Decay of Correlations for Slowly Mixing Flows

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Abstract

We show that polynomial decay of correlations is prevalent for a class of nonuniformly hyperbolic flows. These flows are the continuous time analogue of a class of nonuniformly hyperbolic diffeomorphisms for which Young proved polynomial decay of correlations. Roughly speaking, in situations where the decay rate $O(1/n^\beta)$ has previously been proved for diffeomorphisms, we establish the decay rate $O(1/t^\beta)$ for flows. Applications include certain classes of semidispersing billiards, as well as dispersing billiards with vanishing curvature.

In addition, we obtain results for suspension flows with unbounded roof functions. In particular, the classical planar Lorentz flow with a doubly periodic array of circular scatterers has decay rate $1/t$ as anticipated by physicists.

1 Introduction

Dolgopyat [11] has shown that uniformly hyperbolic (Axiom A) flows typically mix rapidly, faster than any polynomial rate, for sufficiently smooth observables. The restriction to typical flows is necessary; there exist uniformly hyperbolic flows that mix but at an arbitrarily slow rate [24, 22]. We note that so far, exponential decay of correlations has been proved only in very special cases [10, 17, 23].

In previous work [20], we extended Dolgopyat's results to a class of nonuniformly hyperbolic flows. These flows are the continuous time analogue of a class of discrete time nonuniformly hyperbolic systems that are known, by the results of Young [27], to have exponential decay of correlations. In this context, we proved that again the flows typically mix faster than any polynomial rate.

In this paper, we consider nonuniformly hyperbolic flows for which the analogous class of discrete time system is known, by Young [28], to have polynomial decay of correlations. We show that the flows typically have polynomial decay of correlations too, with the same polynomial rate (as upper bound).

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The general set up is that $T : M \to M$ is a nonuniformly hyperbolic diffeomorphism in the sense of Young [27] but with a polynomial return time function r as in [28]. In particular, $T : M \to M$ is modelled by a tower $f : \Delta \to \Delta$ constructed over a "uniformly hyperbolic" base $Y \subset M$. The degree of nonuniformity is measured by the return time function $r : Y \to \mathbb{Z}^+$ to the base. It is assumed that Λ intersects its unstable manifolds in positive Lebesgue measure sets and that $\int r \, d\mu^u < \infty$ where μ^u denotes Lebesgue measure on unstable manifolds. Then there exists a physical (SRB) T-invariant ergodic probability measure ν.

Given a Hölder continuous roof function $h : M \to \mathbb{R}^+$, form the suspension $M^h = \{(x,u) \in M \times \mathbb{R} : 0 \le u \le h(x)\}/\sim$ where $(x, h(x)) \sim (Tx, 0)$. The suspension flow $\phi_t : M^h \to M^h$ is given by $\phi_t(x,u) = (x, u+t)$ computed modulo identifications with ϕ_t-invariant ergodic probability measure $\nu^h = \nu \times \text{lebesgue}/\int_M h \, d\nu$.

Suppose that the return time function satisfies the polynomial tails condition

$$\mu^u(y \in Y : r(y) > n) = O(1/n^{\beta+1}), \quad \beta > 0.$$

Under this assumption, Young [28] obtained the decay rate

$$\int_M v \, w \circ T^n \, d\nu - \int_M v \, d\nu \int_M w \, d\nu = O(1/n^\beta), \tag{1.1}$$

for the discrete time dynamics and Hölder observables $v, w : M \to \mathbb{R}$. We prove that typically the decay of correlations for the suspension flow satisfies

$$\rho_{v,w}(t) = \int_{M^h} v \, w \circ \phi_t \, d\nu^h - \int_{M^h} v \, d\nu^h \int_{M^h} w \, d\nu^h = O(1/t^\beta), \tag{1.2}$$

provided $v, w : M^h \to \mathbb{R}$ are sufficiently regular.

Remark 1.1 (i) In certain situations, estimate (1.1) is sharp [14, 15, 25] and it seems likely that estimate (1.2) is also sharp in the generality that it is proved.

(ii) As in [11, 20], the results in this paper hold only for observables that are sufficiently smooth in the flow direction. In particular, our results do not apply to the position variable in the Lorentz flow examples below.

(iii) The approach in this paper works for general decay rates of $\mu^u(y \in Y : r(y) > n)$, see Sections 3 and 4. We note that the calculations are considerably simpler in the special case $\mu^u(y \in Y : r(y) > n) = O(1/n^{\beta+1})$, $\beta > 1$.

(iv) Throughout this paper, we require that the roof function h is bounded below away from zero. (Such an assumption was not required in [20].) Intuitively, one expects that the violation of this condition may actually accelerate mixing (for example in the case of Sinaĭ billiards with cusps, see [5, p. 15] or [6, Section 5.6]).

Our methods apply also to suspension flows with unbounded roof functions. We assume an exponential tails condition $\mu^u(y \in Y : r(y) > n) = O(\gamma^n)$, $\gamma \in (0,1)$ for the return time r, and consider an unboundedness assumption of the type

$$\mu^u(x \in M : h(x) > n) = O(1/n^{\beta+1}), \quad \beta > 0,$$

for the roof function h. (For technical reasons, the actual assumption is slightly more complicated, see Section 2.3 for the precise statement.) We prove that typically, for $\epsilon > 0$ arbitrarily small

$$\rho_{v,w}(t) = O(1/t^{\beta-\epsilon}). \tag{1.3}$$

Remark 1.2 In future work, we expect to improve estimate (1.3) to

$$\rho_{v,w}(t) = O((\ln t)^{\beta+1}/t^{\beta}). \tag{1.4}$$

Indeed we obtain this improved estimate for nonuniformly expanding semiflows, see Theorem 2.7.

It is possible that the logarithmic factor in (1.4) is an artifact of our method, but it seems more likely that additional assumptions on r and h jointly are required to remove it. This is discussed in Section 6.4. It may be possible to verify such conditions in specific situations, such as in Example 1.6.

We now list some applications of our results. A good source of examples are provided by billiards and the associated Lorentz flows [5, 6]. However, it should be emphasized that the results apply generally to nonuniformly hyperbolic systems modelled by Young towers. In particular, whereas decay of correlations holds at the specified rate for *all* Lorentz flows in the examples below, it is well-known even in the uniformly hyperbolic context, and hence certainly in the generality of this paper, that positive results can be expected only for *typical* flows. As in [20], it suffices that any four periodic orbits intersecting the base Y have periods satisfying a Diophantine-type condition, see Corollary 2.4. (In the uniformly hyperbolic case, it suffices to consider any pair of periodic orbits [11].)

Example 1.3 (Intermittency-type semiflows) Various authors including [15, 18, 28] have studied intermittency (Pomeau-Manneville) maps of the type $T : [0,1] \to [0,1]$ given by

$$Tx = \begin{cases} x(1+2^{\alpha}x^{\alpha}) & 0 \le x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} \le x < 1 \end{cases}$$

for $0 < \alpha < 1$, where there is an indifferent fixed point at 0. There is a unique absolutely continuous ergodic invariant probability measure ν and for $\eta > 0$ there is a constant C such that

$$\Big|\int_{[0,1]} v\, w \circ T^n \, d\nu - \int_{[0,1]} v \, d\nu \int_{[0,1]} w \, d\nu\Big| \le C\|v\|_{C^\eta}|w|_\infty/n^\beta, \qquad \beta = \tfrac{1}{\alpha} - 1,$$

for all $v \in C^\eta([0,1])$, $w \in L^\infty([0,1])$, $n \ge 1$. The decay rate $1/n^\beta$ is optimal [14, 15, 25]. Furthermore, the upper bound $O(1/n^\beta)$ was obtained in [28] via the construction of a Young tower with tail decay rate $1/n^{\beta+1}$.

Now construct the suspension semiflow $\phi_t : [0,1]^h \to [0,1]^h$ where $h : [0,1] \to \mathbb{R}^+$ is a Hölder continuous roof function. Assume that $v : [0,1]^h \to \mathbb{R}$ is C^∞ along the flow direction with C^η derivatives for some $\eta > 0$. For typical roof functions h, it follows from Theorem 2.2 below that there exists a constant C_v such that

$$|\rho_{v,w}(t)| \le C_v |w|_\infty / t^\beta,$$

for all $w \in L^\infty([0,1]^h)$, $t > 0$.

Example 1.4 (Semidispersing Lorentz flows) Chernov & Zhang [7] consider a class of semidispersive billiards with tables of the form $R - \{B_1 \cup \cdots \cup B_r\}$ where R is a rectangle and $B_1, \dots, B_r \subset \operatorname{Int} R$ are disjoint strictly convex scatterers with C^3 boundaries (see [7, Figure 2(a)]). Building upon ideas of Markarian [19], it is shown in [7, 9] that the correlation function for the billiard map (for Hölder observables) decays as $O(1/n)$. A byproduct of the proof (see [7, Section 3] or [9, Section 2]) is the existence of a Young tower with tails decaying as $O(1/n^2)$. Hence, it follows from Theorem 2.6 below (noting Remark 2.3 and Section 5.3) that the corresponding Lorentz flows have decay rates $\rho_{v,w}(t) = O(1/t)$ for observables v, w sufficiently smooth in the flow direction. (Theorem 2.6 guarantees the decay rate for typical configurations of scatterers, but the results in Section 5.3 guarantee this for all configurations.)

Example 1.5 (Dispersing Lorentz flows with vanishing curvature) Chernov & Zhang [8] study a class of finite horizon planar periodic dispersing billiards where the scatterers have smooth strictly convex boundary with nonvanishing curvature, except that the curvature vanishes at two points. Moreover, it is assumed that there is a periodic orbit that runs between the two flat points, and that the boundary near these flat points has the form $\pm(1 + |x|^b)$ for some $b > 2$. The correlation function for the billiard map decays as $O((\ln n)^{\beta+1}/n^\beta)$ where $\beta = (b+2)/(b-2) \in (1, \infty)$. Again, a byproduct of the proof is the existence of a Young tower with tails decaying as $O((\ln n)^{\beta+1}/n^{\beta+1})$. Hence, it follows from Theorem 2.6 and Section 5.3 that the corresponding Lorentz flows have decay rates $O((\ln t)^{\beta+1}/t^\beta)$. (It is anticipated in [8] that the logarithmic factors for the billiard map, and hence for the flow, may be artifacts of the method.)

Example 1.6 (Infinite horizon planar periodic Lorentz gas) The planar periodic Lorentz gas is a class of examples introduced by Sinaĭ [26]. The billiard map $T : M \to M$ has exponential decay of correlations, as shown by Young [27] in the finite horizon case and Chernov [3] in the infinite horizon case. In both cases, the map is modelled by a Young tower with exponential tails.

In the finite horizon case, Chernov [4] has recently proved that correlations for the Lorentz flow decay at least stretched exponentially. (Previously [20] showed that the decay is typically faster than any polynomial rate.) For the infinite horizon case, it is widely expected that the decay rate is $1/t$, see [13]. A calculation shows that $\mu(x \in M : h(x) > n) = O(1/n^2)$, and it follows from Theorem 2.7 below (noting Remark 2.9 and Section 5.3) that the Lorentz flow has decay rate $O(t^{1-\epsilon})$ where $\epsilon > 0$ is arbitrarily small.

Example 1.7 (Classical infinite horizon Lorentz gas) For a certain subclass of the infinite horizon Lorentz gases considered in Example 1.6, we obtain exactly the expected decay rate $1/t$. Let $R \subset \mathbb{R}^2$ be a rectangle with scatterers strictly contained inside the rectangle. Suppose that the configuration of scatterers inside R is preserved by the two reflection symmetries of the rectangle. Form a periodic array of scatterers by tiling the plane with this configuration of scatterers. Then restricting to observables that respect the reflection symmetries (and the spatial periodicity), the resulting planar periodic Lorentz gas has infinite horizon with decay rate $1/t$. This follows from the conclusion of Example 1.4, since such examples are equivalent to the class of Lorentz gases just described via the standard technique of reflecting the rectangle in Example 1.4.

Clearly, this construction includes the classical case of a doubly periodic array of circular scatterers.

The remainder of the paper is organised as follows. In Section 2, we state the main results first in the simpler context of nonuniformly expanding semiflows, and then for nonuniformly hyperbolic flows. Also, we present an outline of the strategy of the proof. The proof for nonuniformly expanding semiflows is carried out in Section 3 and 4. The modifications for nonuniformly hyperbolic flows are described in Section 5. The modifications for (semi)flows with unbounded roof function are described in Section 6.

2 Statement of the main results

In this section, we state our main results. In Subsection 2.1, we consider the technically simpler case of nonuniformly expanding semiflows. In Subsection 2.2, we consider nonuniformly hyperbolic flows. The case of unbounded roof function is discussed in Subsection 2.3. In Subsection 2.4, we describe the strategy of the proof, focusing for simplicity on the result in Subsection 2.1.

2.1 Nonuniformly expanding semiflows

Let (X, d) be a locally compact separable bounded metric space with Borel probability measure μ_0 and let $T : X \to X$ be a nonsingular transformation for which μ_0 is ergodic. Let $Y \subset X$ be a measurable subset with $\mu_0(Y) > 0$, and let $\{Y_j\}$ be an at most countable measurable partition of Y with $\mu_0(Y_j) > 0$. We suppose that there is an L^1 *return time* function $r : Y \to \mathbb{Z}^+$, constant on each Y_j with value $r(j) \geq 1$, and constants $\lambda > 1$, $\eta \in (0, 1)$, $C \geq 1$ such that for each $j \geq 1$,

(1) $F = T^{r(j)} : Y_j \to Y$ is a bijection.

(2) $d(Fx, Fy) \geq \lambda d(x, y)$ for all $x, y \in Y_j$.

(3) $d(T^\ell x, T^\ell y) \leq C d(Fx, Fy)$ for all $x, y \in Y_j$, $0 \leq \ell < r(j)$.

(4) $g_j = \frac{d(\mu_0|Y_j \circ F^{-1})}{d\mu_0|Y}$ satisfies $|\log g_j(x) - \log g_j(y)| \le Cd(x,y)^\eta$ for all $x, y \in Y$.

Such a map $T : X \to X$ is called *nonuniformly expanding.* There is a unique T-invariant probability measure ν equivalent to μ_0 (see for example [28, Theorem 1]).

