MAGNETIC DYNAMOS IN ROTATING CONVECTION — A DYNAMICAL SYSTEMS APPROACH

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Abstract. In this paper, we consider the existence of self-sustained magnetic dynamos in rotating Bénard convection. Dynamical systems techniques are used to bridge the gap between the kinematic dynamo problem and the full rotating magnetohydrodynamic equations.

Building upon the solution of Childress and Soward, proposed in 1972, to the kinematic dynamo problem, we show that secondary bifurcation from purely convective states leads to solutions to the full nonlinear dynamo problem. In particular, we address issues that arise from the Küppers-Lortz instability in rotating convection. We obtain steady, periodic, traveling and intermittent weak-field dynamos.

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1 Introduction

Dynamo theory investigates the creation of self-sustained magnetic fields in electrically conducting fluids. See the review of Roberts and Soward [17]. The aim is to explain magnetic fields in the Earth and the Sun. Since fluid motion in the Earth's core and in the Sun are driven by convection, it is natural to study convecting fluids.

The underlying rotating magnetohydrodynamic equations are given as follows. Consider an incompressible, uniformly rotating fluid between two infinite horizontal plates and suppose that the lower plate is heated uniformly. At low temperatures, the fluid is in a purely conductive, stationary state. Convection sets in at a critical temperature that is determined by the linear stability of the conducting state. The departure from the pure conduction state is governed by the Boussinesq equations for rotating convection. We suppose that the fluid is electrically conducting, so that the Boussinesq equations are coupled to the magnetic induction equation by the Lorentz force.

Let $\Omega = \mathbb{R}^2 \times [0,1]$ denote the domain of the fluid. We shall denote points in the domain Ω as (x,z) where $x = (x_1,x_2) \in \mathbb{R}^2$ and $z \in [0,1]$. Writing $V : \Omega \to \mathbb{R}^3$ for the velocity field of the fluid, $B : \Omega \to \mathbb{R}^3$ for the magnetic field, $p : \Omega \to \mathbb{R}$ for the pressure and $\theta : \Omega \to \mathbb{R}$ for the deviation of the temperature from the conduction state, the governing partial differential equation (PDE) has the form [3]

$$\begin{aligned}
\partial V/\partial t &= -(V\cdot\nabla)V - \nabla p + \sqrt{T}V\times k + \Delta V + \sqrt{R}\theta k - B\times \operatorname{curl} B \\
\partial\theta/\partial t &= -(V\cdot\nabla)\theta + \mathrm{Pr}^{-1}(\Delta\theta + \sqrt{R}V\cdot k) \\
\partial B/\partial t &= \beta\Delta B + \operatorname{curl}(V\times B) \\
\operatorname{div} V &= \operatorname{div} B = 0, \qquad \int_\Omega B_1 = \int_\Omega B_2 = 0.
\end{aligned} \tag{1.1}$$

(We have omitted the centrifugal term which can be incorporated into the pressure.) The constants R, T, Pr and β are dimensionless (Rayleigh number, Taylor number, Prandtl number and magnetic Prandtl number) and $k = (0,0,1)$.

Homogeneous boundary conditions are imposed at the top and bottom of Ω. For definiteness, we consider the specific boundary conditions

$$\frac{\partial V_1}{\partial z} = \frac{\partial V_2}{\partial z} = \frac{\partial B_1}{\partial z} = \frac{\partial B_2}{\partial z} = 0, \qquad V_3 = B_3 = \theta = 0, \tag{1.2}$$

when $z = 0$ and $z = 1$.

Now, V can be written as the sum of a conservative vector field and a divergence free vector field. The equation for $\partial V/\partial t$ splits up into two components one of which can be solved for the pressure p. The remaining system of equations (which we continue to label by (1.1) and which we do not write out explicitly) is an evolution equation in (V,θ,B) where V and B are divergence free vector fields and θ is a scalar field.

In this paper, we shall say that a *magnetic dynamo* is an ω-limit set containing points (V,θ,B) with $B \neq 0$. If the ω-limit set is an attractor, then we speak of an *attracting magnetic dynamo*. However, it should be understood that even for convective solutions, the notion of attractivity is highly problematic when considering stability to all possible perturbations in the infinite planar model. A standard remedy (see [18] for example) is to consider stability only to perturbations with a certain kind of spatial periodicity.

One approach to proving the existence of magnetic dynamos is to use bifurcation theory. There is a convenient 'trivial' solution, namely the pure conduction state $(V,\theta,B) = (0,0,0)$. It is easy to see that this state cannot lose stability directly to a magnetic dynamo: Observe that equations (1.1) possess the symmetry

$B \to -B$ and hence the convection solutions (V, θ) lie in the dynamically-invariant subspace $\{B = 0\}$. Obtaining a bifurcation out of this subspace is equivalent to requiring a bifurcation from the solution $B = 0$ in the $\partial B/\partial t$ equation. For the pure conduction state we have $V = 0$ and the right-hand-side of the $\partial B/\partial t$ equation reduces to $\beta \Delta B$ which is negative definite.

