Dynamics on unbounded domains; co-solutions and inheritance of stability

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Abstract

We consider the dynamics of semiflows of patterns on unbounded domains that are equivariant under a noncompact group action. We exploit the unbounded nature of the domain in a setting where there is a strong 'global' norm and a weak 'local' norm. Relative equilibria whose group orbits are closed manifolds for a compact group action need not be closed in a noncompact setting; the closure of a group orbit of a solution can contain 'co-solutions'.

The main result of the paper is to show that co-solutions inherit stability in the sense that co-solutions of a Lyapunov stable pattern are also stable (but in a weaker sense). This means that the existence of a single group orbit of stable relative equilibria may force the existence of quite distinct group orbits of relative equilibria, and these are also stable. This is in contrast to the case for finite dimensional dynamical systems where group orbits of relative equilibria are typically isolated.

1 Introduction

There has been much effort devoted to trying to understand the dynamics of spatially extended systems; that is, dynamical systems that have not just unbounded time but also unbounded space dependence. Most of this work has progressed by restricting to parabolic partial differential equations such as the Ginzburg-Landau and Swift-Hohenberg equations (see for example [9, 5]) used to model 'generic' instabilities with nontrivial spatial dependence. This has been successful in characterising solutions of specific types in specific systems of equations; for example wave-like solutions, fronts between them [3], spirals and defects. Related to this approach there have been attempts to find a 'qualitative theory' of partial differential equations (see for example [4]) where careful reductions to ODE models can explain many universal features of patterns in unbounded systems, for example the stability of fronts and defects [11].

There remain fundamental problems in trying to characterise what sort of attractors 'generically' appear in initial value problems for partial differential equations with unbounded domains. Related to this is the usual problem of deciding which space of functions and which topology or norm is appropriate. As is well recognised, a change of choice of norm can lead to qualitatively very different behaviour [10, 9]. For example, consider a solution $u(x,t)$ to the heat equation $u_t = u_{xx}$ with initial condition $u_0(x) \geq 0$ satisfying $\int |u_0(x)|\, dx < \infty$. Then $u(x,t)$ decays to zero in sufficiently 'weak' norms; for example a weighted L^1 norm $\|u\| = \int \rho(x)|u(x,t)|\, dx$ with weight $\rho > 0$ such that $\rho(x) \to 0$ as $|x| \to \infty$. By contrast in a more 'global' norm such as the L^1 norm, the solution remains bounded away from zero.

Rather than being discouraged by what might be seen as the arbitrary nature of the choice of topology, in this paper we wish to use it to our advantage. In an attempt to move away from specific systems of equations we consider a general setting where the dynamics is given by a semiflow on a space of patterns. A key assumption is that there are two important topologies; a weak one which characterises local changes and a strong one which characterises global changes. We assume that the semiflow is continuous for both of the topologies and note that this is typical in many systems.

If there is a stable propagating front between states A and B it seems reasonable to ask whether A and B inherit the stability of the front. We describe a setting in which one can make such deductions and extend them to understand the stability in general for 'far field' stable patterns.

In many models for dynamics on unbounded domains there are translational or Euclidean group symmetries in the model. We make an assumption of this form to allow us to discuss a range of different solutions. We assume that the semiflow commutes with the action of a noncompact group Γ, meaning that we can characterise the unboundedness by group symmetries.

Section 2 gives the details of the setting and some motivating examples. Section 3 shows that under quite general assumptions on the semiflow, solutions often force the existence of a variety of new solutions that we call *co-solutions*. These co-solutions are patterns that are in the closure of the group orbit in the weak topology. In Proposition 3.3 we give conditions such that if the original solution is a relative equilibrium then the co-solution is also a relative equilibrium. The main result in Section 4 shows that the stability of a solution implies stability of its co-solutions. Section 5 discusses the results and suggests some further questions that may be usefully addressed using this approach.