Remark 2.1 Discarding sets of zero measure, we have assumed without loss that the induced map $F : Y \to Y$ is defined everywhere on Y. This simplifies the formulation below of certain hypotheses involving periodic points.

Let $h : X \to \mathbb{R}^+$ be a roof function such that for all $j \ge 1$,

(5) $h, \frac{1}{h} \in L^\infty(X)$ and $|h(x) - h(y)| \le Cd(x,y)^\eta$ for all $x, y \in T^\ell Y_j$, $0 \le \ell < r(j)$.

Define the suspension semiflow $T_t : X^h \to X^h$ with invariant ergodic measure ν^h as in the introduction. Let $\rho_{v,w}(t)$ denote the correlation function corresponding to observables $v, w : X^h \to \mathbb{R}$.

For $m \ge 1$, $\eta > 0$, let $C^{m,\eta}(X^h)$ consist of those $v : X^h \to \mathbb{R}$ for which $\|v\|_{m,\eta} = \|v\|_\eta + \|\partial_t v\|_\eta + \cdots + \|\partial_t^m v\|_\eta < \infty$, where ∂_t denotes the derivative in the flow direction and

$$\|v\|_\eta = |v|_\infty + \sup_{(x,u)\neq(y,u)} |v(x,u) - v(y,u)|/d(x,y)^\eta.$$

Suppose that $Z \subset Y$ is a finite union of partition elements Y_j. Let $p \in Z$ be a periodic point for $F : Y \to Y$ such that $F^i p \in Z$ for all $i \ge 1$. We associate to p the triple $(\tau, d, q) \in \mathbb{R}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ where τ is the period of p under the semiflow T_t, d is the period under the map T, and q is the period under the induced map F (so $d = \sum_{i=0}^{q-1} r(F^i p)$ and $\tau = \sum_{i=0}^{d-1} h(T^i p)$). Let $\mathcal{T}_Z$ denote the set of such triples.

Theorem 2.2 *Let $T : X \to X$ be a nonuniformly expanding map and $h : X \to \mathbb{R}^+$ a roof function satisfying properties (1)–(5). Assume that $\mu_0(y \in Y : r(y) > n) = O((\ln n)^\gamma n^{-(\beta+1)})$, for some $\beta > 0$, $\gamma \ge 0$. Let $Z \subset Y$ be a finite union of partition elements Y_j.*

Suppose that there do **not** *exist constants $C, m \ge 1$ such that*

$$|\rho_{v,w}(t)| \le C\|v\|_{m,\eta}|w|_\infty(\ln t)^\gamma t^{-\beta},$$

for all $t > 0$, $v \in C^{m,\eta}(X^h)$, $w \in L^\infty(X^h)$.

Then there exist sequences $b_k \in \mathbb{R}$ with $|b_k| \to \infty$, and ω_k, $\varphi_k \in [0, 2\pi)$; and constants $\alpha > 0$ arbitrarily large, $C, \beta_0 \ge 1$; such that

$$\operatorname{dist}(b_k n_k \tau + \omega_k n_k d + q\varphi_k, 2\pi\mathbb{Z}) \le Cq|b_k|^{-\alpha}, \tag{2.1}$$

for all $k \ge 1$ and all $(\tau, d, q) \in \mathcal{T}_Z$, where $n_k = [\beta_0 \ln |b_k|]$.

Remark 2.3 It is easy to relax the condition that $1/h \in L^\infty$ to the requirement that there exists an $n_0 \ge 0$ such that $1/h_{n_0} \in L^\infty$ where $h_{n_0} = h + h \circ T + \cdots + h \circ T^{n_0 - 1}$. In Example 1.4, this condition is satisfied for $n_0 = 2$ even though h is arbitrarily small near the four corner points.

Corollary 2.4 *Let $T : X \to X$ be a nonuniformly expanding map and $h : X \to \mathbb{R}^+$ a roof function satisfying properties (1)–(5). Assume that $\mu_0(y \in Y : r(y) > n) = O((\ln n)^\gamma n^{-(\beta+1)})$ for some $\beta > 0$, $\gamma \geq 0$.*

There exists an integer m with the following property: Fix four periodic solutions for $T_t : X^h \to X^h$ that each intersect Y, and let $\tau_1, \dots, \tau_4$ be the periods. For Lebesgue almost all $(\tau_1, \cdots, \tau_4) \in (\mathbb{R}^+)^4$, there exists a constant $C \geq 1$ such that

$$|\rho_{v,w}(t)| \leq C\|v\|_{m,\eta}|w|_\infty (\ln t)^\gamma t^{-\beta},$$

for all $t > 0$, $v \in C^{m,\eta}(X^h)$, $w \in L^\infty(X^h)$.

Proof See [20, Corollary 2.4]. ∎

Remark 2.5 Similarly, it suffices that there is a sequence of periodic orbits in Z with *good asymptotics* in the sense of [12]. By [12], good asymptotics is an open-dense condition for smooth systems. Hence results on *stable* rates of mixing reduce to stability of the partition $\{Y_j\}$. We do not explore this issue further in this paper.

2.2 Nonuniformly hyperbolic flows

Let (M, d) be a Riemannian manifold. Young [27] introduced a class of nonuniformly hyperbolic diffeomorphisms $T : M \to M$ (possibly with singularities) with the property that there is an ergodic T-invariant SRB measure ν for which exponential decay of correlations holds for Hölder observables. We refer to [27] for precise definitions, but some of the notions and notation are required to state our main results. (The further structure from [27] required for our proofs is made explicit in Section 5.) In particular, there is a "uniformly hyperbolic" subset $Y \subset M$ with partition $\{Y_j\}$ and a return time function $r : Y \to \mathbb{Z}^+$ constant on partition elements such that, modulo uniformly contracting directions, the induced map $F = T^r : Y \to Y$ is nonuniformly expanding.

The statement of our main result is completely analogous to that of Theorem 2.2. Given a roof function $h : M \to \mathbb{R}^+$, the suspension flow $T_t : M^h \to M^h$ and ergodic measure ν^h is defined as before. Suppose that $Z \subset Y$ is a finite union of partition elements Y_j. Again, we define the set $\mathcal{T}_Z$ consisting of triples (τ, d, q) corresponding to periodic orbits for $F : Y \to Y$ lying entirely in Z.

Theorem 2.6 *Let $T : M \to M$ be nonuniformly hyperbolic in the sense of Young [27] with $\mu^u(y \in Y : r(y) > n) = O((\ln n)^\gamma n^{-(\beta+1)})$ for some $\beta > 0$, $\gamma \geq 0$. Let $h : M \to \mathbb{R}^+$ be a roof function with $h, \frac{1}{h} \in L^\infty(M)$ and $|h(x) - h(y)| \leq Cd(x, y)^\eta$ for all $x, y \in T^\ell Y_j$, $0 \leq \ell < r(j)$. Let $Z \subset Y$ be a finite union of partition elements Y_j.*

Suppose that there do **not** *exist constants $C, m \geq 1$ such that*

$$|\rho_{v,w}(t)| \leq C\|v\|_{m,\eta}\|w\|_{m,\eta} (\ln t)^\gamma t^{-\beta},$$

for all $t>0$, $v,w\in C^{m,\eta}(M^h)$.

Then condition (2.1) *holds as in Theorem 2.2.*

2.3 Unbounded roof functions

Suppose now that T is nonuniformly expanding as in Subsection 2.1, except that the roof function $h:X\to\mathbb{R}^+$ may be unbounded. Let $X(n)=\bigcup\{T^\ell Y_j:\|h1_{T^\ell Y_j}\|_{C^\eta}\ge n\}$. Condition (5) is relaxed to

(6) $\frac{1}{h}\in L^\infty(X)$,

(7) $\mu_0(X(n))\le Cn^{-(\beta+1)}$.

Theorem 2.7 *Let $T:X\to X$ be a nonuniformly expanding map satisfying properties (1)–(4) and $h:X\to\mathbb{R}^+$ a roof function satisfying properties (6), (7) where $\beta>0$. Assume that $\mu_0(y\in Y:r(y)>n)=O(e^{-cn})$ for some $c>0$. Let $Z\subset Y$ be a finite union of partition elements Y_j.*

Suppose that there do **not** *exist constants $C,m\ge 1$ such that*

$$|\rho_{v,w}(t)|\le C\|v\|_{m,\eta}|w|_\infty(\ln t)^{\beta+1}t^{-\beta},$$

for all $t>0$, $v\in C^{m,\eta}(X^h)$, $w\in L^\infty(X^h)$.

Then condition (2.1) *holds as in Theorem 2.2.*

Remark 2.8 The methods in this paper can handle general decay rates for $\mu_0(r>n)$ and $\mu_0(X(n))$. In the absence of motivating examples, we do not consider this generality. Again, a joint estimate of these quantities, if available in a specific application, might lead to improved results, as in Remark 1.2.

Remark 2.9 The analogous result holds for $T:M\to M$ nonuniformly hyperbolic, except that presently we obtain the weaker decay rate (1.3). This is proved in Section 6.5.

2.4 Strategy of the proof

There are a number of steps in proving Theorem 2.2.

Step 1 We model the nonuniformly expanding map $T:X\to X$ by a tower map $f:\Delta\to\Delta$. Recall that $F=T^r:Y\to Y$ is the induced map. Define $\Delta=\{(y,\ell)\in Y\times\mathbb{N}:0\le\ell\le r(y)\}/\sim$ where $(y,r(y))\sim(Fy,0)$. Define the tower map $f:\Delta\to\Delta$ by setting $f(y,\ell)=(y,\ell+1)$ computed modulo identifications. The projection $\pi:\Delta\to X$, $\pi(y,\ell)=T^\ell y$ defines a semiconjugacy, $\pi\circ f=T\circ\pi$.

There is a unique invariant ergodic probability measure μ_Y equivalent to $\mu_0|Y$ for the induced map $F:Y\to Y$. Moreover, the density is bounded above (and below) so that μ_Y inherits the property $\mu_Y(r>n)=O((\ln n)^\gamma n^{-(\beta+1)})$.

We obtain an invariant probability measure on Δ given by $\mu_\Delta = \mu_Y \times \mu_C / \int_Y r\, d\mu_Y$ where μ_C denotes counting measure, and $\pi : \Delta \to X$ is measure-preserving, carrying μ_Δ to ν. Given $h : X \to \mathbb{R}^+$ Hölder, let $\tilde{h} = h \circ \pi : \Delta \to \mathbb{R}^+$. We obtain invariant measures ν^h and $\mu_{\Delta^{\tilde{h}}} = (\mu_\Delta)^{\tilde{h}}$ for the suspension flows on X^h and $\Delta^{\tilde{h}}$. The projection $\pi : \Delta \to X$ induces a projection $\pi : \Delta^{\tilde{h}} \to X^h$ which carries $\mu_{\Delta^{\tilde{h}}}$ to ν^h.

If $x, y \in Y$, let $s(x,y)$ be the least integer $n \geq 0$ such that $F^n x, F^n y$ lie in distinct partition elements in Y. If $x, y \in Y_j \times \{\ell\}$, then there exist unique $x', y' \in Y_j$ such that $x = f^\ell x'$ and $y = f^\ell y'$. Set $s(x,y) = s(x',y')$. For all other pairs x, y, set $s(x,y) = 0$. This defines a separation time $s : \Delta \times \Delta \to \mathbb{N}$ and hence a metric $d_\theta(x,y) = \theta^{s(x,y)}$ on Δ. Let $F_\theta(\Delta)$ denote the Banach space of Lipschitz functions $v : \Delta \to \mathbb{R}$ with norm $\|v\|_\theta = |v|_\infty + |v|_\theta$ where $|v|_\theta = \sup_{x \neq y} |v(x) - v(y)|/d_\theta(x,y)$. We can choose $\theta \in (0,1)$ so that $v \circ \pi \in F_\theta(\Delta)$ for all $v \in C^\eta(X)$. It follows that $v \circ \pi \in F_{m,\theta}(\Delta^{\tilde{h}})$ for all $v \in C^{m,\eta}(X^h)$ where $\tilde{h} = h \circ \pi$ and $F_{m,\theta}(\Delta^{\tilde{h}})$ is defined in the obvious way.

Hence, we may reduce to the situation where $f_t : \Delta^h \to \Delta^h$ is a suspension flow over a tower map $f : \Delta \to \Delta$ and $h \in F_\theta(\Delta)$. It suffices to consider decay of correlations for observables $v \in F_{m,\theta}(\Delta^h)$, $w \in L^\infty(\Delta^h)$.

Step 2 We truncate the return time function r so that $r \leq N$. This produces an error $O\big((\ln N)^\gamma N^{-\beta} + t(\ln N)^\gamma N^{-(\beta+1)}\big)$ and reduces the problem to a suspension flow over a truncated tower $f' : \Delta' \to \Delta'$ with bounded return time r'. (All constructions from $F : Y \to Y$, μ_Y, r and h, are repeated with r replaced by r'.)

Step 3 Let $\rho'(t)$ denote the correlation function on $(\Delta')^h$ and let $\hat{\rho}(s)$ denote the Laplace transform of $\rho'(t)$. Modulo an analytic term,

$$\hat{\rho}(s) \sim \sum_{n \geq 1} \int_{\Delta'} e^{-sh_n \circ \pi} v_s\, w_s \circ (f')^n \, d\mu_{\Delta'},$$

where $v_s(x) = \int_0^{h(x)} e^{su}\, v(x,u)\, du$ and $w_s(x) = \int_0^{h(x)} e^{-su}\, w(x,u)\, du$. Decay of $\rho'(t)$ reduces to analyticity properties of $\hat{\rho}(s)$.

Step 4 Let $L : L^1(\Delta') \to L^1(\Delta')$ be the (Perron-Frobenius) transfer operator for $f' : \Delta' \to \Delta'$ (so $\int_{\Delta'} v\, w \circ f' \, d\mu_{\Delta'} = \int_{\Delta'} Lv\, w \, d\mu_{\Delta'}$ for $v \in L^1(\Delta')$, $w \in L^\infty(\Delta')$). For $s \in \mathbb{C}$, define the twisted transfer operator L_s by $L_s v = L(e^{sh} v)$. Then

$$\hat{\rho}(s) \sim \sum_{n \geq 1} \int_{\Delta'} L_{-s}^n v_s \, w_s \, d\mu_{\Delta'}.$$

Via the technique of operator renewal sequences, estimates on $|L_s^n v|_1$ for $v \in F_{m,\theta}((\Delta')^h)$ are related to estimates for the transfer operator of the (fixed) induced map $F : Y \to Y$.