Similarly, for fixed $\beta > 0$ and V small, all solutions to the $\partial B/\partial t$ equation are exponentially damped to 0 as $t \to \infty$. When V is large, the possibility of instability depends on the precise form of V and there are antidynamo theorems [17] that rule out instabilities when V is not sufficiently complicated. It is thus necessary to find a bifurcation from the pure conduction state to a sufficiently complicated convective state which, on reaching large enough amplitude, can undergo a secondary bifurcation to a 'weak-field' magnetic dynamo.

The possibility or impossibility of this secondary magnetic instability is called the *kinematic dynamo problem*. This concerns the linear stability of 0 in the $\partial B/\partial t$ equation given the V component of the convection solution. To obtain a solution to the full *hydromagnetic dynamo problem*, that is, to find a magnetic dynamo in the coupled nonlinear PDE (1.1), it is necessary to show that the magnetic instability is saturated by the Lorentz force (the so-called *dynamo* effect [17]).

In this paper, we show how techniques from equivariant bifurcation theory and dynamical systems can be used to turn a solution to the kinematic dynamo problem into a full-fledged solution to the hydromagnetic dynamo problem. We concentrate on rotating convection. In nonrotating convection ($T = 0$) the pure conduction solution typically gives rise to a bifurcation of rolls solutions and the desired secondary magnetic instability is ruled out by the antidynamo theorems. However, it is widely believed since Childress and Soward [4, 18] that rolls solutions in rotating convection ($T \neq 0$) are sufficiently complicated to undergo a magnetic instability. The required linear stability calculations turn out to be highly nontrivial and have led to many spurious solutions. Recent calculations of Matthews [14] appear to overcome these difficulties.

Assuming that solutions to the kinematic dynamo problem in rotating convection exist, the nature of the corresponding solutions to the hydromagnetic dynamo problem depends on two main factors:

- The primary bifurcation may lead to stable rolls or unstable rolls.
- The secondary magnetic instability of convection rolls may occur via a steady-state bifurcation or a Hopf bifurcation.

The stability of the primary branch of rolls in rotating convection was studied by Küppers and Lortz [10]. Rolls are stable provided T is small enough, but once T exceeds some critical value, rolls are unstable to convective perturbations consisting of rolls oriented at angle close to 58°. As shown by Busse and Heikes [2, 9], there is a homoclinic cycle corresponding to an intermittency phenomenon observed in fluid experiments where, locally, systems of rolls form but are then replaced as time goes on by new systems of rolls inclined at an angle of roughly 60° to the old rolls.

The existence of the Busse-Heikes cycle can be made completely rigorous by performing a reduction to a six-dimensional center manifold consisting of solutions that are spatially-periodic with respect to a hexagonal lattice. There is a three-dimensional flow-invariant submanifold [1] that contains the three sets of rolls. The homoclinic cycle connecting the three sets of rolls is robust (due to the symmetry) and is asymptotically stable to perturbations preserving the hexagonal lattice. For mathematical details of the analysis in the three-dimensional subspace, see [8]. We refer also to [15, 6, 12]

There are thus four cases that we analyze in this paper:

(i) Steady-state bifurcation from stable rolls,
(ii) Hopf bifurcation from stable rolls,
(iii) Steady-state bifurcation from the Busse-Heikes cycle,
(iv) Hopf bifurcation from the Busse-Heikes cycle.

Cases (i) and (ii) reduce to bifurcations analyzed in [7], whereas cases (iii) and (iv) are *transverse bifurcations of homoclinic cycles* [5].

The remainder of this paper is organized as follows. In Section 2, we describe the symmetries of equations (1.1). In Section 3, we recall the primary, purely convective, bifurcation to rolls and we describe the Busse-Heikes cycle. The four secondary bifurcations (i)–(iv) are analyzed in Section 4 and conclusions for the magnetic dynamo problem are presented in Section 5.

2 The Symmetries of the Problem

Equation (1.1) is equivariant under the *special Euclidean group* $\mathbf{SE}(2)$ of rotations and translations in the plane. The action is effectively given by translation and rotation of the unbounded domain variables $x \in \mathbb{R}^2$. In addition, there is an up-down symmetry τ which transforms z to $1-z$ and a field-reversal symmetry ρ which transforms B to $-B$. The up-down symmetry is a consequence of the identical boundary conditions at $z=0$ and $z=1$.

Altogether, we have an action of the group $\mathbf{SE}(2)\times\mathbb{Z}_2(\tau)\times\mathbb{Z}_2(\rho)$ on functions $(V,\theta,B):\mathbb{R}^2\times[0,1]\to\mathbb{R}^7$.

Reduction to a compact group

There are technical problems in dealing with noncompact symmetry groups such as the one just described. A standard method for avoiding these difficulties is to restrict to a class of functions that is doubly spatially-periodic in the plane [7]. For our purposes, it is sufficient to restrict to the *hexagonal lattice*.