2 Semiflows with noncompact domain symmetries

We consider the behaviour of patterns that evolve on an infinite domain for a semiflow equivariant under a group Γ acting on this domain. By a space of *patterns* we mean a vector space $\mathcal{B}$ of functions $u : D \to \mathbb{R}^s$ where for example $D = \mathbb{R}^n$. We suppose that Γ is a (noncompact) Lie group acting on $\mathcal{B}$ and $L\Gamma$ is the Lie algebra of Γ. Our main applications have $D = \mathbb{R}^n$ and $\Gamma = \mathbf{E}(n)$ or $\mathbf{SE}(n)$.

2.1 The strong and weak topologies

We consider semiflows on $\mathcal{B}$ subject to two topologies that express closeness in a weak (local) or in a strong (global) senses. More precisely, we assume:

(H1) (a) The strong norm $\|.\|_s$ is Γ-invariant, i.e. $\|\gamma u\|_s = \|u\|_s$ for all $u \in \mathcal{B}$ and $\gamma \in \Gamma$.

(b) For each fixed $\gamma \in \Gamma$, the linear map $u \mapsto \gamma u$ is bounded in the weak norm. (So $\|\gamma u\|_w \leq K\|u\|_w$ for all $u \in \mathcal{B}$, where $K = \|\gamma\|_w < \infty$ is the operator norm of γ.[1])

(c) The map $\gamma \to \|\gamma\|_w$ is continuous.

(H2) There is a family of functions $\Lambda = \{\lambda\}$ of functions $\lambda : D \to (0, \infty)$ such that

(a) $\|u\|_w \leq \|\lambda u\|_s \leq \|u\|_s$, and

(b) For all $M > 0$ and $\epsilon > 0$, there exists $\lambda \in \Lambda$ such that

$$\|u\|_s \leq M \text{ implies that } \|(1-\lambda)u\|_w < \epsilon.$$

(H3) For all $u \in \mathcal{B}$, $v \in \mathrm{clos}_w(\Gamma u)$, $\lambda \in \Lambda$ and $\epsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$\|\lambda(\gamma u - v)\|_s < \epsilon.$$

We give an example of a setting of weighted norms where these hypotheses can be verified in Section 2.3 below. In our example, $\mathcal{B}$ is a Banach space under the strong norm but not the weak norm. This is the typical situation for the applications we have in mind, but completeness is not used in this paper.

2.2 Dynamics on $\mathcal{B}$

Suppose that we have an evolution given by a semiflow $\Phi_t : \mathcal{B} \to \mathcal{B}$ such that

$$\Phi_t \circ \Phi_s = \Phi_{s+t}, \quad \Phi_0 = \mathrm{Id}.$$

We suppose that Φ_t is continuous in both the weak norm $\|.\|_w$ and the strong norm $\|.\|_s$ on the function space $\mathcal{B}$. We suppose also that Φ_t is Γ-equivariant.

[1] $\|\gamma\|_w = \sup_{u\in\mathcal{B}} \|\gamma u\|_w/\|u\|_w$.

We say $u \in \mathcal{B}$ has symmetry or *isotropy* $\Sigma_u = \{\gamma \in \Gamma \ : \ \gamma u = u\}$ (see for example [7]). A subgroup Σ of Γ is *cocompact* if the coset space Γ/Σ is compact.

We say $u_0 \in \mathcal{B}$ is a *relative equilibrium* if $\Phi_t(u_0) = u(t) = e^{\eta t}u_0$ for some $\eta \in L\Gamma$ which is called the *drift* of u_0. Note that for any $\gamma \in \Gamma$, γu_0 is also a relative equilibrium with drift η_γ, where η_γ is given by the adjoint action of γ on η (i.e. $e^{\eta_\gamma t} = \gamma e^{\eta t}\gamma^{-1}$ for all $t > 0$.)