Step 5 Choosing $N = N(t)$ appropriately, the estimates in Steps 2 and 4 yield the required result.

Steps 1 and 3 are standard. See for example [20, Section 4.1] for Step 1, and [10, 22] or specifically [11, Section 10] for Step 3. The truncation in Step 2 is the main new idea in this paper and is carried out in Section 3. In Section 4, we carry out Step 4 following [20] but keeping careful track of the dependence of estimates on N. We then specify $N = N(t)$ to obtain the final result.

Remark 2.10 (a) In Step 1, the subset $Y \subset X$ is identified with the base $\{\ell = 0\}$ of the tower Δ. The induced map $F : Y \to Y$ becomes a first return map for $f : \Delta \to \Delta$. In particular, $f : \Delta \to \Delta$ is Markov, even though no such assumption is made on $T : X \to X$.

(b) The induced map $F : Y \to Y$ is a full shift on a countable alphabet with good distortion properties (guaranteed by condition (4) in Subsection 2.1). Such maps are often called *Gibbs-Markov* and are studied extensively in [1].

(c) Note that h is unchanged in Step 2, except that it is restricted to Δ'. Similarly for v, w, except that a further approximation is required to ensure that v remains inside $F_{m,\theta}(\Delta^h)$, see Section 3.1.

(d) The truncation in Step 2 seems at first sight to make uncontrollable changes to $f : \Delta \to \Delta$ and hence to the suspension flow on Δ^h. However, it should be noted that the induced map $F : Y \to Y$ and the F-invariant measure μ_Y are unchanged by the truncation. The techniques in [20] based on operator renewal sequences [25, 14, 2] are hence well-suited to this situation, see Section 4.

3 Truncation of the roof function

Let $f : \Delta \to \Delta$ be a tower map, modelling the underlying nonuniformly expanding map $T : X \to X$, as discussed in Section 2.4. Let $h : \Delta \to \mathbb{R}^+$ be a Lipschitz roof function and let $f_t : \Delta^h \to \Delta^h$ be the suspension flow. Recall that Δ is itself a discrete suspension over the induced map $F : Y \to Y$ with ergodic invariant probability measure μ_Y and return time $r : Y \to \mathbb{Z}^+$. The tower map $f : \Delta \to \Delta$ has an ergodic invariant probability measure $\mu_\Delta = \mu_Y \times \text{counting}/\bar{r}$ where $\bar{r} = \int_Y r \, d\mu_Y$. Similarly, $f_t : \Delta^h \to \Delta^h$ has an ergodic invariant probability measure $\mu_{\Delta^h} = \mu_\Delta \times \text{lebesgue}/\bar{h}$ where $\bar{h} = \int_\Delta h \, d\mu_\Delta$.

For fixed $N \geq 1$, we define the truncated return time function $r' = \min\{r, N\} : Y \to \mathbb{Z}^+$. Then we form the truncated tower map $f' : \Delta' \to \Delta'$ over Y with measure $\mu_{\Delta'} = \mu_Y \times \text{counting}/\bar{r'}$. Restricting h to Δ', we obtain the truncated suspension flow $f'_t : (\Delta')^h \to (\Delta')^h$ with measure $\mu_{(\Delta')^h} = \mu_{\Delta'} \times \text{lebesgue}/\bar{h}'$ where $\bar{h}' = \int_{\Delta'} h \, d\mu_{\Delta'}$.

Write $\Delta = \Delta_{\text{left}} \dot{\cup} \Delta_{\text{right}}$ where

$$\Delta_{\text{left}} = \{(y, \ell) \in \Delta : r(y) < N\}, \qquad \Delta_{\text{right}} = \{(y, \ell) \in \Delta : r(y) \geq N\}.$$

Proposition 3.1 *(i)* $\bar{r} - \bar{r'} = \sum_{n>N} \mu_Y(r \geq n)$.

(ii) $\mu_\Delta(\Delta_{\text{right}}) = (1/\bar{r})\{N\mu_Y(r \geq N) + \sum_{n>N} \mu_Y(r \geq n)\}$.

Proof This is a standard computation. ∎

Proposition 3.2 *For $k \geq 1$, define*

$$E_k = \{x \in \Delta : f^j x \in \Delta_{\text{right}} \text{ for at least one } j \in \{0, 1, \ldots, k\}\}.$$

Then $\mu_\Delta(E_k) \leq (1/\bar{r})\{\sum_{n>N} \mu_Y(r \geq n) + (N+k)\mu_Y(r \geq N)\}$.

Proof Write E_k as the disjoint union $E_k = \bigcup_{j=0}^k G_j$ where

$$G_j = \{f^i x \in \Delta_{\text{left}} \text{ for } i \in \{0, 1, \ldots, j-1\} \text{ and } f^j x \in \Delta_{\text{right}}\}.$$

In particular $\mu_\Delta(G_0) = \mu_\Delta(\Delta_{\text{right}})$. For $j \geq 1$, it follows from the definition that if $x \in G_j$, then $f^j x \in \Delta_{\text{right}} \cap Y$ (the base of the tower). Hence $\mu_\Delta(G_j) \leq \mu_\Delta(f^{-j}(\Delta_{\text{right}} \cap Y)) = \mu_\Delta(\Delta_{\text{right}} \cap Y) = (1/\bar{r})\mu_Y(r \geq N)$. ∎

For notational convenience, we write $\Omega = \Delta^h$ and $\Omega' = (\Delta')^h$ throughout the remainder of this section. Throughout the paper C denotes a universal constant, varying from line to line, dependent only on the suspension semiflow $T_t : X^h \to X^h$ and the regularity exponents m, η.

Lemma 3.3 *Suppose that $h, \frac{1}{h} : \Delta \to \mathbb{R}^+$, $v, w : \Omega \to \mathbb{R}$ all lie in L^∞. Let*

$$\begin{aligned}
\rho(t) &= \int_\Omega v\, w \circ f_t \, d\mu_\Omega - \int_\Omega v \, d\mu_\Omega \int_\Omega w \, d\mu_\Omega, \\
\rho'(t) &= \int_{\Omega'} v\, w \circ f'_t \, d\mu_{\Omega'} - \int_{\Omega'} v \, d\mu_{\Omega'} \int_{\Omega'} w \, d\mu_{\Omega'}.
\end{aligned}$$

Then there exists N_0, t_0 (depending only on μ, r and h) such that for all $N \geq N_0$, $t \geq t_0$,

$$|\rho(t) - \rho'(t)| \leq C|v|_\infty |w|_\infty \{\textstyle\sum_{n>N} \mu_Y(r \geq n) + (N+t)\mu_Y(r \geq N)\}.$$

Proof We choose $N \geq N_0$ sufficiently large that $1/\bar{r}' \leq 2/\bar{r}$, $1/\bar{h}' \leq 2/\bar{h}$. It follows that $\frac{1}{\bar{r}'} - \frac{1}{\bar{r}} \leq \frac{2}{\bar{r}^2}(\bar{r} - \bar{r}')$ and $|\frac{1}{\bar{h}'} - \frac{1}{\bar{h}}| \leq \frac{2}{\bar{h}^2}|\bar{h} - \bar{h}'|$. Further, $|\bar{h} - \bar{h}'| \leq 4|h|_\infty(\bar{r} - \bar{r}')/\bar{r}$. By Proposition 3.1(i),

$$\frac{1}{\bar{r}'} - \frac{1}{\bar{r}} \leq C \sum_{n>N} \mu_Y(r \geq n), \quad \Big|\frac{1}{\bar{h}'} - \frac{1}{\bar{h}}\Big| \leq \sum_{n>N} \mu_Y(r \geq n). \tag{3.1}$$

Let $A = \int_\Omega v\, w\circ f_t\, d\mu_\Omega$, $A' = \int_{\Omega'} v\, w\circ f_t'\, d\mu_{\Omega'}$. By definition, $A = (1/\bar h)(1/\bar r)B$, $A' = (1/\bar h')(1/\bar{r'})B'$ where

$$B = \int_Y \sum_{\ell=0}^{r(y)-1} \int_0^{h(y,\ell)} v(y,\ell,u)\, w\circ f_t(y,\ell,u)\, du\, d\mu_Y,$$

$$B' = \int_Y \sum_{\ell=0}^{r'(y)-1} \int_0^{h(y,\ell)} v(y,\ell,u)\, w\circ f_t'(y,\ell,u)\, du\, d\mu_Y.$$

Note that $B = \bar r \int_\Delta \int_0^h v\, w\circ f_t\, du\, d\mu_\Delta$ so that $|B| \le \bar r \bar h |v|_\infty |w|_\infty$. Hence by (3.1),

$$|A - A'| \le C\{|v|_\infty |w|_\infty \textstyle\sum_{n>N} \mu_Y(r\ge n) + |B - B'|\}.$$

Write $|B - B'| \le I + II$ where

$$I = |v|_\infty |w|_\infty \int_Y \sum_{\ell=N}^{r(y)-1} h(y,\ell) 1_{\{r(y)>N\}} d\mu_Y \le C|v|_\infty |w|_\infty \sum_{n>N} \mu_Y(r\ge n),$$

and

$$\begin{aligned} II &= |v|_\infty \int_Y \sum_{\ell=0}^{r'(y)-1} \int_0^{h(y,\ell)} |w\circ f_t(y,\ell,u) - w\circ f_t'(y,\ell,u)| du\, d\mu_Y \\ &= |v|_\infty \bar{r'} \int_{\Delta'} \int_0^{h(x)} |w\circ f_t(x,u) - w\circ f_t'(x,u)| du\, d\mu_{\Delta'} \\ &= |v|_\infty \bar r \int_{\Delta'} \int_0^{h(x)} |w\circ f_t(x,u) - w\circ f_t'(x,u)| du\, d\mu_\Delta. \end{aligned}$$

(Starting from the last expression, we are regarding Δ' as a subset of Δ; for measurable sets $E \subset \Delta' \subset \Delta$ note that $\bar{r'}\mu_{\Delta'}(E) = \bar r \mu_\Delta(E)$.)

Now $f_t(x,u) = f_t'(x,u)$ provided N is sufficiently large that $f_s(x,u)$ lies in the part of the suspension over $\Delta_{\rm left}$ for $s\in[0,t]$. Note also that the flow reaches the roof at most $t|\frac{1}{h}|_\infty + 1$ times by time t so it suffices that $f^j x \in \Delta_{\rm left}$ for $0\le j\le [t|\frac{1}{h}|_\infty]+2$. Hence

$$II \le C|v|_\infty |w|_\infty \mu_\Delta(E_k),$$

where $k = [t|\frac1h|_\infty]+2$. By Proposition 3.2,

$$II \le C|v|_\infty |w|_\infty \{\textstyle\sum_{n>N} \mu_Y(r\ge n) \;+\; (N+k)\mu_Y(r\ge N)\},$$

and so

$$|A - A'| \le C|v|_\infty |w|_\infty \{\textstyle\sum_{n>N} \mu_Y(r\ge n) \;+\; (N+t)\mu_Y(r\ge N)\}.$$

A similar (but simpler) calculation shows that

$$\big|\textstyle\int_\Omega v\, d\mu_\Omega \int_\Omega w\, d\mu_\Omega - \int_{\Omega'} v\, d\mu_{\Omega'} \int_{\Omega'} w\, d\mu_{\Omega'}\big| \le C|v|_\infty |w|_\infty \sum_{n>N} \mu_Y(r\ge n),$$

and the result follows. ∎

Remark 3.4 If $\mu_Y(r \geq n) = O((\ln n)^\gamma n^{-(\beta+1)})$, $\beta > 0$, $\gamma \geq 0$ then

$$\sum_{n>N} \mu_Y(r \geq n) \leq C \sum_{n \geq N} \frac{(\ln n)^\gamma}{n^{\frac{1}{2}\beta}} \frac{1}{n^{\frac{1}{2}\beta+1}} \leq C \frac{(\ln N)^\gamma}{N^{\frac{1}{2}\beta}} \sum_{n \geq N} \frac{1}{n^{\frac{1}{2}\beta+1}} \leq C \frac{(\ln N)^\gamma}{N^\beta}.$$

Hence

$$|\rho(t) - \rho'(t)| \leq C|v|_\infty |w|_\infty \{(\ln N)^\gamma N^{-\beta} + t(\ln N)^\gamma N^{-(\beta+1)}\}.$$

3.1 Regularity of observables

The original observable $v : \Delta^h \to \mathbb{R}$ is assumed to be smooth in the flow direction, but after restriction to $(\Delta')^h$ this condition is typically violated since the identifications are different. Namely, we now have

$$(y, r'(y), h(y)) \sim (Fy, 0, 0),$$

whereas v is smooth respect to the old identifications

$$(y, r(y), h(y)) \sim (Fy, 0, 0).$$

This problem is resolved by using the top level of the truncated tower as a *buffer*. That is, we modify v on the strip $\{(y, N, u) : r'(y) = N, u \in [0, h(y, \ell)]\}$ to obtain a new observable $\tilde{v}$ that is as regular in the flow direction on $(\Delta')^h$ as v was on Δ^h. Since v is bounded and h is bounded below, we can make this modification in such a way that $\|\tilde{v}\|_{m,\eta} \leq C\|v\|_{m,\eta}$. The resulting error in the correlation function is at most $C|v|_\infty |w|_\infty \mu_Y(r = N)$ which is smaller than the error in Lemma 3.3. Hence without loss we may suppose that the observable v retains its smoothness in the flow direction when restricted to $(\Delta')^h$.

4 Decay for nonuniformly expanding semiflows

In this section, we complete the proof of Theorem 2.2.