Define the subgroup of translations $\mathcal{L}\subset\mathbf{SE}(2)$ generated by translations

$$\ell_1 = c(0,1), \qquad \ell_2 = c(\tfrac{\sqrt{3}}{2},\tfrac{1}{2}),$$

where $c>0$ is a unit of length to be chosen later. Then

$$\mathcal{L} = \{m_1\ell_1 + m_2\ell_2 : m_1, m_2 \in \mathbb{Z}\} \cong \mathbb{Z}^2$$

is a discrete subgroup of the group of translations. Consider the space $\mathrm{Fix}\,\mathcal{L}$ of functions that are spatially periodic with respect to $\mathcal{L}$, that is, $V(x+\ell, z) = V(x, z)$ for all $\ell \in \mathcal{L}$ and similarly for θ and B. The equivariance of equations (1.1) implies that the space $\mathrm{Fix}\,\mathcal{L}$ is invariant under the semiflow induced by the PDE.

The *holohedry* of $\mathcal{L}$ is the subgroup $\mathbb{Z}_6 \subset \mathbf{SO}(2)$ of rotations that preserve the hexagonal lattice $\mathcal{L}$. This is the largest subgroup of $\mathbf{SO}(2)$ that preserves $\mathrm{Fix}\,\mathcal{L}$. Thus the residual group of symmetries in $\mathbf{SE}(2)$ is a semidirect product $\mathbb{Z}_6 \dot{+} \mathbb{R}^2$ of certain rotations together with all translations. We quotient out the translations in $\mathcal{L}$ (which act trivially on $\mathrm{Fix}\,\mathcal{L}$) and obtain a two-torus of translations $\mathbf{T}^2 \cong \mathbb{R}^2/\mathcal{L}$. This results in a compact symmetry group Γ acting on $\mathrm{Fix}\,\mathcal{L}$:

$$\Gamma = (\mathbb{Z}_6 \dot{+} \mathbf{T}^2) \times \mathbb{Z}_2(\tau) \times \mathbb{Z}_2(\rho).$$

Equation (1.1) defines a Γ-equivariant dynamical system on $\mathrm{Fix}\,\mathcal{L}$.

We note that $\mathrm{Fix}\,\mathcal{L}$ consists of functions (V, θ, B) with Fourier expansions

$$V(x, z) = \sum V_k(z) e^{ik\cdot x},$$

where the sum is over all vectors $k \in \mathbb{R}^2$ such that $k \cdot \ell \equiv 0 \bmod 2\pi$, and similarly for θ and B. Equivalently, $k \in \mathcal{L}^*$ where $\mathcal{L}^*$ is the *dual lattice* generated by

$$\tfrac{4\pi}{\sqrt{3}c}(1, 0), \qquad \tfrac{4\pi}{\sqrt{3}c}(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}).$$

3 Primary Bifurcation to Convection Rolls

As discussed in the introduction, the primary bifurcation from the trivial (pure conduction) solution $(V, \theta, B) = (0, 0, 0)$ takes place inside the flow-invariant subspace $\{B = 0\}$ and leads to pure convection solutions $(V, \theta, 0)$. Hence, we may restrict to the Boussinesq equations (equation (1.1) with $B = 0$). We review the results for linear stability of the trivial solution and the bifurcations to convection rolls and to the Busse-Heikes cycle.

Linear stability of the trivial solution

If all eigenvalues of the linearization of the Boussinesq equations around the pure conduction solution $(V, \theta) = (0, 0)$ have negative real part, then this solution is asymptotically stable. As the Rayleigh number R is increased, the pure conduction state may lose stability as certain eigenvalues cross the imaginary axis.

For fixed values of $\mathrm{Pr} > 1$ and T, the initial loss of stability occurs via a *steady-state* bifurcation as the parameter R is increased [3]. That is, an eigenvalue of the linear terms passes through zero. At criticality, the (complexified) kernel W_0 of the linearized PDE is spanned by eigenfunctions of the form $d(z)e^{ik\cdot x}$, where the wave vectors k satisfy $|k| = k_c$ for some fixed $k_c > 0$ (the *critical wavenumber*). Since $k \in \mathcal{L}^*$, the dimension of W_0 is finite and depends on the unit of length c that was introduced in the definition of $\mathcal{L}$. For the choice $c = 4\pi/\sqrt{3}k_c$, we have

$\dim W_0 = 6$ and a basis for W_0 is obtained by applying rotations in the holohedry $\mathbb{Z}_6$ to the eigenfunction

$$e_0(x,z) = d_0(z)e^{ik_cx_1} + \bar{d}_0(z)e^{ik_cx_1}. \tag{3.1}$$

A computation using the boundary conditions (1.2) shows that d_0 has the form

$$d_0(z) = (A\cos\pi z, B\cos\pi z, C\sin\pi z, D\sin\pi z), \tag{3.2}$$

for some constants $A, B, C, D \in \mathbb{C}$. (In general, the form of d_0 depends on the choice of boundary conditions.)