Remark 2.1 Associated to the relative equilibrium $u(t) = e^{\eta t}u_0$ is the closed Lie subgroup

$$K(\eta) = \overline{\{\exp \eta t \ : \ t \in \mathbb{R}\}}.$$

Recall that for Γ compact and any η the group $K(\eta)$ is a torus and for generic η this torus is maximal [6, 8]. For Γ non-compact, $K(\eta)$ is either isomorphic to $\mathbb{R}$ or to a torus; generically $K(\eta)$ is either a maximal torus or $\mathbb{R}$, see [2]. For $\mathbf{E}(n)$, generically $K(\eta)$ is a torus for n even, and generically $K(\eta) \cong \mathbb{R}$ for n odd.

2.3 An example in $\mathbb{R}^n$ satisfying (H1)–(H3)

Let $\mathcal{B}$ be a set of functions $u : \mathbb{R}^n \to \mathbb{R}^s$. We assume that Γ is a closed subgroup of $\mathbf{E}(n)$ acting as Euclidean isometries in the domain variables $\mathbb{R}^n$, possibly coupled with a norm-preserving action in the range $\mathbb{R}^s$. More precisely, let $\gamma = (A, a) \in \mathbf{E}(n) = \mathbf{O}(n) \ltimes \mathbb{R}^n$ act on $x \in \mathbb{R}^n$ by $\gamma x = Ax + a$, and let $\chi : \Gamma \to O(s)$ be an orthogonal action of Γ on $\mathbb{R}^s$. Then we assume that the action of γ on functions $u \subset \mathcal{B}$ is given by

$$(\gamma u)(x) = (\chi_\gamma u)(\gamma^{-1}x).$$

Define the strong norm $\|u\|_s = \sup_x |u(x)|$ and the weak norm $\|u\|_w = \|\rho u\|_s$ where $\rho(x) = (1 - |x|^2)^{-1}$ and $|x|^2 = x_1^2 + \cdots + x_n^2$. We also define the family of weights $\lambda_\alpha(x) = \rho(\alpha x)$, $0 < \alpha \leq 1$.

Lemma 2.2 *In this setting, hypotheses (H1)–(H3) are satisfied with* $\Lambda = \{\lambda_\alpha : 0 < \alpha \leq 1\}$.

Proof Let $\gamma \in \Gamma$, $u \in \mathcal{B}$. It follows from orthogonality of the action χ on $\mathbb{R}^s$ that $|(\gamma u)(x)| = |u(\gamma^{-1}x)|$. In particular, $\|\gamma u\|_s = \|u\|_s$ proving (H1)(a).

For any $a \in \mathbb{R}^n$, $\lim_{|x|\to\infty} \rho(x+a)/\rho(x) = 1$ so we can define $C(a) = \sup_x \rho(x+a)/\rho(x) < \infty$. Writing $\gamma x = Ax + a = A(x + A^{-1}a)$, we claim that $\|\gamma\|_w = C(A^{-1}a)$. It suffices to show that $\|\gamma\|_w = 1$ for $\gamma x = Ax$ and $\|\gamma\|_w = C(a)$ for $\gamma x = x + a$.

Note that $\|\gamma u\|_w = \sup_x \rho(x)|u(\gamma^{-1}x)| = \sup_x \rho(\gamma x)|u(x)|$. If $\gamma x = Ax$, then $\rho(\gamma x) = \rho(x)$ and $\|\gamma\|_w = 1$. If $\gamma x = x + a$ is a translation, then it follows from the definition of $C(a)$ that $\|\gamma u\|_w = C(a)$. This completes the proof of (H1)(b).

It is clear that $C(a)$ depends continuously on a so that $\|\gamma\|_w$ depends continuously on γ, proving (H1)(c).

Let $\alpha > 0$ and note that for any x

$$|\rho(x)u(x)| \leq |\lambda_\alpha u(x)| \leq |u(x)|$$

proving (H2)(a). To verify (H2)(b), suppose that $\|u\|_s < M$ so that

$$\|(1-\lambda_\alpha)u\|_w = \sup_x |\rho(x)(1-\lambda_\alpha(x))u(x)| \le P_\alpha M$$

where

$$P_\alpha = \sup_r \frac{1}{1+r^2}\,\frac{\alpha^2 r^2}{1+\alpha^2 r^2} \le \alpha^2.$$

Hence $P_\alpha \to 0$ as $\alpha \to 0$, and so we can choose $\lambda = \lambda_\alpha$ with α sufficiently small.