Define the induced roof function $H : Y \to \mathbb{R}$ by $H(y) = \sum_{\ell=0}^{r(y)-1} h \circ f^\ell(y)$. For $b \in \mathbb{R}, \omega \in [0, 2\pi)$, we define $M_{b,\omega} : L^\infty(Y) \to L^\infty(Y)$,

$$M_{b,\omega} v = e^{-ibH} e^{-i\omega r} v \circ F.$$

Definition 4.1 A subset $Z_0 \subset Y$ is a *finite subsystem* of Y if $Z_0 = \bigcap_{n \geq 0} F^{-n} Z$ where Z is the union of finitely many elements from the partition $\{Y_j\}$. (Note that $F|_{Z_0} : Z_0 \to Z_0$ is a a full one-sided shift on finitely many symbols.)

Definition 4.2 We say that $M_{b,\omega}$ has an *approximate eigenfunction* on a subset $Z \subset Y$ if there exist constants $\alpha, \beta_0 > 0$ arbitrarily large and $C \geq 1$, and sequences

$|b_k| \to \infty$, $\omega_k \in [0, 2\pi)$, $\varphi_k \in [0, 2\pi)$, $u_k \in F_\theta(Y)$ with $|u_k| \equiv 1$ and $\|u_k\|_\theta \le C|b_k|$, such that setting $n_k = [\beta_0 \ln |b_k|]$,

$$|(M_{b_k,\omega_k}^{n_k} u_k)(y) - e^{i\varphi_k} u_k(y)| \le C|b_k|^{-\alpha},$$

for all $y \in Z$ and all $k \ge 1$.

Define $d_N = \sum_{k=1}^N k\mu_Y(r \ge k)$. The main result of this section is:

Theorem 4.3 *Let $Z_0 \subset Y$ be a finite subsystem and suppose that $M_{b,\omega}$ has no approximate eigenfunctions on Z_0. Choose N sufficiently large that $r|Z_0 \le N$.*

Let d, $p > 0$. There exists C, $m \ge 1$, $\epsilon > 0$ such that

$$|\rho'_{v,w}(t)| \le C\|v\|_{m,\theta}|w|_\infty\{d_N N^{1+d} e^{-\epsilon N^{-1} \ln N\, t} + (d_N N^d)^{p+2} t^{-p}\},$$

for all $t > 0$, $v \in F_{m,\theta}((\Delta')^h)$, $w \in L^\infty((\Delta')^h)$.

Corollary 4.4 *Let $Z_0 \subset Y$ be a finite subsystem and suppose that $M_{b,\omega}$ has no approximate eigenfunctions on Z_0.*

If $\mu_Y(r \ge n) = O((\ln n)^\gamma n^{-(\beta+1)})$, $\beta > 0$, $\gamma \ge 0$, then there exist constants $C, m \ge 1$ such that

$$|\rho_{v,w}(t)| \le C\|v\|_{m,\theta}|w|_\infty(\ln t)^\gamma t^{-\beta},$$

for all $t > 0$, $v \in F_{m,\theta}(\Delta^h)$, $w \in L^\infty(\Delta^h)$.

Proof Choose N sufficiently large that $r|Z_0 \le N$. Combining Lemma 3.3 (specifically Remark 3.4) and Theorem 4.3, we obtain

$$\rho(t) = O\{(\ln N)^\gamma N^{-\beta} + t(\ln N)^\gamma N^{-(\beta+1)} + d_N N^{1+d} e^{-\epsilon N^{-1}\ln N\, t} + (d_N N^d)^{p+2} t^{-p}\}.$$

We compute that $d_N \le C$ for $\beta > 1$, $d_N \le C(\ln N)^{\gamma+1}$ for $\beta = 1$, and $d_N \le C(\ln N)^\gamma N^{1-\beta}$ for $\beta \in (0, 1)$.

Set $N = [t/q]$. Then the first two terms in $\rho(t)$ are $O((\ln t)^\gamma t^{-\beta})$. The third term is $O(d_N t^{1+d-\epsilon q}) = O(t^{-\beta})$ for q sufficiently large.

If $\beta > 1$, the fourth term is $O(t^{d-p}) = O(t^{-\beta})$ for $p > \beta$ and d sufficiently small. The case $\beta = 1$ differs only by a logarithmic factor so the same choices of p and d suffice. If $\beta < 1$, the fourth term, ignoring a logarithmic factor, is $O(t^{(1-\beta+d)(p+2)-p}) = O(t^{-\beta})$ for $p > (2-\beta)/\beta$ and d small. ∎

Theorem 2.2 is immediate from Corollary 4.4 since it is known that the existence of approximate eigenfunctions implies the periodic data criterion (2.1), see [12, Theorem 1.8].

4.1 Estimates for the Gibbs-Markov map $F : Y \to Y$

Let R denote the transfer operator for the Gibbs-Markov map $F : Y \to Y$. We continue to let $f' : \Delta' \to \Delta'$ denote the truncation of $f : \Delta \to \Delta$ with return time $r' = \min\{r, N\}$. Note that the return map $F = f^r = (f')^{r'} : Y \to Y$ is independent of N and so the operator R is fixed throughout. Define $H'(y) = \sum_{\ell=0}^{r'(y)-1} h \circ f^j(y)$.

For $s, z \in \mathbb{C}$, define the twisted transfer operator $R_{s,z}$ to be $R_{s,z}v = R(e^{sH'}e^{zr'}v)$.

Proposition 4.5 $\sum_{j\ge1} |1_{Y_j}H'|_\theta \mu_Y(Y_j) \le |h|_\theta\, \bar{r}$.

Proof This follows from the estimate $|1_{Y_j}H'|_\theta \le (r'|Y_j)\,|h|_\theta \le (r|Y_j)|h|_\theta$. ∎

It follows from Proposition 4.5 that the estimates in [20, Proposition 3.7] hold independent of N. In particular,

$$|R^n_{ib,i\omega}v|_\theta \le C\{|b||v|_\infty + \theta^n|v|_\theta\}, \tag{4.1}$$

for all $n, N \ge 1$, $|b| > 1$, $\omega \in [0, 2\pi)$, $v \in F_\theta(Y)$.

Define the norm $\|v\|_b = \max\{|v|_\infty, |v|_\theta/(2C|b|)\}$ where C is the constant in (4.1).

Lemma 4.6 *(cf. [20, Lemma 3.5]). Assume no approximate eigenfunctions on Z_0 and choose N sufficiently large that $r|Z_0 \le N$. Then there exist $\alpha > 0$, $C \ge 1$ independent of N such that*

$$\|(I - R_{ib,i\omega})^{-1}\|_b \le C|b|^\alpha,$$

for all $|b| > 1$, $\omega \in [0, 2\pi)$.

Proof Since $r|_{Z_0} \le N$, it makes no difference whether we define $M_{b,\omega}$ using H or H' for the assumption that there are no approximate eigenfunctions on Z_0.

When $\omega = 0$, it remains to verify that the proof of [20, Lemmas 3.12 and 3,13] goes through unchanged. The main issue is the dependence on the constant called C_6 in [20] which potentially depends on N. However, this constant is shown to be uniform in (4.1). The remaining arguments in [20] indeed go through without change proving the result for $\omega = 0$.

As in [20, Section 3.3], the case $\omega \ne 0$ presents no additional complications. ∎

Proposition 4.7 *(cf. [20, Proposition 3.10]).*

$$\|R_{s,z} - R_{ib,i\omega}\|_b \le Cd_N(|a| + |\sigma|)e^{(|a||h|_\infty+|\sigma|)N},$$

for all $s = a + ib$, $z = \sigma + i\omega \in \mathbb{C}$.

Proof The key estimate is [20, Proposition 3.9(d)] which states that

$$\|(R_s - R_{ib})1_{Y_j}\|_b \le C|a|\|1_{Y_j}H'\|_\theta(1 + |1_{Y_j}H'|_\theta)e^{|a||1_{Y_j}H'|_\infty}\mu_Y(Y_j).$$

Similarly,

$$\|(R_{s,z}-R_{ib,i\omega})1_{Y_j}\|_b \le C(|a|\|1_{Y_j}H'\|_\theta+|\sigma|r'(j))(1+|1_{Y_j}H'|_\theta)e^{|a||1_{Y_j}H'|_\infty}e^{|\sigma|r'(j)}\mu_Y(Y_j).$$

It follows that for each $j\ge 1$,

$$\|(R_{s,z}-R_{ib,i\omega})1_{Y_j}\|_b \le C(|a|\|h\|_\theta+|\sigma|)(1+|h|_\theta)e^{(|a||h|_\infty+|\sigma|)N}\,r'(j)^2\mu_Y(Y_j).$$

Now $\sum_{j\ge1} r'(j)^2\mu_Y(Y_j)=\sum_{k=1}^N k^2\mu_Y(r'=k)\le 2d_N$, so the result follows from the fact that $R_{s,z}-R_{ib,i\omega}=\sum_{j\ge1}(R_{s,z}-R_{ib,i\omega})1_{Y_j}$. ∎

In the sequel, s always denotes $s=a+ib\in\mathbb{C}$, similarly $z=\sigma+i\omega\in\mathbb{C}$. All constants C, ϵ, etc are uniform in $|b|>1$ and $\omega\in[0,2\pi)$ but we suppress the domain of b and ω.

Lemma 4.8 *(cf. [20, Lemma 3.14]). Assume no approximate eigenfunctions on Z_0 and choose N sufficiently large that $r|Z_0\le N$. Let $d>0$ and set $\widetilde{d}_N=d_NN^d$. There exist $\alpha>0$, $\epsilon>0$ and $C\ge1$ independent of N, such that*

$$\|(I-R_{s,z})^{-1}\|_b\le C|b|^\alpha, \tag{4.2}$$

for all a, $\sigma\in U_b$, where

$$U_b=\{a\in\mathbb{R}:|a|<\epsilon\min\{N^{-1}\ln N,\widetilde{d}_N^{-1}|b|^{-\alpha}\}\}.$$

Proof Choose $\epsilon<d(|h|_\infty+1)^{-1}$. It follows from Proposition 4.7 that for s,z in the stipulated region,

$$\||R_{s,z}-R_{ib,i\omega}\|_b\le C\epsilon N^{-d}|b|^{-\alpha}e^{(|h|_\infty+1)\epsilon\ln N}\le C\epsilon|b|^{-\alpha}.$$

By Lemma 4.6, $\|R_{s,z}-R_{ib,i\omega}\|_b\|(I-R_{ib,i\omega})^{-1}\|_b\le\frac12$ say for ϵ sufficiently small. Using a resolvent inequality as in [11, Section 2], we obtain $\|(I-R_{s,z})^{-1}\|_b\le 2\|(I-R_{ib,i\omega})^{-1}\|_b$ giving the required result. ∎

4.2 Operator renewal sequences

Let L denote the transfer operator for the truncated tower map $f':\Delta'\to\Delta'$. Recall that for $s\in\mathbb{C}$, the twisted transfer operator L_s is defined to be $L_sv=L(e^{sh'}v)$. Hence $(L_s^nv)(x)=\sum_{(f')^nz=x}g_n'(z)e^{sh_n'(z)}v(z)$ where $h_n'(z)=h(z)+h(f'z)+\cdots+h((f')^{n-1}z)$ and $g_n'(z)$ is the inverse of the Jacobian of $(f')^n$ at z.

Let $Z_n=\{y\in Y:r'=n\}$. Then $\{Z_1,\dots,Z_N\}$ is a finite partition of Y.

For $s\in\mathbb{C}$, define the operator renewal sequences

$$T_{s,n}=1_YL_s^n1_Y,\quad R_{s,n}=1_YL_s^n1_{Z_n},$$

and the Fourier series $T, R : \mathbb{C} \to L(F_\theta(Y))$ given by

$$T_s(z) = \sum_{n=0}^{\infty} T_{s,n} e^{zn}, \quad R_s(z) = \sum_{n=1}^{N} R_{s,n} e^{zn}.$$

We have the renewal equation $T_s(z) = (I - R_s(z))^{-1}$. Note also that $R_s(z)v = R_{s,z}v = R(e^{sH'} e^{zr'} v)$.

Lemma 4.9 *(cf. [20, Lemma 4.3]) Assume no approximate eigenfunctions on Z_0 and choose N sufficiently large that $r|Z_0 \le N$. There exist constants $\epsilon, \delta > 0, \alpha > 0$, $C \ge 1$ independent of N such that*

$$\|T_{s,n}\|_b \le C|b|^\alpha e^{-n\delta \min\{N^{-1} \ln N, \widetilde{d}_N^{-1} |b|^{-\alpha}\}}, \tag{4.3}$$

for all $n \ge 1$, and $\sigma \in U_b$.

Proof By the renewal equation and (4.2),

$$\|T_s(z)\|_b = \|(I - R_s(z))^{-1}\|_b \le C|b|^\alpha,$$

for a, $\sigma \in U_b$.

By definition, $R_s(z)$ is a polynomial of degree N in e^z and hence analytic in z. It follows that $T_s(z)$ is analytic in z on the domain of $(I - R_s(z))^{-1}$, namely U_b. Hence the Fourier coefficients $T_{s,n}$ decay at the required rate for any $\delta < \epsilon$. ∎

Lemma 4.10 *(cf. [20, Lemma 4.4]) Assume no approximate eigenfunctions on Z_0 and choose N sufficiently large that $r|Z_0 \le N$. Let $d > 0$ and set $\widetilde{d}_N = d_N N^d$. There exist constants $\epsilon, \delta > 0, \alpha > 0$, $C \ge 1$ independent of N such that*

$$\sum_{n \ge 1} |L_s^n v|_1 \le C\|v\|_b \, \widetilde{d}_N |b|^\alpha \max\{N(\ln N)^{-1}, \widetilde{d}_N |b|^\alpha\}, \tag{4.4}$$

for all $v \in F_\theta(\Delta')$, $n \ge 1$, and $a \in U_b$.

