A., Finite bandwidth, finite amplitude convection, *J. Fluid Mech.*, **38**, pp. 279-303, 1969.

[30] Segel, L.A., Distant side-walls cause slow amplitude modulation of cellular convection, *J. Fluid Mech.*, **38**, pp. 203-224, 1969.

[31] Cross, M.C. & Newell, A.C., Convection patterns in large aspect ratio systems, *Physica D*, **10**, pp. 299-320, 1984.

[32] Gunaratne, G., Complex spatial patterns on planar continua, *Phys. Rev. Lett.*, **71**, pp.1367-1370, 1993.

[33] Schneider, G., Hopf bifurcation in spatially extended reaction-diffusion systems, *J. Nonlin.Sci.*, **8**, pp. 17-41, 1998.

[34] Knobloch, E. & De Luca, J., Amplitude equations for travelling wave convection, *Nonlinearity*, **3**, pp. 975-980, 1990.

[35] Pierce, R.D. and Wayne, C.E., On the validity of mean-field amplitude equations for counterpropogating wavetrains, preprint, 1994.

[36] Schneider, G., Justification of mean-field coupled modulation equations, *Proceedings of the Royal Society of Edinburgh A*, **127**, pp. 639-650, 1997.

[37] Guckenheimer, J. & Holmes, P., Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. *Appl. Math. Sci.*, **42**, Springer, New York, 1983.

[38] Coullet, P., Elphick, C. & Repaux, D., Nature of spatial chaos, *Phys. Rev. Lett.*, **58**, pp. 431-434, 1987.

[39] Iooss, G. & Los, J., Bifurcation of spatially quasiperiodic solutions in hydrodynamic stability problems, *Nonlinearity*, **3**, pp. 851-871, 1990.