For (H3), let $M = \|u\|_s + \|v\|_s$. Since $v \in \operatorname{clos}_w(\Gamma u)$, there is a sequence γ_n such that $\gamma_n u - v \to_w 0$. In particular, $\gamma_n u - v \to 0$ pointwise. Moreover, $\sup_x |\gamma_n u(x) - v(x)| \le M$ and so it is easy to verify that $\lambda(\gamma_n u - v) \to 0$ uniformly for each fixed $\lambda \in \Lambda$ as required. ∎

It is routine to extend this result to the case of a C^k norm, $k \ge 1$. As a trivial example of a semiflow that evolves continuously on $\mathcal{B}$, take any semiflow that evolves continuously according to its local value, i.e. such that

$$(\Phi_t(u))(x) = F_t(u(x))$$

where F_t is a continuous semiflow on $\mathbb{R}^s$. Less trivial examples are given by solutions of reaction-diffusion systems.

3 Co-solutions and relative equilibria

Suppose that Φ_t is a semiflow on $\mathcal{B}$ that is continuous in strong and weak norms satisfying the hypotheses in Section 2.

For continuous action of compact groups, relative equilibria are compact and hence closed. As noted in [2], this is not true for noncompact groups unless one makes further assumptions. Generally speaking, in the situations of interest in this paper, the relative equilibria are closed in the strong topology but not in the weak topology.

Definition 3.1 Let $u_0, v_0 \in \mathcal{B}$. We say that v_0 is a *co-solution* of u_0 if $v_0 \in \operatorname{clos}_w(\Gamma u_0)$.

If $u_0, v_0 \in \mathcal{B}$ and $u(t) = \Phi_t(u_0)$, $v(t) = \Phi_t(v_0)$ are the corresponding solutions, then we say that $v(t)$ is a co-solution of $u(t)$ if v_0 is a co-solution of u_0. It follows from weak-continuity of the flow that the property $v(t) \in \operatorname{clos}_w(\Gamma u(t))$ holds for one value of t if and only if it holds for all t. Hence the set of co-solutions of any given solution is also an invariant set.

Remark 3.2 (a) Note that v_0 being a co-solution for u_0 means that one can find arbitrarily large patches of u_0 that resemble v_0 arbitrarily closely, up to transformation by elements of Γ.

(b) Our definition of co-solution is in terms of the weak topology. We can also define a strong notion of co-solution. However in many situations of interest, the notion is vacuous. Indeed, suppose that u_0 is a relative equilibrium with isotropy Σ. Following [2, Definition 5.2], we say that a sequence $\{\gamma_n\} \subset \Gamma/\Sigma$ is an *approximate symmetry* of u_0 if γ_n has no convergent subsequences and $\|\gamma_n u_0 - u_0\|_s \to 0$. If no such approximate symmetries exist, then in the strong topology Γu_0 is a closed submanifold diffeomorphic to Γ/Σ (see [2, Proposition 5.3]).

(c) Arguing as in (b), we note that relative equilibria with cocompact isotropy subgroup are compact and hence closed in both the strong and weak topologies. In particular, such relative equilibria cannot have nontrivial co-solutions.

It is clear the co-solutions of equilibria are themselves equilibria. In certain cases, co-solutions of relative equilibria are also relative equilibria.

Proposition 3.3 *Suppose u_0 is a relative equilibrium with drift ξ. If there is a sequence $\gamma_n \in \Gamma$, an $\eta \in L\Gamma$ and a $v_0 \in \mathcal{B}$ such that $\|\gamma_n u_0 - v_0\|_w \to 0$ and*

$$\|(e^{\xi_n t} - e^{\eta t})\gamma_n u_0\|_w \to 0$$

as $n \to \infty$ (where $\xi_n = \xi_{\gamma_n}$ is the drift of $\gamma_n u_0$) for all $t > 0$, then v_0 is a relative equilibrium with drift η.