Proof Recall that $(L_s^n v)(x) = \sum_{(f')^n z = x} g_n'(z) e^{s h_n'(z)} v(z)$. Following the proof and notation of [20, Lemma 4.4], we write

$$L_s^n = \textstyle\sum_{i+j+k=n} A_{s,i} T_{s,j} B_{s,k} \ + \ E_{s,n},$$

where

$$(T_{s,n}v)(x) = \sum_{\substack{f^n z = x \\ x, z \in Y}} \quad , \qquad (A_{s,n}v)(x) = \sum_{\substack{f^n z = x \\ z \in Y;\ fz \notin Y, \ldots, f^n z \notin Y}} \quad ,$$

$$(E_{s,n}v)(x) = \sum_{\substack{f^n z = x \\ z \notin Y, \ldots, f^n z \notin Y}} \quad , \qquad (B_{s,n}v)(x) = \sum_{\substack{f^n z = x \\ z \notin Y, \ldots, f^{n-1} z \notin Y;\ f^n z \in Y}} \quad ,$$

and we have suppressed the summands $g'_n(z)e^{sh'_n(z)}v(z)$. We view these as operators $L_s^n : F_\theta(\Delta') \to L^1(\Delta')$, $T_{s,n} : F_\theta(Y) \to L^\infty(Y)$, $A_{s,n} : L^\infty(Y) \to L^1(\Delta')$, $B_{s,n} : F_\theta(\Delta') \to F_\theta(Y)$, $E_{s,n} : F_\theta(\Delta') \to L^1(\Delta')$, with the $\| \|_b$ norm on $F_\theta(Y)$ and $F_\theta(\Delta')$. In the corresponding operator norms, we have

$$\|L_s^n\| \le \sum_{i+j+k=n} \|A_{s,i}\| \, \|T_{s,j}\| \, \|B_{s,k}\| \;+\; \|E_{s,n}\|.$$

Due to truncation, $A_{s,n} = B_{s,n} = E_{s,n} = 0$ for $n > N$. We claim further that

$$\begin{aligned} &\|A_{s,n}\| \le CN^{\epsilon'}\mu_Y(r' \ge n), \quad \|B_{s,n}\| \le CN^{\epsilon'} n\mu_Y(r' \ge n), \\ &\|E_{s,n}\| \le CN^{\epsilon'} \textstyle\sum_{k=n}^N \mu_Y(r' \ge k), \end{aligned} \tag{4.5}$$

where $\epsilon' = \epsilon|h|_\infty$, for all $n \ge 1$, $|a| \le \epsilon N^{-1}\ln N$, $b \in \mathbb{R}$.

Let $u_n = \mu_Y(r' \ge n)$ and $v_n = e^{-cn}$ where $0 < c \le \delta N^{-1}\ln N$. Then $(u * v)_n = \sum_{k=1}^N \mu_Y(r' \ge k)e^{-c(n-k)} \le N^\delta e^{-cn}\sum_{k=1}^N \mu_Y(r' \ge k) \le N^\delta \bar{r} e^{-cn}$. Similarly, if $u'_n = n\mu_Y(r' \ge k)$, then $(u' * v)_n \le N^\delta e^{-cn}\sum_{k=1}^N k\mu_Y(r' \ge k) \le N^\delta d_N e^{-cn}$. Using this calculation and estimates (4.3), (4.5), we obtain

$$\|L_s^n\| \le Cd_N N^{2\epsilon'+2\delta}|b|^\alpha e^{-n\delta \min\{N^{-1}\ln N, \tilde{d}_N^{-1}|b|^{-\alpha}\}} + \|E_{s,n}\|. \tag{4.6}$$

Moreover, $\sum_{n\ge1}\|E_{s,n}\| \le CN^\epsilon \sum_{n=1}^N\sum_{k=n}^N \mu_Y(r' \ge k) = CN^\epsilon d_N$. Shrink ϵ and δ if necessary so that $2\epsilon' + 2\delta < d$. Since $(1-e^{-x})^{-1} \le 2x^{-1}$ for $x > 0$ small, we obtain the required estimate for $\sum_{n\ge1} L_s^n$,

It remains to verify estimates (4.5). Note that the support of $A_{s,n}v$ is contained in level $n \le N$ of the tower and has measure at most $\sum_{r'(j)>n}\mu_{\Delta'}(Y_j) \le (1/\bar{r}')\mu_Y(r' \ge n)$. For x in level n, we have $(A_{s,n}v)(x) = e^{sh'_n(z)}v(z)$ where z is the unique point in Y with $(f')^n z = x$, and so $|A_{s,n}v|_\infty \le e^{\epsilon N^{-1}\ln N\, n|h|_\infty}|v|_\infty \le N^{\epsilon|h|_\infty}|v|_\infty$. Hence $|A_{s,n}v|_1 \le (1/\bar{r}')N^{\epsilon'}\mu_Y(r' \ge n)$. Similarly,

$$|E_{s,n}v|_1 \le N^{\epsilon'} \sum_{\substack{r'(j)>n \\ n<\ell<r'(j)}} \mu_{\Delta'}(\Delta_{j,\ell})|v|_\infty \le (1/\bar{r}')N^{\epsilon'}\sum_{k=n+2}^N \mu_Y(r' \ge k)\|v\|_b.$$

Finally, if $y \in Y$, then $(B_{s,n}v)(y) = \sum_{r'(j)>n} g'_n(z'_j)e^{sh'_n(z'_j)}v(z'_j)$ where z'_j is the unique preimage of y in $\Delta_{j,r'(j)-n}$. Since $f' : \Delta_{j,\ell} \to \Delta_{j,\ell+1}$ is an isometry for $\ell < r' - 1$, we can write $g'_n(z'_j) = g(z_j)$ where z_j is the unique point satisfying $z_j \in Y_j$, $Fz_j = y$, and g is the Jacobian in part (4) of the definition of nonuniform expansion in Section 2.1. (Alternatively, g is the weight in the definition $(Rv)(x) = \sum_{Fy=x} g(y)v(y)$ of the transfer operator R for the Gibbs-Markov map $F : Y \to Y$.) The log-Hölder condition on g implies that $|g(y)| \le C\mu_Y(Y_j)$ and $|g(y)/g(\hat{y}) - 1| \le Cd_\theta(y, \hat{y})$ for all $y, \hat{y} \in Y_j$. Hence $|B_{s,n}v|_\infty \le \sum_{r'(j)>n} C\mu_Y(Y_j)N^{\epsilon'}|v|_\infty \le CN^{\epsilon'}\mu_Y(r' \ge n)|v|_\infty$ and $|(B_{s,n}v)(y) - B_{s,n}v)(\hat{y})| \le \sum_{r'(y)>n}|g(z_j)e^{sh'_n(z'_j)}v(z'_j) - g(\hat{z}_j)e^{sh'_n(\hat{z}'_j)}v(\hat{z}'_j)| \le$

$d_\theta(y,\hat y)\sum_{r'(y)>n}\Big(C\mu_Y(Y_j)N^{\epsilon'}|v|_\theta + C\mu_Y(Y_j)N^{\epsilon'}|s|n|h|_\theta|v|_\infty + C\mu_Y(Y_j)N^{\epsilon'}|v|_\infty\Big)$ so that $|B_{s,n}v|_\theta \le CN^{\epsilon'}|b|n\mu_Y(r'\ge n)\|v\|_b$. It follows that $\|B_{s,n}\|_b \le CN^{\epsilon'}n\mu_Y(r'\ge n)$ completing the verification of estimates (4.5). ∎

Proof of Theorem 4.3 By the formula in Step 4, Section 2.4, it follows from Lemma 4.10 that

$$|\hat\rho(s)| \le C\|v\|_\theta|w|_\infty\widetilde d_N|b|^\alpha\max\{N(\ln N)^{-1},\widetilde d_N|b|^\alpha\}, \tag{4.7}$$

for

$$s=a+ib\in U_b=\{|a|<\epsilon\min\{N^{-1}\ln N,\widetilde d_N^{-1}|b|^{-\alpha}\}.$$

To recover $\rho'(t)$ via a contour integral $\int e^{st}\hat\rho(s)ds$, we integrate along a contour $s=a+ib$, $a=a_N(b)$, $-\infty<b<\infty$ in the left-half-plane; in the range of validity of the estimate (4.7). Specifically, we choose $a=-\epsilon\min\{N^{-1}\ln N,\widetilde d_N^{-1}|b|^{-\alpha}\}$ for $|b|>1$ (and decreased ϵ). Integrating by parts m times as in [11], we can replace the contour integral by $\int s^{-m}e^{st}\hat\rho_m(s)ds$ where $\hat\rho_m$ is defined in the same way as $\hat\rho$ but with v replaced by $\partial_t^m v$. Focusing on the part of the contour with $b>1$, it remains to estimate

$$\begin{aligned}
&\Big|\int_1^\infty(a_N(b)+ib)^{-m}e^{(a_N(b)+ib)t}\hat\rho_m(a_N(b)+ib)db\Big|\\
&\quad\le\int_1^\infty b^{-m}e^{-\epsilon\min\{N^{-1}\ln N,\widetilde d_N^{-1}b^{-\alpha}\}t}C\|\partial_t^m v\|_\theta|w|_\infty\widetilde d_N b^\alpha\max\{N(\ln N)^{-1},\widetilde d_N b^\alpha\}db\\
&\quad\le C\|v\|_{m,\theta}|w|_\infty(I+II),
\end{aligned}$$

where

$$I=\int_1^\infty\widetilde d_N N e^{-\epsilon N^{-1}\ln N\,t}b^{\alpha-m}\,db,\quad II=\int_1^\infty e^{-\epsilon\widetilde d_N^{-1}b^{-\alpha}t}\widetilde d_N^2 b^{2\alpha-m}\,db.$$

Taking $m>\alpha+1$ yields $I\le C\widetilde d_N N e^{-\epsilon N^{-1}\ln N\,t}$. A change of variables yields

$$II\le\alpha^{-1}\epsilon^{-p}t^{-p}\widetilde d_N^{p+2}\int_0^\infty e^{-y}y^{p-1}\,dy=\alpha^{-1}\epsilon^{-p}(p-1)!\,\widetilde d_N^{p+2}t^{-p},$$

with $m=(p+2)\alpha+1$. ∎

5 Decay for nonuniformly hyperbolic flows

In this section we prove Theorem 2.6. The main steps are the same as for Theorem 2.2, but there is an additional step between Steps 3 and 4 where we pass from the Young tower to a nonuniformly expanding quotient tower $\bar f':\bar\Delta'\to\bar\Delta'$ by quotienting along stable manifolds.

In Subsection 5.1, we include the necessary background material and notation from Young [27, 28] on nonuniformly hyperbolic diffeomorphisms and towers. In Subsection 5.2, we use approximation arguments to reduce the nonuniformly hyperbolic case to the nonuniformly expanding case studied in Section 4. In Subsection 5.3, we give an alternative criterion for decay of correlations using the temporal distance function instead of periodic data; this gives stronger results for the Lorentz gas examples.

5.1 Background on nonuniformly hyperbolic systems

Let $T : M \to M$ be a nonuniformly hyperbolic diffeomorphism in the sense of Young [27, 28]. As described in Section 2.2, there is a partition $\{Y_j\}$ of $Y \subset M$ with return time function $r : Y \to \mathbb{Z}^+$, constant on partition elements $\{Y_j\}$, and induced return map $F : Y \to Y$ given by $F(y) = T^{r(y)}(y)$. There exists an ergodic T-invariant probability measure ν that is an SRB measure.

Let $\Delta = \{(y, \ell) : y \in Y,\ \ell = 0, \dots, r(y) - 1\}$ and define the tower map $f : \Delta \to \Delta$ by setting $f(y, \ell) = (y, \ell + 1)$ for $0 \le \ell < r(y) - 1$ and $f(y, r(y) - 1) = (Fy, 0)$. The projection $\pi : \Delta \to M$ given by $\pi(y, \ell) = T^\ell y$ is a semiconjugacy between $f : \Delta \to \Delta$ and $T : M \to M$.

The subset Y is covered by families of *stable disks* $\{W^s(y),\ y \in Y\}$ and *unstable disks* $\{W^u(y),\ y \in Y\}$ such that each stable disk intersects each unstable disk in exactly one point. For $p = (x, \ell)$, $q = (y, \ell) \in \Delta$, we write $q \in W^s(p)$ if $y \in W^s(x)$ (and $q \in W^u(p)$ if $y \in W^u(x)$).

Quotienting out the stable directions, we obtain the quotient maps $\bar f : \bar\Delta \to \bar\Delta$ and $\bar F : \bar Y \to \bar Y$.

Proposition 5.1 ([27, 28]) *The quotient tower map $\bar f : \bar\Delta \to \bar\Delta$ is a nonuniformly expanding tower map of the type considered in Section 4. In particular, there are $\bar F$ and $\bar f$-invariant measures $\bar\mu$ and $\bar\mu \times \mu_C / \int_{\bar Y} r\, d\bar\mu$ on $\bar Y$ and $\bar\Delta$ respectively, such that $\bar F : \bar Y \to \bar Y$ is Gibbs-Markov with respect to the quotient partition $\{\bar Y_j\}$. Moreover, there is a f-invariant measure μ on Δ such that the natural projection $\bar\pi : \Delta \to \bar\Delta$ and the projection $\pi : \Delta \to M$ are measure-preserving semiconjugacies.* ■

In Step 1, we defined $s : \bar\Delta \times \bar\Delta \to \mathbb{N}$ relative to returns under $\bar F$ to the partition $\{\bar Y_j\}$. This lifts to a separation time $s : \Delta \times \Delta \to \mathbb{N}$ given by $s(p, q) = s(\bar\pi p, \bar\pi q)$. Note that s is defined on both $\bar\Delta$ and Δ, but the metric $d_\theta(p, q) = \theta^{s(p,q)}$ is defined only on $\bar\Delta$.

We assume that there exists $\gamma \in (0, 1)$ such that

(P1) If $q \in W^s(p)$, then $d(\pi f^n p, \pi f^n q) \le C\gamma^n$ for all $n \ge 1$.

(P2) If $q \in W^u(p)$, then $d(\pi f^n p, \pi f^n q) \le C\gamma^{s(p,q)-n}$ for $0 \le n < s(p, q)$.

This means that there is exponential contraction along stable disks but nonuniform expansion along unstable disks.