Bifurcation of rolls

To discuss the bifurcation to rolls, it is useful to work in the abstract framework of [7]. The six-dimensional kernel is given by

$$W_0 = \{w_1e_0 + w_2R_{2\pi/3}\cdot e_0 + w_3R_{4\pi/3}\cdot e_0 : w_1, w_2, w_3 \in \mathbb{C}\} \cong \mathbb{C}^3,$$

where $R_\theta \in \mathbf{SO}(2)$ denotes rotation through angle θ. The center manifold theorem reduces the underlying PDE (1.1) to a Γ-equivariant six-dimensional ordinary differential equation on W_0 while preserving the local dynamics (near the pure conduction solution) close to criticality ($R = R_c$).

The action of Γ on functions (V, θ, B) induces on W_0 the action

$$\begin{aligned} \varphi = (\varphi_1, \varphi_2) \in \mathbf{T}^2 &: (w_1, w_2, w_3) \mapsto (e^{i\varphi_1}w_1, e^{i\varphi_2}w_2, e^{-i(\varphi_1+\varphi_2)}w_3) \\ R_{\pi/3} \in \mathbb{Z}_6 &: (w_1, w_2, w_3) \mapsto (\overline{w}_2, \overline{w}_3, \overline{w}_1) \\ \tau : w \mapsto -w, &\qquad \rho : w \mapsto w\ . \end{aligned}$$

Remark 3.1 Note that the action of τ follows from the form (3.2) of the eigenfunctions, whereas the remainder of the action of Γ follows from the general structure (3.1) and hence is independent of the choice of boundary conditions. For general boundary conditions that are identical on top and bottom, τ can act as plus or minus the identity on W_0. However, in practice only the minus identity action seems to be realized, cf [6]. (Of course, if there are different boundary conditions on top and bottom, then there is no up-down symmetry τ and $\Gamma = (\mathbb{Z}_6\dot{+}\mathbf{T}^2)\times\mathbb{Z}_2(\rho)$.)

Points of the form $(\alpha_1, 0, 0) \in W_0$, $\alpha_1 \in \mathbb{R}$, correspond to rolls

$$\xi_1 = \alpha_1\big(d_0(z)e^{ik_cx_1} + \bar{d}_0(z)e^{-ik_cx_1}\big).$$

The isotropy subgroup $\Sigma_{\text{rolls}} = \Sigma_{\xi_1} = \{\gamma \in \Gamma : \gamma\xi_1 = \xi_1\}$ is given by

$$\Sigma_{\text{rolls}} = (\mathbb{Z}_2(R_\pi)\dot{+}\mathbf{T}^1) \times \mathbb{Z}_2(\tilde{\tau}) \times \mathbb{Z}_2(\rho),$$

where $\mathbf{T}^1 \subset \mathbf{T}^2$ consists of translations $\varphi = (0, \varphi_2)$ parallel to the x_2-axis and $\tilde{\tau} = \tau\circ(\pi,\pi)$ is the glide reflection corresponding to the up-down symmetry τ coupled with the translation $\varphi = (\pi,\pi)$ by half a period.

Observe that $\dim(\mathrm{Fix}\,\Sigma_{\mathrm{rolls}} \cap W_0) = 1$. It follows from general theory [7] that there is generically a pitchfork bifurcation to a branch of rolls solutions. The branching equation has the form

$$0 = \lambda\alpha_1 + b\alpha_1^3 + \text{high order terms},$$

where $\lambda = \sqrt{R} - \sqrt{R_c}$ and $b \in \mathbb{R}$ is a constant that can be computed via the center manifold reduction. For the boundary conditions (1.2) (and quite generally for reasonable boundary conditions) it turns out that $b < 0$ and hence we obtain the supercritical branch of solutions

$$\xi_1 = \sqrt{\tfrac{-\lambda}{b}}\big(d_0(z)e^{ik_c x_1} + \bar{d}_0(z)e^{-ik_c x_1}\big) + O(\lambda^{3/2}).$$

By equivariance, there is a group orbit $\Gamma\xi_1$ of rolls solutions — three different orientations compatible with the hexagonal lattice together with translates. The three orientations correspond to the subspaces $\{(w_1, 0, 0)\}$, $\{(0, w_2, 0)\}$ and $\{(0, 0, w_3)\}$. In the remainder of the paper, we shall let ξ_1 refer both to the particular roll solution constructed above and also to the continuous group orbit of translates. Similarly, the notation $\xi_2 = R_{2\pi/3}\xi_1$ and $\xi_3 = R_{4\pi/3}\xi_1$ will denote both a particular rolls solution and also the continuous group orbit of translates.

The Busse-Heikes cycle

The rolls solutions ξ_1, ξ_2 and ξ_3 are related by the rotations in $\mathbb{Z}_6$ and hence are mutually oriented at $60°$. When the Taylor number is large enough, we have the Küppers-Lortz instability whereby the supercritical rolls are saddles in W_0 with one neutral eigenvalue, three stable eigenvalues (including the branching direction) and two unstable eigenvalues.