Proof We set $u_n = \gamma_n u_0$ and calculate

$$\begin{aligned} \|e^{\eta t} v_0 - \Phi_t(v_0)\|_w &= \|e^{\eta t} v_0 - e^{\eta t} u_n + e^{\eta t} u_n - e^{\xi_n t} u_n + e^{\xi_n t} u_n - \Phi_t(v_0)\|_w \\ &\leq \|e^{\eta t} v_0 - e^{\eta t} u_n\|_w + \|e^{\eta t} u_n - e^{\xi_n t} u_n\|_w + \|\Phi_t(u_n) - \Phi_t(v_0)\|_w. \end{aligned}$$

In the limit $n \to \infty$ for fixed t, by continuity of the group action the first term goes to zero, by the hypothesis the second term goes to zero and by continuity of the flow the third term goes to zero. Hence

$$\Phi_t(v_0) = e^{\eta t} v_0$$

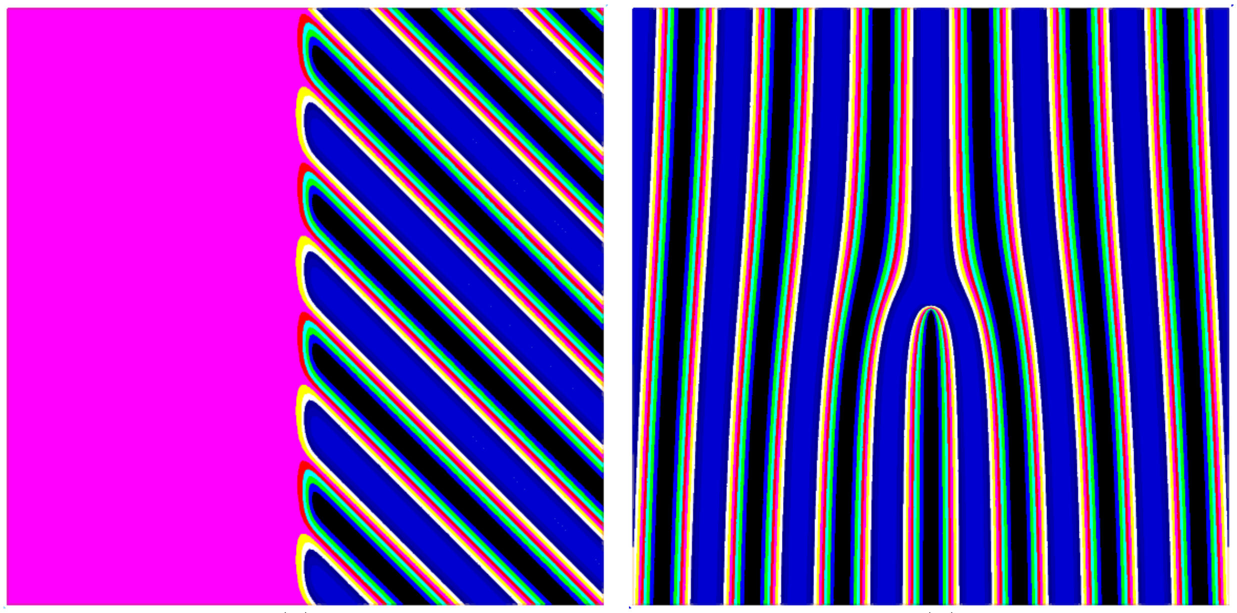
meaning that v_0 is a relative equilibrium with drift η. ∎

Remark 3.4 (a) A special case where Proposition 3.3 applies is when $v_0 \in \mathrm{clos}_w(\Gamma u_0)$ and v has a drift that is in the centre of Γ. In such a case $\xi_n = \xi$ and we can choose $\eta = \xi$ to satisfy the hypotheses.