Proposition 5.2 *$d(T^n\pi p, T^n\pi q) \le C\gamma^{\min\{n,s(p,q)-n\}}$ for all $p, q \in \Delta$, $0 \le n \le s(p,q)$.*

Proof Define $z = W^s(p) \cap W^s(q)$. By (P1), $d(\pi f^n p, \pi f^n z) \le C\gamma^n$. Moreover, $s(z,q) = s(p,q)$ and so by (P2), $d(\pi f^n z, \pi f^n q) \le C\gamma^{s(p,q)-n}$. ∎

5.2 Proof of Theorem 2.6

We continue to assume that $T : M \to M$ is a nonuniformly hyperbolic diffeomorphism, modelled by a Young tower $f : \Delta \to \Delta$ as in Subsection 5.1. We have the measure-preserving semiconjugacy $\pi : \Delta \to M$.

Let $h : M \to \mathbb{R}^+$ be a η-Hölder roof function with associated suspension flow $T_t : M^h \to M^h$. Define $\tilde h = h \circ \pi$ with suspension flow $f_t : \Delta^{\tilde h} \to \Delta^{\tilde h}$. The projection $\pi : \Delta^{\tilde h} \to M^h$ defined by $\pi(p,u) = (\pi p, u)$ is a measure-preserving semiconjugacy. Given $v, w \in C^\eta(M^h)$, let $\tilde v = v \circ \pi$, $\tilde w = w \circ \pi$. It suffices to prove decay of correlations for the observables $\tilde v, \tilde w : \Delta^{\tilde h} \to \mathbb{R}$.

We now introduce the truncation $r' = \min\{r, N\}$ and the truncated tower map $f' : \Delta' \to \Delta'$. The argument in Section 3 for the nonuniformly expanding case applies equally to the nonuniformly hyperbolic case.

To simplify notation, in the remainder of this section we write $f : \Delta \to \Delta$ for the truncated tower map and μ for the measure on Δ'. Note that estimate (P2) is unaffected by truncation since the return map to Y is unchanged. Also, estimate (P1) can only be improved by truncation. In particular, Proposition 5.2 remains valid. We note that many of the objects defined below, such as χ, $v_{s,k}$ and so on, depend on N. However, the estimates involve universal constants independent of N.

As in the uniformly expanding case, the significant part of the Laplace transform of the correlation function for the truncated flow has the form

$$\hat\rho(s) = \sum_{n\ge1} \int_\Delta e^{-s\tilde h_n} v_s \, w_s \circ f^n \, d\mu,$$

where $v_s(p) = \int_0^{\tilde h(p)} e^{su} \tilde v(p,u) du$ and $w_s(p) = \int_0^{\tilde h(p)} e^{-su} \tilde w(p,u) du$.

To estimate $\hat\rho(s)$, the first step is to write $\tilde h$ as a coboundary plus a roof function that “depends only on future coordinates”.

Lemma 5.3 *There exist functions $\bar h, \chi : \Delta \to \mathbb{R}$ such that*

(i) $\tilde h = \bar h + \chi - \chi \circ f$,

(ii) $\chi \in L^\infty(\Delta)$ *and* $|\chi|_\infty \le C$ *(independent of N),*

(iii) If $s(p,q) \ge 3k$, *then* $|\chi(f^k p) - \chi(f^k q)| \le C\gamma_1^k$, *where* $\gamma_1 = \gamma^\eta$,

(iv) $\bar h(p) = \bar h(q)$ *for all* $p \in W^s(q)$,

(v) $\bar{h} : \bar{\Delta} \to \mathbb{R}$ *is Lipschitz with respect to the metric* d_θ*, for* $\theta = \gamma_1^{1/3}$.

Proof We modify the proof of [20, Lemma 5.4]. Instead of the two separation times s and s_1 in [20], we have only the separation time s. Most of [20, Lemma 5.4] goes through word for word with s substituted for s_1. The proof only differs in part (v): instead of choosing $p, q \in \Delta$ with $s_1(p,q) \ge 2k+1$, we require that $s(p,q) \ge 3k+1$. Since $\bar{\Delta}$ is Markov, we can choose $\bar{p}' \in \bar{f}^{-k}\bar{p}$, $\bar{q}' \in \bar{f}^{-k}\bar{q}$ with $s(p',q') \ge 3k+1$. (Unlike in [20], the separation of p, q does not necessarily increase with each backward iterate.) By (i), (iii) and the Hölder continuity of h, we have that $|\bar{h}(p) - \bar{h}(q)| = |\bar{h}(f^k p') - \bar{h}(f^k q')| \le C\gamma_1^k$ as required. ∎

(The proof of Lemma 5.3 shows that the introduction of s_1 in [20] is unnecessary.)

By Lemma 5.3, we can write $\hat{\rho}(s) = \sum_{n\ge1} \int_\Delta e^{-s\bar{h}_n}(e^{-s\chi}v_s)\,(e^{s\chi}w_s)\circ f^n\,d\mu$. Next we approximate $e^{-s\chi}v_s$ and $e^{s\chi}w_s$ by functions that "depend only on finitely many coordinates". For $k \ge 1$, define $v_{s,k}(p) = \inf\{(e^{-s\chi}v_s)(f^k q) : s(p,q) \ge 3k\}$.

Lemma 5.4 *The function* $v_{s,k} : \Delta \to \mathbb{R}$ *lies in* $L^\infty(\Delta)$ *and projects down to a Lipschitz observable* $\overline{v_{s,k}} : \bar{\Delta} \to \mathbb{R}$*. Within the region* $s = a + ib$, $|a| \le 1$, $|b| \ge 1$,

(a) $|\overline{v_{s,k}}|_\infty = |v_{s,k}|_\infty \le e^{|\chi|_\infty}|v_s|_\infty \le C|\tilde{v}|_\infty = C|v|_\infty$.

(b) $|\overline{v_{s,k}}|_\theta \le C|v|_\infty\,\theta^{-3k}$.

(c) $|(e^{-s\chi}v_s)\circ f^k - v_{s,k}|_\infty \le C\|v\|_\eta |b|\gamma_1^k$.

Proof The proof is unchanged from [20, Lemma 5.5] except that s is again substituted for s_1. ∎

Write $\int_\Delta e^{-s\bar{h}_n}(e^{-s\chi}v_s)\,(e^{s\chi}w_s)\circ f^n\,d\mu = \int_\Delta e^{-s\bar{h}_n\circ f^k}(e^{-s\chi}v_s)\circ f^k\,(e^{s\chi}w_s)\circ f^k \circ f^n\,d\mu = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= \textstyle\int_\Delta e^{-s\bar{h}_n\circ f^k}(e^{-s\chi}v_s)\circ f^k\,((e^{s\chi}w_s)\circ f^k - w_{s,k})\circ f^n\,d\mu, \\ I_2 &= \textstyle\int_\Delta e^{-s\bar{h}_n\circ f^k}((e^{-s\chi}v_s)\circ f^k - v_{s,k})\,w_{s,k}\circ f^n\,d\mu, \\ I_3 &= \textstyle\int_\Delta e^{-s\bar{h}_n\circ f^k}v_{s,k}\,w_{s,k}\circ f^n\,d\mu. \end{aligned}$$

By Lemma 5.4,

$$|I_1| \le e^{|a||\bar{h}_n|_\infty}e^{|\chi|_\infty}|v_s|_\infty|(e^{s\chi}w_s)\circ f^k - w_{s,k}|_\infty \le C|v|_\infty\|w\|_\eta|b|e^{n|a||h|_\infty}\gamma_1^k,$$

and similarly $|I_2| \le C\|v\|_\eta|w|_\infty|b|e^{n|a||h|_\infty}\gamma_1^k$. Hence

$$|I_1|, |I_2| \le C\|v\|_\eta\|w\|_\eta|b|e^{n\epsilon\min\{N^{-1}\ln N,\,\tilde{d}_N^{-1}|b|^{-\alpha}\}|h|_\infty}\gamma_1^k,$$

for all $a \in U_b$.

The integrand in I_3 projects down to $\bar\Delta$ and $\bar h_n \circ \bar f^k = \bar h_n + \bar h_k \circ \bar f^n - \bar h_k$, so

$$I_3 = \int_{\bar\Delta} e^{-s\bar h_n}[e^{s\bar h_k}\overline{v_{s,k}}]\,[e^{-s\bar h_k}\overline{w_{s,k}}]\circ \bar f^n\,d\bar\mu = \int_{\bar\Delta} L_{-s}^n[e^{s\bar h_k}\overline{v_{s,k}}]\,[e^{-s\bar h_k}\overline{w_{s,k}}]\,d\bar\mu.$$

Here, L is the transfer operator for the truncated quotient tower map $\bar f : \bar\Delta \to \bar\Delta$, and $L_s u = L(e^{s\bar h}u)$. By (4.6),

$$\|L_s^n\| \le C\widetilde d_N |b|^\alpha e^{-n\delta \min\{N^{-1}\ln N,\, \widetilde d_N^{-1}|b|^{-\alpha}\}} + \|E_{s,n}\|$$

on $F_\theta(\bar\Delta)$. Hence,

$$\begin{aligned}|I_3| &\le \|L_{-s}^n\|_b \|e^{s\bar h_k}\|_\theta \|\overline{v_{s,k}}\|_\theta |e^{-sh_k}|_\infty |\overline{w_{s,k}}|_\infty \\ &\le C|v|_\infty |w|_\infty (\widetilde d_N|b|^\alpha e^{-n\delta\min\{N^{-1}\ln N,\, \widetilde d_N^{-1}|b|^{-\alpha}\}} + \|E_{s,n}\|)|b|\theta^{-4k}e^{2k|h|_\infty}.\end{aligned}$$

Choose $k = k(b,n,N)$ so that

$$(e^{2|h|_\infty}\theta^{-4})^k \sim e^{\frac12 n\delta\min\{N^{-1}\ln N,\, \widetilde d_N^{-1}|b|^{-\alpha}\}}.$$

Then there exists $\delta' > 0$ (depending on γ_1 and θ) such that

$$\begin{aligned}&I_1, I_2 = O(e^{-n(\delta'-\epsilon)\min\{N^{-1}\ln N,\, \widetilde d_N^{-1}|b|^{-\alpha}\}|h|_\infty}|b|),\\ &I_3 = O(\widetilde d_N e^{-\frac12 n\delta\min\{N^{-1}\ln N,\, \widetilde d_N^{-1}|b|^{-\alpha}\}}|b|^{\alpha+1}) + O(N^{\delta/2}\|E_{s,n}\||b|).\end{aligned}$$

Here, we have used the fact that $E_{s,n} = 0$ for $n > N$. Choosing ϵ small enough, we obtain a new $\delta > 0$ such that

$$\begin{aligned}&\left|\int_\Delta e^{-s\bar h_n}(e^{-s\chi}v_s)\,(e^{s\chi}w_s)\circ f^n d\mu\right| \\ &\qquad \le C\|v\|_\eta \|w\|_\eta (\widetilde d_N e^{-n\delta\min\{N^{-1}\ln N,\, \widetilde d_N^{-1}|b|^{-\alpha}\}} + N^\delta\|E_{s,n}\|)|b|^{\alpha+1}.\end{aligned}$$

Summing over n as in Lemma 4.10, we obtain

$$|\hat\rho(s)| \le C\|v\|_\eta\|w\|_\eta\,\widetilde d_N|b|^{\alpha+1}\max\{N(\ln N)^{-1}, \widetilde d_N|b|^\alpha\}.$$

This is almost identical to the estimate (4.7) obtained in the nonuniform expanding case, and so we recover the required decay of correlation result for $\rho'(t)$ in Theorem 4.3 as before (but with $m = (p+2)\alpha + 2$) and hence for $\rho(t)$. ∎

5.3 Temporal distance function

In this subsection, we follow Dolgopyat [11, Appendix]. In the situation of Subsection 5.1, let $y_1, y_4 \in Y$ and use the hyperbolic product structure on Y to uniquely define $y_2, y_3 \in Y$ by setting $y_2 \in W^s(y_1) \cap W^u(y_4)$, $y_3 \in W^u(y_1) \cap W^s(y_4)$. By exponential contraction along stable and unstable disks and Hölder continuity of h, the *temporal distance function*

$$D(y_1, y_4) = \sum_{n=-\infty}^{\infty} h(T^n y_1) - h(T^n y_2) - h(T^n y_3) + h(T^n y_4)$$

is a well-defined continuous map $D : Y \times Y \to \mathbb{R}$.

Writing $\tilde{h} = h \circ \pi$ and $p_j = (y_j, 0) \in \Delta$, we have $h(T^n y_j) = \tilde{h}(f^j p_j)$. By Lemma 5.3, we can write $\tilde{h} = \bar{h} + \chi - \chi \circ f$ where $\bar{h}$ depends only on future coordinates. Hence

$$\begin{aligned} D(y_1, y_4) &= \sum_{n=-\infty}^{-1} \bar{h}(f^n p_1) - \bar{h}(f^n p_2) - \bar{h}(f^n p_3) + \bar{h}(f^n p_4) \\ &= \sum_{n=-\infty}^{-1} \bar{H}(F^n y_1) - \bar{H}(F^n y_2) - \bar{H}(F^n y_3) + \bar{H}(F^n y_4). \end{aligned}$$

Note that $D(y_1, y_4) = D_M(y_1, y_4) + O(\gamma_1^M)$, where

$$\begin{aligned} D_M(y_1, y_4) &= \sum_{n=-M}^{-1} \bar{H}(F^n y_1) - \bar{H}(F^n y_2) - \bar{H}(F^n y_3) + \bar{H}(F^n y_4) \\ &= \bar{H}_M(F^{-M} y_1) - \bar{H}_M(F^{-M} y_2) - \bar{H}_M(F^{-M} y_3) + \bar{H}_M(F^{-M} y_4). \end{aligned}$$

Hence,

$$\exp\{ibD_M(y_1, y_4)\} = \frac{\exp\{ib\bar{H}_M(F^{-M} y_1)\} \exp\{ib\bar{H}_M(F^{-M} y_4)\}}{\exp\{ib\bar{H}_M(F^{-M} y_2)\} \exp\{ib\bar{H}_M(F^{-M} y_3)\}}. \tag{5.1}$$

This should be viewed as being defined on the quotient $\bar{Y}$ with the interpretation that $\{F^{-M} y_j\}$ denotes a fixed inverse branch, such that the inverse branch of y_1 is compatible with that for y_3 and similarly for y_2 and y_4.