We introduce the subgroup $\Sigma_{\mathrm{BH}} \cong \mathbb{Z}_2^3$ generated by R_π, $\tilde{\tau}$ and ρ. Observe that $\mathrm{Fix}\,\Sigma_{\mathrm{BH}} \cap W_0 = \{(\alpha_1, \alpha_2, 0)\}$ is a two-dimensional flow-invariant subspace. A calculation based on the Poincaré-Bendixson Theorem shows that there is a saddle-sink connection in $\{(\alpha_1, \alpha_2, 0)\}$ joining ξ_1 to ξ_2. By symmetry, there are connections joining ξ_2 to ξ_3 and ξ_3 to ξ_1. The resulting homoclinic cycle is robust to perturbations that preserve the symmetries. In addition, the homoclinic cycle is asymptotically stable to perturbations that preserve the hexagonal lattice. Again we refer to [8, 6, 12] for details.

4 Secondary Bifurcation from Convection Rolls

We have seen that the pure conduction state loses stability to a supercritical branch of convection rolls. For rolls of large enough amplitude, there may be a secondary instability to magnetic perturbations as predicted in [4].

We assume that such a magnetic instability occurs for suitable choices of the parameters in (1.1). Let Y_0 denote the critical eigenspace of rolls in the $\partial B/\partial t$ component of (1.1). The isotropy subgroup Σ_{rolls} acts on Y_0 and the dynamics associated with the bifurcation depends on the action of Σ_{rolls}. There is some *a priori* information that we can deduce. First, the field-reversal symmetry ρ acts as $-I$ on Y_0. Second, the translations $\mathbf{T}^1$ acts nontrivially on Y_0. Otherwise, the rolls ξ_1 would be unstable to magnetic perturbations with no x_2-dependence and this is ruled out by Cowling's antidynamo theorem [17, p. 465].

(i) Steady-state bifurcation from stable rolls

Suppose that we have a primary bifurcation of asymptotically stable convection rolls, and that these rolls subsequently undergo a secondary magnetic instability. Suppose further that this is a steady-state bifurcation, so eigenvalues of the linearized PDE around rolls pass through zero. It follows from [7] that generically Σ_{rolls} acts irreducibly on the kernel Y_0. (This genericity property has to be checked for the particular PDE (1.1) as part of the linear stability calculation.) Now $\Sigma_{\text{rolls}} = (\mathbb{Z}_2(R_\pi)\dot{+}\mathbf{T}^1) \times \mathbb{Z}_2(\tilde{\tau}) \times \mathbb{Z}_2(\rho) \cong \mathbf{O}(2) \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and it follows that Y_0 is an irreducible representation of $\mathbf{O}(2)$. Since $\mathbf{T}^1 \subset \mathbf{O}(2)$ acts nontrivially, we deduce that $Y_0 \cong \mathbb{C}$ is two-dimensional and that coordinates $y \in Y_0$ can be chosen so that the action of $\mathbb{Z}_2(R_\pi)\dot{+}\mathbf{T}^1$ is given by

$$R_\pi \in \mathbb{Z}_2 : y \mapsto \bar{y}, \qquad (0, \varphi_2) \in \mathbf{T}^1 : y \mapsto e^{in\varphi_2} y, \tag{4.1}$$

for some positive integer n. In addition, we have

$$\tilde{\tau} y = \pm y, \qquad \rho y = -y.$$

Without loss of generality, we may suppose that $\tilde{\tau} y = y$ (redefining $\tilde{\tau}$ to be $\tilde{\tau}\circ\rho$ if necessary — this does not change the action of $\tilde{\tau}$ on the purely convective subspace $\{B = 0\}$, for example we still have that the subgroup Σ_{BH} defined in Section 3 is generated by R_π, $\tilde{\tau}$ and ρ).

It follows again from standard theory [7] that there is a pitchfork bifurcation from equilibrium convection rolls ξ_1 to equilibrium dynamos ξ_1' that have a large (V, θ)-component with symmetry Σ_{rolls} together with a small B-component with Σ_{bif}-symmetry. Here, Σ_{bif} is the proper subgroup of Σ_{rolls} generated by R_π together with the kernel K of the action of Σ_{rolls} on Y_0. We compute that $K = \mathbb{Z}_{2n} \times \mathbb{Z}_2(\tilde{\tau})$ where $\mathbb{Z}_{2n} \subset \mathbf{T}^1 \times \mathbb{Z}_2(\rho)$ is generated by $\tilde{\rho} = \rho\circ(0, \pi/n)$. Hence

$$\Sigma_{\text{bif}} = \mathbb{D}_{2n} \times \mathbb{Z}_2(\tilde{\tau}),$$

where $\mathbb{D}_{2n}$ denotes the dihedral group of order $4n$ generated by R_π and $\tilde{\rho}$. In particular, the dynamo solutions have discrete symmetry and hence depend nontrivially on all spatial variables in accordance with the antidynamo theorems.