(b) Another special case satisfying these hypotheses is where v has full symmetry, in which case $\eta = 0$.

3.1 Examples of co-solutions

To motivate the results we give a few examples of patterns that have a nontrivial set of co-solutions, building on ideas in [1]. Figure 1 shows two examples. Figure 1(a) shows a front between a spatially periodic pattern for $x > 0$ and a uniform state for $x < 0$. If this pattern is a relative equilibrium for a flow that fits our setting then there are two families of co-solutions; the uniform pattern for $x \to -\infty$ and the periodic pattern for $x \to +\infty$.



(a) (b)

Figure 1: (a) A relative equilibrium for a Euclidean-equivariant system with two co-solutions that are relative equilibria; a uniform state (by taking translations to the right) and a family of spatially periodic states (by taking translations to the left). (b) A relative equilibrium that is a defect of this form has stripe co-solutions that are relative equilibira on taking translations in any direction. In both cases the co-solutions have additional symmetries.

Figure 2: A spiral relative equilibrium for a Euclidean-equivariant system with co-solutions given by rolls that drift if the spiral rotates; these are be found by considering translates of the spiral in any direction.

Figure 1(b) shows a defect solution that implies the existence of stripe solutions as well. Note that in both cases all co-solutions have cocompact symmetry, so the co-solutions have no further co-solutions.

Another example is illustrated in Figure 2; this shows one component of reaction diffusion system with a spiral relative equilibrium that rotate anticlockwise. By taking limits of large translations in any direction we obtain weak co-solutions that are propagating spatially periodic stripe patterns.

3.2 Symmetries of co-solutions

In spite of the fact that co-solutions are generated by symmetries of the system there is not a simple relationship between the symmetries of a relative equilibrium v and the symmetries of a co-solution. Figures 1 and 2 show cases where the co-solutions have more symmetry than the original pattern. By contrast Figure 3 has a reflection symmetry in the vertical axis that is missing on the cosolutions obtained by translating to the left or right. Hence symmetry may be gained or lost in passing from a relative equilibrium to a co-solution.

4 Inheritance of stability

Let $u_0 \in \mathcal{B}$. We say that u_0 is (Lyapunov) *sw-stable* if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|u - u_0\|_s < \delta \text{ implies that } \|\Phi_t(u) - \Phi_t(u_0)\|_w < \epsilon \text{ for all } t > 0. \tag{1}$$

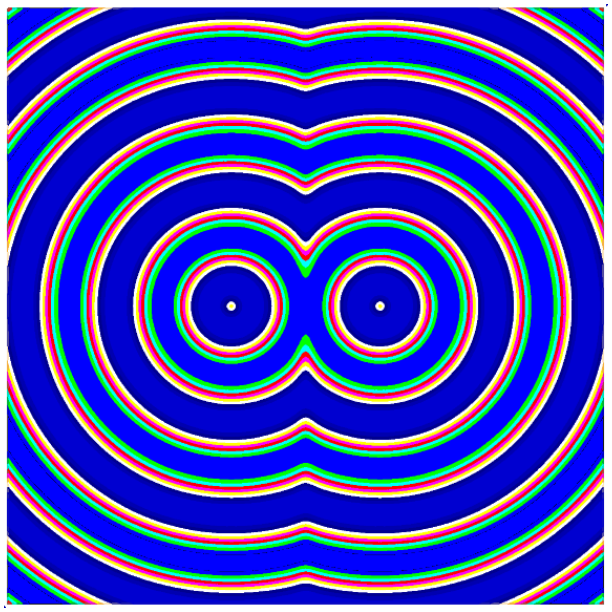


Figure 3: Consider an equilibrium for a Euclidean-equivariant system that has two target patterns anchored near each other. Observe that this has co-solutions that are spatially periodic stripe patterns; moreover the vertical axis reflection symmetry of the original pattern is missing in taking the stripe pattern cosolutions.

Similarly, we say u_0 is *ss-stable* if (1) holds with $\|.\|_w$ replaced by $\|.\|_s$. This corresponds to the usual notion of stability. Observe that ss-stability clearly implies sw-stability.