Theorem 5.5 *Assume that the hypotheses of Theorem 2.6 are valid. (So decay of correlations does* not *hold at the required rate.) Then for every finite subsystem $Z_0 = \bigcap_{n=0}^{\infty} F^{-n} Z \subset Y$, for all $\alpha > 0$, there is a sequence $b_k \in \mathbb{R}$ with $|b_k| \to \infty$ such that*

$$|e^{ib_k D(y_1, y_4)} - 1| \le C|b_k|^{-\alpha}$$

for all $y_1, y_4 \in Z_0$.

Proof If decay of correlations fails, then it follows from Corollary 4.4 that there are approximate eigenfunctions u_k on $\bar{Z}_0 \subset \bar{Y}$. In particular,

$$|e^{-ib_k \bar{H}_{n_k}} e^{-i\omega_k r_{n_k}} u_k \circ \bar{F}^{n_k} - e^{i\varphi_k} u_k| \le 1/|b_k|^\alpha. \tag{5.2}$$

Recall that $|u_k| \equiv 1$ and $\|u_k\|_\theta \le C|b_k|$.

Substituting (5.2) into (5.1) leads to a significant amount of cancellation. The factors $e^{i\varphi_k}$ cancel, as do the r_{n_k} terms since $r_M(F^{-M}y_1) = r_M(F^{-M}y_3)$ and $r_M(F^{-M}y_2) = r_M(F^{-M}y_4)$. Hence

$$\exp\{ib_k D_{n_k}(y_1, y_4)\} = \frac{u_k(F^{-n_k}y_3)}{u_k(F^{-n_k}y_1)} \frac{u_k(F^{-n_k}y_2)}{u_k(F^{-n_k}y_4)} + O(|b_k|^{-\alpha}).$$

Since we have chosen identical inverse branches for y_1 and y_3,

$$\Big|\frac{u_k(F^{-n_k}y_3)}{u_k(F^{-n_k}y_1)} - 1\Big| \le \|u_k\|_\theta d_\theta(F^{-n_k}y_3, F^{-n_k}y_1) \le C|b_k|\theta^{n_k} \le C|b_k|^{\beta_0 \ln\theta+1},$$

and similarly for the remaining quotient. Choosing β_0 sufficiently large, we obtain $\exp\{ib_k D_{n_k}(y_1, y_4)\} = 1 + O(|b_k|^{-\alpha})$ as required. ∎

Corollary 5.6 *In the situation of Theorem 5.5, the range of $D|Z_0 \times Z_0$ has lower box dimension zero.*

Proof The condition $|e^{ib_k D(y_1,y_4)} - 1| \le |b_k|^{-\alpha}$ implies that the range of D restricted to $Z_0 \times Z_0$ lies inside $\bigcup_{m\in\mathbb{Z}}(\frac{2\pi m}{b_k} - \frac{2}{|b_k|^{\alpha+1}}, \frac{2\pi m}{b_k} + \frac{2}{|b_k|^{\alpha+1}})$, for all k. Hence $\underline{\mathrm{BD}}(D(Z_0 \times Z_0)) \le 1/(1+\alpha)$. The result follows since α is arbitrarily large. ∎

Example 5.7 (Lorentz gas examples) We are now in a position to explain why the decay rates for the Lorentz gas examples in Section 1 hold always rather than typically. Such examples possess a *contact structure*: there is a differential 1-form α on the odd-dimensional $(2n+1)$-manifold M (in our examples $n = 1$) with $\alpha \wedge (d\alpha)^n$ non-vanishing. Moreover, the Lorentz flow is a *contact flow* (the contact form is preserved by the flow). Liverani [17, Appendix B] is a good reference for basic and nonbasic facts about contact flows.

Let $x \in M$, $y^s \in W^s(x)$, $y^u \in W^u(x)$. Then a formula of Katok & Burns [16, Lemma 3.2] (see also Liverani [17, Lemma B.7]) states that

$$D(y^s, y^u) = d\alpha(v^s, v^u) + o(|v^s|^2 + |v^u|^2), \tag{5.3}$$

where $y^{s,u} = \exp_x v^{s,u}$, $v^{s,u} \in E^{s,u}(x)$.

We apply formula (5.3) by fixing $y_2 = x \in Z_0$ and $y_1 = y^s \in W^s(x) \cap Z_0$, and varying $y_4 = y^u \in W^u(x) \cap Z_0$. It follows from (5.3) that $v^u \to D(y_1, y_4)$ is linear in v^u at lowest order. We claim that the unstable disk $W^u(x) \cap Z_0$ has positive lower box dimension. It then follows that the range of D has positive lower box dimension, ruling out the existence of approximate eigenfunctions on Z_0 by Corollary 5.6. (The construction of Young [27] guarantees that $W^u(x) \cap Y$ has positive measure with respect to the one-dimensional Lebesgue measure induced on $W^u(x)$ and so $W^u(x) \cap Y$

certainly has positive dimension. Our claim is that the zero measure set $W^u(x) \cap Z_0$ still has positive dimension.)

Clearly, it suffices to work at the level of the quotient tower, so it remains to show that $\underline{\mathrm{BD}}(\bar{Z}_0) > 0$. From now on, we write Z_0 to mean $\bar{Z}_0$ and so on. Recall that $Z_0 = \bigcap_{n=0}^{\infty} F^{-n}Z$ where Z consists of finitely many partition elements of Y mapped bijectively by F with bounded distortion onto Y. If the partition elements were intervals, then Z_0 would be a *dynamically-defined Cantor set* with $\mathrm{BD}(Z_0) = \mathrm{HD}(Z_0) \in (0,1)$ (see for example [21]). Now it follows from the construction of Young [27] that we can extend Z_0 to such a dynamically-defined Cantor set, choosing intervals $I_1, \ldots, I_m$ containing the partition elements in Z, and an interval $K \supset Y$ such that F extends to a uniformly expanding bijection $F : I_j \to K$ with bounded distortion for each j. Then $K_0 = \bigcap_{n=0}^{\infty} F^{-n}K$ is a dynamically-defined Cantor set with $d = \mathrm{HD}(K_0) \in (0,1)$. Moreover, the d-dimensional Hausdorff measure of K_0 satisfies $m_d(K_0) \in (0,\infty)$. Let $\mathcal{B}_n$ be the collection of intervals of the form

$$I_{j_0,\ldots,j_n} = \bigcap_{\ell=0}^{n} F^{-\ell} I_{j_\ell}, \quad j_0, \ldots, j_n \in \{1, \ldots, r\}.$$

There are constants $0 < a \le b < 1$ such that $a^n \le |I| \le b^n$ for all $I \in \mathcal{B}_n$ and n large enough. We claim that $\mathrm{BD}(Z_0) \ge c = d \ln b / \ln a$.

It remains to verify the claim. Let $\delta > 0$ and suppose that $\{U\}$ is a cover of Z_0 by intervals of diameter $|U| = \delta$. Choose $n = [-\gamma \ln \delta] + 1$ where $\gamma = -1/\ln a$. Let $\mathcal{B}_n^U = \{I \in \mathcal{B}_n : I \cap U \ne \emptyset\}$ and let N be the number of such intervals that lie entirely inside U. Then $Na^n \le \sum_{I \in \mathcal{B}_n^U} |I| \le |U|$, and hence $N \le \delta/a^n$. It follows that

$$\begin{aligned} \sum_{I \in \mathcal{B}_n^U} |I|^d &\le (\delta/a^n + 2) b^{nd} \le \delta (b^d/a)^{-\gamma \ln \delta} + 2b^{-d\gamma \ln \delta} \\ &\le \delta^{1-\gamma \ln(b^d/a)} + 2\delta^{-d\gamma \ln b} = 3\delta^c = 3|U|^c. \end{aligned} \tag{5.4}$$

Note that $\mathcal{B}_n = \bigcup_U \mathcal{B}_n^U$ (since every basic interval intersects Z_0 and hence intersects at least one U). It follows that

$$\sum_{I \in \mathcal{B}_n} |I|^d \le \sum_U \sum_{I \in \mathcal{B}_n^U} |I|^d \tag{5.5}$$

Summing over U, and using estimates (5.4) and (5.5),

$$0 < m_d(K_0) \le \sum_{I \in \mathcal{B}_n} |I|^d \le \sum_U \sum_{I \in \mathcal{B}_n^U} |I|^d \le 3 \sum_U |U|^c.$$

Hence $\underline{\mathrm{BD}}(Z_0) \ge c > 0$ as required.

6 Decay for flows with unbounded roof functions

In this section, we prove Theorem 2.7. The strategy is similar to the previous sections but now two truncations are required. Let

$$U_b = \big\{a \in \mathbb{R} : |a| < \epsilon \min\{N^{-1}, \widetilde{d}_N^{-1}|b|^{-\alpha}\}\big\},$$
$$V_b = \big\{\sigma \in \mathbb{R} : |\sigma| < \epsilon \widetilde{d}_N^{-1}|b|^{-\alpha}\big\},$$

where $\widetilde{d}_N$ is defined below in Lemma 6.6. In the semiflow case, the steps are as follows:

(a) Model by a tower.

(b) Truncate h to $h' = \min\{h, N\}$. This leads to an error $O(N^{-\beta} + tN^{-(\beta+1)})$ in the correlation function.

(c) Truncate r to $r' = \min\{r, [q \ln N]\}$. The truncation error takes the form $O(tN^{-(cq-1)})$. Choosing $q > (\beta + 2)/c$ ensures that this is dominated by the error in (b).

(d) $\|(I - R_{s,z})^{-1}\|_b \le C|b|^\alpha$ for $a \in U_b$, $\sigma \in V_b$.

(e) $|L_s^n v|_1 \le C\|v\|_b N|b|^\alpha e^{-n\delta \widetilde{d}_N^{-1}|b|^{-\alpha}}$ for $a \in U_b$.

(f) $|\hat{\rho}(s)| \le C\|v\|_\theta |w|_\infty N^3 \widetilde{d}_N |b|^{2\alpha}$ for $a \in U_b$, where $\hat{\rho}(s)$ is the Laplace transform of the (doubly) truncated correlation function $\rho'(t)$.

(g) $|\rho'(t)| \le C\|v\|_{m,\theta}|w|_\infty\{N^3\widetilde{d}_N e^{-\epsilon N^{-1}t} + N^3\widetilde{d}_N^{p+1}t^{-p}\}$.

(h) Specify $N = N(t)$.

Step (a) is identical to Step 1 in Section 2.4. We may assume from now on that the nonuniformly expanding map is a tower map $f : \Delta \to \Delta$ and that $h : \Delta \to \mathbb{R}^+$ is a (nonuniformly) Lipschitz roof function.

6.1 Truncation of h

In this subsection, we carry out Step (b). Let $\Delta(n) = \bigcup\{\Delta_{j,\ell} : \|h1_{\Delta_{j,\ell}}\|_\theta \ge n\}$. Condition (7) on h in Section 2.3 guarantees that $\mu_\Delta(\Delta(n)) \le Cn^{-(\beta+1)}$.

Fix $N \ge 1$ and let $h' = \min\{h, N\}$. We form the suspension flows $f_t : \Delta^h \to \Delta^h$ and $f'_t : \Delta^{h'} \to \Delta^{h'}$. Observables v, w on Δ^h restrict to observables on $\Delta^{h'}$ and we define the correlation functions $\rho(t)$ and $\rho'(t)$.

Write $\Delta^h = \Delta^h_{\rm left} \dot\cup \Delta^h_{\rm right}$ where

$$\Delta^h_{\rm left} = \{(x, u) \in \Delta^h : h(x) \le N\}, \qquad \Delta^h_{\rm right} = \{(x, u) \in \Delta^h : h(x) > N\}.$$

As in Proposition 3.1, we obtain $\bar{h} - \bar{h}' \le CN^{-\beta}$ and $\mu_{\Delta^h}(\Delta^h_{\rm right}) \le CN^{-\beta}$.

Proposition 6.1 *For $k \ge 1$, define*

$$E_k = \{p \in \Delta^h : f_t p \in \Delta^h_{\rm right} \ \textit{for some}\ t \in [0,k]\}.$$

Then $\mu_{\Delta^h}(E_k) \le C\{N^{-\beta} + kN^{-(\beta+1)}\}$ for all $N \ge 2$.

Proof Write E_k as the disjoint union $E_k = \bigcup_{j=1}^k G_j$ where

$$G_j = \{f_t p \in \Delta^h_{\rm left} \ \text{ for } t \in [0, j-1) \ \text{ and } \ f_t p \in \Delta^h_{\rm right} \ \text{ for some } t \in [j-1, j]\}.$$

For $j \ge 2$, it follows from the definition that if $p \in G_j$, then $f_j p \in \Delta^1_{\rm right}$ where $\Delta^1_{\rm right} = \{(x,u) \in \Delta \times [0,1] : h(x) > N\}$. Hence $\mu_{\Delta^h}(G_j) \le \mu_{\Delta^h}(f_j^{-1}(\Delta^1_{\rm right})) = \mu_{\Delta^h}(\Delta^1_{\rm right}) = (1/\bar h)\mu_\Delta(h > N) \le CN^{-(\beta+1)}$.