Again, it follows from equivariance that there is a group orbit of equilibrium dynamos ξ_1' obtained by translating in the plane and reversing the direction of the magnetic field. In principle, there could be nontrivial drift along this group orbit. However, a calculation using results of [11] shows that the symmetry $R_\pi \in \Sigma_{\text{bif}}$ obstructs any drift. Hence, the bifurcating dynamos ξ_1' are genuine equilibria.

Finally, we note that the usual exchange of stability holds in this context. Since the convection rolls are asymptotically stable, the dynamos are (orbitally) asymptotically stable if and only if the bifurcation is supercritical.

(ii) Hopf bifurcation from stable rolls

The standard theory here is Hopf bifurcation with $\Sigma_{\rm rolls}$ symmetry which is analogous to Hopf bifurcation with $\mathbf{O}(2)$ symmetry [7]. There are pitchfork bifurcations to two families of periodic magnetic dynamos: traveling waves and standing waves. If both of these branches of dynamos bifurcate supercritically, then precisely one of the branches is asymptotically stable (which one depends on a coefficient which can be computed from the PDE). Otherwise, both dynamos are unstable. The standing wave has spatial symmetries $\Sigma_{\rm bif}$ together with a space-time symmetry where shifting by half a period in time is the same as reversing the direction of the magnetic field. The traveling wave breaks the R_π rotation symmetry in $\Sigma_{\rm bif}$ and (neglecting drift) time evolution corresponds to translation with uniform speed parallel to the x_2-axis.

Again, there is the possibility of nontrivial drift along the group orbits of periodic solutions. This possibility is realized for the traveling wave (but not the standing wave). Indeed, the traveling wave solutions are quasiperiodic relative to the hexagonal lattice with generically two independent frequencies: a fast frequency from the Hopf bifurcation together with a slow drift frequency. Taking this drift into account, we conclude that time evolution of the traveling wave corresponds to translation with uniform speed almost parallel to the x_2-axis.

(iii) Steady-state bifurcation from the Busse-Heikes cycle

We now consider the case where the primary instability of the pure conduction solution leads to a branch of asymptotically stable homoclinic cycles with trajectories connecting rolls solutions. Suppose as in case (i) that the rolls undergo a secondary magnetic instability in the form of a steady-state bifurcation. An analysis in the neighborhood of a single rolls solution leads to a pitchfork bifurcation of $\Sigma_{\rm bif}$-symmetric equilibrium dynamos as before, except that now the bifurcating equilibria are automatically unstable. Hence, it is necessary to analyze the bifurcation in a neighborhood of the full homoclinic cycle as in [5].

Our analysis follows Type B in [5] but there are added complications. In particular, our results depend on the positive integer n introduced in (4.1). Let S denote the flow-invariant subspace $S = \operatorname{Fix}\mathbb{Z}_2^2(R_\pi,\tilde\tau)$. (This subspace corresponds to the subspace Q' in [5, Proof of Theorem 3.2].)

Proposition 4.1 *The convection rolls ξ_1, ξ_2 (and the heteroclinic connection from ξ_1 to ξ_2) lie in S. In addition, the bifurcating equilibrium dynamo ξ_1' lies in S. However, ξ_2' lies in S if and only if n is even.*

Proof Observe that $\mathbb{Z}_2^2(R_\pi,\tilde\tau) = \Sigma_{\rm bif}\cap\Sigma_{\rm BH}$ and hence it is immediate that ξ_1, ξ_2, the heteroclinic connection between ξ_1 and ξ_2, and ξ_1' lie in S.

It remains to determine whether $\xi_2'\in S$. Clearly, R_π fixes ξ_2', so $\xi_2'\in S$ if and only if $\tilde\tau\in\Sigma_{\xi_2'} = R_{2\pi/3}\Sigma_{\rm bif}R_{2\pi/3}^{-1}$. That is, $R_{2\pi/3}^{-1}\tilde\tau R_{2\pi/3}\in\Sigma_{\rm bif}$.

A computation shows that $R^{-1}_{2\pi/3}(\pi,\pi)R_{2\pi/3} = (\pi,0)$. (An easy way to check this is to use the (faithful) representation of $(\mathbb{Z}_6\dot{+}\mathbf{T}^2)$ on W_0.) It follows that $R^{-1}_{2\pi/3}\tilde{\tau}R_{2\pi/3} = \tilde{\tau}\circ(0,\pi)$. (This is true regardless of whether $\tilde{\tau} = \tau\circ(\pi,\pi)$ or $\tilde{\tau} = \tau\circ(\pi,\pi)\circ\rho$). But $\tilde{\tau} \in \Sigma_{\mathrm{bif}}$ so it remains to determine whether $(0,\pi) \in \Sigma_{\mathrm{bif}}$. Now $\tilde{\rho} = \rho\circ(0,\pi/n) \in \Sigma_{\mathrm{bif}}$ so that $\rho^n\circ(0,\pi) \in \Sigma_{\mathrm{bif}}$. Since $\rho \notin \Sigma_{\mathrm{bif}}$ we deduce that $(0,\pi) \in \Sigma_{\mathrm{bif}}$ if and only if n is even. ∎

Our analysis now splits into the two cases n even and n odd.