If $p \in G_1$, then either $p \in \Delta^h_{\rm right}$ or $f_1 p \in \Delta^1_{\rm right}$. Hence $\mu_{\Delta^h}(G_1) \le CN^{-\beta} + CN^{-(\beta+1)}$. ∎

Lemma 6.2 *Suppose that $v, w : \Delta^h \to \mathbb{R}$ lie in L^∞ and define $\rho(t)$, $\rho'(t)$ as indicated above. For $N \ge 2$, $t > 0$,*

$$|\rho(t) - \rho'(t)| \le C|v|_\infty |w|_\infty \{N^{-\beta} + tN^{-(\beta+1)}\}.$$

Proof For notational convenience, we write $\Omega = \Delta^h$ and $\Omega' = \Delta^{h'}$. Let $A = \int_\Omega v\, w \circ f_t \, d\mu_\Omega$, $A' = \int_{\Omega'} v\, w \circ f'_t \, d\mu_{\Omega'}$. Then

$$\begin{aligned} A - A' &= \int_\Omega (v\, w \circ f_t - v\, w \circ f'_t)\, d\mu_\Omega + \Big(\int_\Omega v\, w \circ f'_t \, d\mu_\Omega - \int_{\Omega'} v\, w \circ f'_t \, d\mu_{\Omega'} \Big) \\ &= I + II. \end{aligned}$$

Using Proposition 6.1, we compute that

$$|I| \le 2|v|_\infty |w|_\infty \mu_\Omega\{f_t \ne f'_t\} \le C|v|_\infty |w|_\infty \{N^{-\beta} + (t+1)N^{-(\beta+1)}\}.$$

Next,

$$II = (1/\bar h) \int_\Delta \int_{h'}^{h} v\, w \circ f'_t \, du\, d\mu_\Delta + \Big((1/\bar h) - (1/\bar h') \Big) \int_\Delta \int_0^{h'} v\, w \circ f'_t \, du\, d\mu_\Delta.$$

Hence

$$\begin{aligned} |II| &\le (1/\bar h)|v|_\infty |w|_\infty (\bar h - \bar h') + (1/\bar h)(1/\bar h')(\bar h - \bar h')|v|_\infty |w|_\infty \bar h' \\ &\le C|v|_\infty |w|_\infty N^{-\beta}. \end{aligned}$$

The result follows. ∎

6.2 Truncation of r

Recall that $\mu_Y(r > n) = O(e^{-cn})$ where $c > 0$. We make the second truncation $r' = \min\{r, [q \ln N]\}$. Following Section 3, we obtain $\bar{r} - \bar{r}' \le CN^{-cq}$, $\mu_\Delta(\Delta_{\text{right}}) \le CN^{-cq}$ and $\mu_\Delta(E_k) \le CkN^{-cq}$. (Note that none of these calculations depends on h.) The proof of Lemma 3.3 proceeds as before except that there is need for care since $|h'|_\infty = N$. This leads to the loss of one factor of N and hence the truncation error $O(tN^{-(cq-1)})$.

6.3 Decay for the semiflow

In this subsection, we carry out Steps (d)-(h), completing the proof of Theorem 2.7. Let $Y(n) = \bigcup\{Y_j : \|1_{Y_j} H\|_\theta \ge n\}$.

Lemma 6.3 $\mu_Y(Y(n)) = O((\ln n)^{\beta+2} n^{-(\beta+1)})$.

Proof Let $Q > 0$ and write $\mu_Y(Y(n)) \le \mu_Y(r > [Q \ln n]) + \sum_{k=1}^{[Q\ln n]} \mu_Y(\{r = k\} \cap Y(n))$. If $Y_j \subset \{r = k\} \cap Y(n)$, then $\|1_{\Delta_{j,\ell}} h\|_\theta > n/k$ for some $\ell < k$, and so $\Delta_{j,\ell} \subset \Delta(n/k)$. Since $\mu_Y(Y_j) = \bar{r}\mu_\Delta(\Delta_{j,\ell})$,

$$\mu_Y(\{r = k\} \cap Y(n)) \le \bar{r}\mu_\Delta(\Delta(n/k)) \le C(k/n)^{\beta+1}.$$

Hence

$$\mu_Y(Y(n)) \le C\Big(e^{-cQ\ln n} + \sum_{k=1}^{[Q\ln n]} (k/n)^{\beta+1}\Big) \le C(e^{-cQ\ln n} + (\ln n)^{\beta+2} n^{-(\beta+1)}).$$

Now choose $Q = (\beta + 1)/c$. ∎

Since $h' = \min\{h, N\}$, we have $\|1_{\Delta_{j,\ell}} h'\|_\theta \le \|1_{\Delta_{j,\ell}} h\|_\theta$ for all partition elements $\Delta_{j,\ell}$, and hence $\|1_{Y_j} H'\|_\theta \le \|1_{Y_j} H\|_\theta$. By Lemma 6.3,

$$\sum_{j\ge1} \|1_{Y_j} H'\|_\theta \, \mu_Y(Y_j) \le \sum_{j\ge1} \|1_{Y_j} H\|_\theta \, \mu_Y(Y_j) < \infty.$$

This corresponds to Proposition 4.5 and guarantees that we obtain a basic inequality

$$|R^n_{ib,i\omega} v|_\theta \le C\{|b||v|_\infty + \theta^n |v|_\theta\},$$

uniformly in N. Define $\|v\|_b = \max\{|v|_\infty, |v|_\theta/(2C|b|)\}$.

Let $Z_0 \subset Y$ be a finite subsystem and let Z_0^r denote the part of the tower Δ over Z_0. Then Z_0^r consists of finitely many partition elements so that $h|Z_0^r$ is bounded. It follows that there exists N_0 such that $h|Z_0^r = h'|Z_0^{r'}$ for all $N \ge N_0$. In particular, $H|Z_0 = H'|Z_0$ for all $N \ge N_0$. We now have the ingredients required for the analogue of Lemma 4.6:

Lemma 6.4 *Assume no approximate eigenfunctions on Z_0 and choose N sufficiently large that $H|Z_0 = H'|Z_0$. Then there exist $\alpha > 0$, $C \geq 1$ independent of N such that*

$$\|(I - R_{ib,i\omega})^{-1}\|_b \leq C|b|^\alpha,$$

for all $|b| > 1$, $\omega \in [0, 2\pi)$. ∎

Proposition 6.5 *Let $d_N = \sum_{k=1}^N k\mu_Y(Y(k))$. Then*

$$\|R_{s,z} - R_{ib,i\omega}\|_b \leq C d_N(|a| + |\sigma|)e^{q(|a|N+|\sigma|)\ln N},$$

for all s, $z \in \mathbb{C}$.

Proof As in the proof of Proposition 4.7, we have

$$\|(R_{s,z} - R_{ib,i\omega})1_{Y_j}\|_b \leq C(|a|\|1_{Y_j}H'\|_\theta + |\sigma|r'(j))(1 + |1_{Y_j}H'|_\theta)e^{|a||1_{Y_j}H'|_\infty}e^{|\sigma|r'(j)}\mu_Y(Y_j).$$

Note that $r'(j) \leq |\frac{1}{h}|_\infty |1_{Y_j}H'|_\infty$. Since $r'_j \leq q\ln N$ and $|1_{Y_j}H'|_\infty \leq qN\ln N$,

$$\|(R_{s,z} - R_{ib,i\omega})1_{Y_j}\|_b \leq C(|a| + |\sigma|)e^{q(|a|N+|\sigma|)\ln N}\|1_{Y_j}H'\|_\theta^2\mu_Y(Y_j).$$

Now sum over $j \geq 1$. ∎

Combining these two results leads to the following analogue of Lemma 4.8, completing Step (d):

Lemma 6.6 *Assume no approximate eigenfunctions on Z_0 and choose N sufficiently large that $H|Z_0 = H'|Z_0$. Let $d > 0$ and set $\widetilde{d}_N = d_N N^d$. Define U_b, V_b as at the beginning of the section. Then there exist $\alpha > 0, \epsilon > 0$ and $C \geq 1$ independent of N such that*

$$\|(I - R_{s,z})^{-1}\|_b \leq C|b|^\alpha,$$

for all $a \in U_b$, $\sigma \in V_b$. ∎

Next we carry out Step (e).

Lemma 6.7 *Assume no approximate eigenfunctions on Z_0 and choose N sufficiently large that $H|Z_0 = H'|Z_0$. Let $d > 0$ and set $\widetilde{d}_N = d_N N^d$. There exist constants $\epsilon, \delta > 0, \alpha > 0$, $C \geq 1$ independent of N such that*

$$|L_s^n v|_1 \leq C\|v\|_b\, N|b|^\alpha e^{-n\delta\widetilde{d}_N^{-1}|b|^{-\alpha}},$$

for all $v \in F_\theta(\Delta')$, $n \geq 1$, and $a \in U_b$.

Proof Define the sequences $T_{s,n}$, $A_{s,n}$, $B_{s,n}$, $E_{s,n}$ as in Section 4.2. Assuming no approximate eigenfunctions, it follows from Lemma 6.6 that $\|T_{s,n}\|_b \le C|b|^\alpha e^{-n\delta \widetilde{d}_N^{-1}|b|^{-\alpha}}$.

By truncation of r, the operators $\|A_{s,n}\|$, $\|B_{s,n}\|$, $\|E_{s,n}\|$ vanish for $n > [q\ln N]$. For $|a| \le \epsilon N^{-1}$, we compute that $\|A_{s,n}\| \le Ce^{-c'n}$, $\|B_{s,n}\| \le CNne^{-c'n}$, and $\|E_{s,n}\| \le Ce^{-c'n}$ where $c' = c - \epsilon$. The result follows. ∎

Consequently, $\sum_{n\ge 1} |L_s^n v|_1 \le C\|v\|_b N\widetilde{d}_N|b|^{2\alpha}$. Moreover, it is easy to check that

$$\|v_s\|_\theta \le C\|v\|_\theta N, \qquad |w_s|_\infty \le C|w|_\infty N.$$

Hence

$$|\hat\rho(s)| \le C\|v\|_\theta |w|_\infty N^3\widetilde{d}_N|b|^{2\alpha},$$

for all $a \in U_b$ completing Step (f). Step (g) is proved as in Section 4, and combining Steps (b) and (g) we obtain

$$|\rho(t)| \le C\|v\|_{m,\theta}|w|_\infty\{N^{-\beta} + tN^{-(\beta+1)} + N^3\widetilde{d}_N e^{-\epsilon N^{-1}t} + N^3\widetilde{d}_N^{p+1}t^{-p}\}.$$

Set $N = [t/(q\ln t)]$. For p, q sufficiently large, $\rho(t) = O((\ln t)^{\beta+1}t^{-\beta})$ as required.

6.4 Logarithmic factors

Lemma 6.3 shows that the decay rates on r and h lead to a decay rate for H. An alternative approach is to make an assumption on H (via $Y(n)$) from the outset. In particular, if we assume that

$$\mu_Y(r > n) = O(\gamma^n), \qquad \mu_Y(Y(n)) = O(n^{-(\beta+1)}),$$

then we obtain typically the estimate $\rho(t) = O(t^{-\beta})$. (The proof proceeds by truncating so that $r' = \min\{r, [q\ln N]\}$ and $H' = \min\{H, N\}$, with U_b modified so that $|a| \le \epsilon N^{-1}\ln N$ as in Section 4.) With the obvious modifications, we can handle general decay rates for $\mu_Y(Y(n))$. Presumably this method gives sharp results, but the assumption on H is more difficult to verify.

One situation where $Y(n)$ decays slower than $\Delta(n)$ is when the values of h are constant up the tower. Write $a_n \sim b_n$ to mean $a_n = O(b_n)$ and $b_n = O(a_n)$. Suppose that $\mu_Y(r = n) \sim e^{-n}$ and let $h = (n^{-1}e^n)^{1/(\beta+2)}$ on all partition elements $\Delta_{j,\ell}$ with $r(j) = n$. By definition,

$$\mu_\Delta(h = (n^{-1}e^n)^{1/(\beta+2)}) \sim ne^{-n}.$$

It follows that $\mu_\Delta(h = n) \sim n^{-(\beta+2)}$ and hence $\mu_\Delta(\Delta(n)) \sim n^{-(\beta+1)}$. On the other hand, $H = n(n^{-1}e^n)^{1/(\beta+2)}$ on all Y_j with $r(j) = n$ so that a similar calculation gives $\mu_Y(Y(n)) \sim (\ln n)^{\beta+1}n^{-(\beta+1)}$ which is one factor of $\ln n$ short of the upper bound in Lemma 6.3. In this situation, we cannot hope to improve Theorem 2.7.

On the other hand, if we modify the previous example so that $h = e^{n/(\beta+2)}$ on partition elements $\Delta_{j,0}$ with $r(j) = n$ and h is uniformly bounded on the remainder of the tower, then again $\mu_\Delta(\Delta(n)) \sim n^{-(\beta+1)}$, but this time $\mu_Y(Y(n)) \sim n^{-(\beta+1)}$ and typically $\rho(t) = O(t^{-\beta})$.

Alternatively, suppose that there is a constant $m \geq 1$ such that for each j with $r(j) \leq q \ln N$ there are at most m values of $\ell < r(j)$ such that $\|1_{\Delta_{j,\ell}} h\|_\theta \geq N \ln N^{-1}$. Then $\|1_{Y_j} H'\|_\theta \leq mN + (q \ln N)(N \ln N^{-1}) = (m+q)N$. In this situation, the truncations of r and h automatically achieve the required truncation of H and so typically $\rho(t) = O(t^{-\beta})$.

6.5 Decay for the flow

Here, we mimic Section 5 but in the context of Subsection 6.3, taking account of the fact that $|h'|_\infty = N$. It is easily verified in [20, Lemma 5.4] that $\bar{h}$ and χ inherit a single factor of N from h' and this is compensated for by the fact that $|a| \leq \epsilon N^{-1}$. Proceeding as in Section 5, we break the n'th term of the series for $\hat{\rho}(s)$ into $I_1 + I_2 + I_3$ where

$$I_1, I_2 = O(|b|\gamma_1^k e^{n\epsilon N \min\{N^{-1}, \widetilde{d}_N^{-1}|b|^{-\alpha}\}}), \quad I_3 = O(|b|^{\alpha+1} N^3 \lambda^k e^{-n\delta \widetilde{d}_N^{-1}|b|^{-\alpha}}),$$

where $\lambda > 1$.

To progress further, we modify the definition of U_b so that $|a| \leq \epsilon N^{-1} \widetilde{d}_N^{-1}|b|^{-\alpha}$. Then

$$I_1, I_2 = O(|b|^{\alpha+1} N^3 \gamma_1^k e^{n'\epsilon}), \quad I_3 = O(|b|^{\alpha+1} N^3 \lambda^k e^{-n'\delta}),$$

where $n' = n\widetilde{d}_N^{-1}|b|^{-\alpha}$. Summing over n, we obtain

$$|\hat{\rho}(s)| \leq C\|v\|_\eta \|w\|_\eta N^3 \widetilde{d}_N |b|^{2\alpha+1},$$

for all $s = a + ib$ with $|a| \leq \epsilon N^{-1} \widetilde{d}_N^{-1}|b|^{-\alpha}$. Taking $N = t^{1-\epsilon'}$ and p sufficiently large, we obtain the result claimed in Remark 2.9.

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