Theorem 4.2 (The case n even) *Suppose that the Busse-Heikes cycle is asymptotically stable and that the rolls solutions ξ_j undergo pitchfork bifurcations to equilibrium dynamos ξ'_j. If the action of Σ_{rolls} on the critical eigenspace Y_0 is given by* (4.1) *with n even, then generically there is a pitchfork bifurcation from the Busse-Heikes cycle to a homoclinic cycle connecting the equilibrium dynamos. This magnetic cycle is asymptotically stable if and only if the pitchfork bifurcation is supercritical.*

Proof The proof is completely analogous to the proof of [5, Theorem 3.2]. (We note that the extension from four dimensions to infinite dimensions is immediate in this case.) We consider the case when the bifurcation is supercritical; the subcritical case is similar.

First, we restrict attention to the invariant subspace S. Before the bifurcation, there is a saddle-sink connection in S from ξ_1 to ξ_2. After the bifurcation, ξ'_1 is a saddle with a one-dimensional unstable manifold in S and ξ'_2 is a sink. (Of course, there is a pair of equilibria in S corresponding to ξ'_1 and similarly for ξ'_2.)

In a neighborhood of ξ_2, we can apply the center manifold theorem. On the center manifold, all trajectories except for ξ_2 are asymptotic to ξ'_2. Moreover, there is a neighborhood $U \subset S$ of ξ_2 so that trajectories in U are attracted to the center manifold. (Note that the remaining unstable directions associated with ξ_2 do not lie in S.) Hence, almost every trajectory in U is asymptotic to ξ'_2.

Before criticality, trajectories starting near ξ_1 are asymptotic to ξ_2. By continuity, after criticality the unstable manifold of ξ'_1 intersects U and hence is generically asymptotic to ξ'_2. Hence we have the desired saddle-sink connection from ξ'_1 to ξ'_2. By symmetry, we obtain connections also from ξ'_2 to ξ'_3 and from ξ'_3 to ξ'_1, resulting in a robust homoclinic cycle connecting the equilibrium dynamos.

Finally, the stability of the Busse-Heikes cycle and of the new cycle is governed by the ratio of the contracting and expanding eigenvalues at ξ_1 and ξ'_1 respectively. It follows easily as in [5] that the bifurcating cycle is asymptotically stable supercritically. ∎

Schematically, we have a pitchfork bifurcation from the Busse-Heikes cycle

$$\xi_1 \longrightarrow \xi_2 \longrightarrow \xi_3 \longrightarrow \xi_1$$

to the magnetic cycle

$$\xi'_1 \longrightarrow \xi'_2 \longrightarrow \xi'_3 \longrightarrow \xi'_1\ . \tag{4.2}$$

Theorem 4.3 (The case n odd) *Suppose that the Busse-Heikes cycle is asymptotically stable and that the rolls solutions ξ_j undergo pitchfork bifurcations to equilibrium dynamos ξ_j'. If the action of Σ_{rolls} on the critical eigenspace Y_0 is given by* (4.1) *with n odd, then generically there is a bifurcation from the Busse-Heikes cycle to a heteroclinic network connecting the rolls and the equilibrium dynamos. This magnetic network is asymptotically stable if and only if the pitchfork bifurcation is supercritical.*

Proof The difference from the case n even is that ξ_2' does not lie in S. Hence ξ_2 remains a sink in S after criticality and we obtain a saddle-sink connection from ξ_1' to ξ_2. Of course, the saddle-sink connection from ξ_1 to ξ_2 persists and in addition there is a connection in the center manifold from ξ_1 to ξ_1'. By symmetry, we have a heteroclinic network connecting the ξ_j and the ξ_j'.

Sufficient conditions for asymptotic stability of heteroclinic cycles are given in [13, Theorem 2.7] and such arguments generalize easily to heteroclinic networks. Moreover, the sufficient conditions for the network near the transverse bifurcation are a small perturbation of the sufficient conditions for the Busse-Heikes cycle. In the case of the Busse-Heikes cycle, the sufficient conditions of [13] are also necessary [13, Theorem 3.1] (see also [12]). Hence the assumption that the Busse-Heikes cycle is stable before the transverse bifurcation guarantees that the sufficient conditions for the cycle and hence the network are valid. ∎

Schematically, we have the heteroclinic network

$$\begin{tikzcd}
\xi_1 \arrow[rr] \arrow[dr] & & \xi_2 \arrow[rr] \arrow[dr] & & \xi_3 \arrow[rr] \arrow[dr] & & \xi_1 \; . \\
& \xi_1' \arrow[ur] & & \xi_2' \arrow[ur] & & \xi_3' \arrow[ur] &
\end{tikzcd}$$

The network is an indecomposable asymptotically stable attractor (indecomposable in the sense that no proper flow-invariant subset is asymptotically stable).

(iv) Hopf bifurcation from the Busse-Heikes cycle

This case is analogous to case (iii), the differences being almost identical to the differences between cases (i) and (ii). For simplicity, we suppose that both the traveling waves and the standing waves bifurcate supercritically.

The analogue of Theorem 4.2 is that there is a pitchfork bifurcation to an asymptotically stable magnetic homoclinic cycle, of the form shown schematically in (4.2). The cycle connects either three sets of traveling waves or three sets of standing waves (depending as in (ii) on a coefficient computed from the PDE).

Next, we describe the analogue of Theorem 4.3. There is a bifurcation to a single heteroclinic network containing equilibrium convection rolls, magnetic traveling waves and magnetic standing waves. On the center manifold in a neighborhood of ξ_j, there is a saddle-sink connection between the traveling waves ξ_j^{TW} and the standing waves ξ_j^{SW} (the direction of this connection depending again on the coefficient computed from the PDE). Schematically, we have

$$\begin{tikzcd} \xi_1 \arrow[rr] \arrow[dr] & & \xi_2 \arrow[rr] \arrow[dr] & & \xi_3 \arrow[rr] \arrow[dr] & & \xi_1 \\ & \xi_1^* \arrow[ur] & & \xi_2^* \arrow[ur] & & \xi_3^* \arrow[ur] & \end{tikzcd}$$

where the notation ξ_j^* stands for either $\xi_j^{TW} \longrightarrow \xi_j^{SW}$ or $\xi_j^{TW} \longleftarrow \xi_j^{SW}$. As in (iii), the heteroclinic network is an indecomposable asymptotically stable attractor.

5 Conclusions

In this paper, we have taken the proposed kinematic dynamo solution of Childress and Soward [4] and deduced some of the implications for the full hydromagnetic dynamo problem. In principle, our results hold rigorously for the PDE (1.1) once the instability of [4] is verified. Following such a verification, it is a relatively straightforward calculation to determine which of the cases (i) to (iv) occur and to determine directions of branching. For ease of exposition, we assume from now on that all bifurcations are supercritical.

Subject to these provisos, and subject also to our restriction to the class of functions that are spatially periodic with respect to the hexagonal lattice, we have the following conclusions for bifurcation of magnetic dynamos from stable rolls:

(i) Pitchfork bifurcation of attracting equilibrium magnetic dynamos.

(ii) Pitchfork bifurcation of periodic and traveling magnetic dynamos. The periodic solutions are standing waves and the magnetic field reverses direction each half-period. The traveling solutions are quasiperiodic relative to the hexagonal lattice and time evolution corresponds to translation almost parallel to the axis of the rolls solution. Precisely one of these solutions is an attracting dynamo.

The conclusions for bifurcation from the Busse-Heikes cycle (cases (iii) and (iv)) depend on the symmetry that is broken in the bifurcation from convection rolls to equilibrium dynamos. Recall that the convection rolls ξ_1 are invariant under translation parallel to the x_2-axis. Most of this symmetry is broken in the bifurcation to equilibrium dynamos, but there remains at least the periodicity due to the translations in the hexagonal lattice $\mathcal{L}$. Additional discrete translation symmetry parallel to the x_2-axis is possible depending on the integer n in (4.1). In particular, there is a half-period translation symmetry parallel to the x_2-axis if and only if n is even.

When the equilibrium dynamos have the half-period translation symmetry parallel to the x_2-axis, we obtain pitchfork bifurcations from the purely convective Busse-Heikes cycle to magnetic homoclinic cycles. These are attracting intermittent magnetic dynamos connecting equilibria in case (iii) and magnetic traveling/standing waves in case (iv).

When the equilibrium dynamos break the half-period translation symmetry parallel to the x_2-axis, bifurcation leads to heteroclinic networks consisting of convection rolls together with equilibrium dynamos in case (iii) and together with traveling/standing waves in case (iv). In Section 4, we showed that these networks are indecomposable asymptotically stable magnetic dynamos. Moreover, the Busse-Heikes cycle, which is the pure convection part of the network, is not asymptotically stable after the secondary bifurcation.

We have not attempted a detailed analysis of the dynamics on these heteroclinic networks. One reason is that there appear to be many distinct situations to consider. However, preliminary investigations suggest that our description above, in terms of asymptotic stability, is somewhat simplistic. There is at least one situation (we omit the details) when the Busse-Heikes cycle X is 'essentially asymptotically stable' [16] — most trajectories in a neighborhood of X are attracted to X. It is thus conceivable that, for realistic definitions of attractor, there is no attracting dynamo in some cases.

Actually, there is a further complication since we should take into account the effects of forced symmetry breaking, where some or all of the symmetry is broken slightly. It is likely (though difficult to prove) that after such a perturbation, there are genuine (though very weak) magnetic dynamos.

Finally, we emphasize that these complications apply only to some subcases of cases (iii) and (iv). In cases (i) and (ii), and in the 'half-period symmetric' subcases of (iii) and (iv), we obtain straightforward pitchfork bifurcations to equilibrium, periodic, traveling and intermittent attracting magnetic dynamos.

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