In a similar way one could define ww-stability but this is probably too weak to be of use and so we do not discuss it further here.

Proposition 4.1 *If $u_0 \in \mathcal{B}$ is sw-stable (resp. ss-stable) then so is γu_0 for all $\gamma \in \Gamma$.*

Proof Suppose that u_0 is sw-stable and $\gamma \in \Gamma$. By (H1)(b), $K = \|\gamma\|_w < \infty$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\|u - u_0\|_s < \delta$ implies that $\|\Phi_t(u) - \Phi_t(u_0)\|_w < \epsilon/K$ for all $t > 0$.

We show that $v_0 = \gamma u_0$ is sw-stable. Suppose that $\|v - v_0\|_s < \delta$. By (H1)(a), $\|\gamma^{-1}v - u_0\|_s < \delta$. Hence $\|\Phi_t(\gamma^{-1}v) - \Phi_t(u_0)\|_w < \epsilon/K$. By (H1)(b) and equivariance of the flow,

$$\|\Phi_t(v) - \Phi_t(v_0)\|_w \leq K\|\gamma^{-1}(\Phi_t(v) - \Phi_t(v_0))\|_w = K\|\Phi_t(\gamma^{-1}v) - \Phi_t(u_0)\| < K\epsilon/K = \epsilon,$$

proving that v_0 is sw-stable.

The proof that ss-stability of u_0 is inherited by v_0 is simpler (with $K = 1$) by (H1)(a). ∎

Theorem 4.2 *Suppose that $u_0, v_0 \in \mathcal{B}$ and $v_0 \in \mathrm{clos}_w(\Gamma u_0)$. If u_0 is ss-stable then v_0 is sw-stable.*

Proof We prove the statement by contradiction, assuming that v_0 is sw-unstable and arguing that u_0 must be ss-unstable.

Since v_0 is sw-unstable, there is an $\epsilon > 0$ such that for all $\delta > 0$ we can find a $T > 0$ and v_1 (both depending on δ) such that

$$\|v_1 - v_0\|_s < \delta, \tag{2}$$

but

$$\|\Phi_T(v_1) - \Phi_T(v_0)\|_w \geq \epsilon. \tag{3}$$

By weak continuity of Φ_t, there exists $\eta \in (0, \delta)$ such that

$$\|\Phi_T(z) - \Phi_T(v_0)\|_w < \frac{\epsilon}{3} \quad \text{for all } z \in \mathcal{B} \text{ with } \|z - v_0\|_w < \eta, \tag{4}$$

$$\|\Phi_T(z) - \Phi_T(v_1)\|_w < \frac{\epsilon}{3} \quad \text{for all } z \in \mathcal{B} \text{ with } \|z - v_1\|_w < \eta. \tag{5}$$

Set $M = \|u_0\|_s + \|v_1\|_s$. Then $\|\gamma u_0 - v_1\|_s \leq M$ for all $\gamma \in \Gamma$ by (H1)(a), and hence by (H2)(b) there exists $\lambda_0 \in \Lambda$ such that

$$\|(1 - \lambda_0)(\gamma u_0 - v_1)\|_w < \eta \quad \text{for all } \gamma \in \Gamma. \tag{6}$$

Since $v_0 \in \mathrm{clos}_w(u_0)$, it follows from (H3) that there exists $\gamma_0 \in \Gamma$ such that

$$\|\lambda_0(\gamma_0 u_0 - v_0)\|_s < \eta < \delta. \tag{7}$$

By hypothesis (H2)(a) and estimate (7),

$$\|\gamma_0 u_0 - v_0\|_w \leq \|\lambda_0(\gamma_0 u_0 - v_0)\|_s < \eta,$$

so it follows from (4) that

$$\|\Phi_T(\gamma_0 u_0) - \Phi_T(v_0)\|_w < \frac{\epsilon}{3}. \tag{8}$$

Now define

$$u_1 = (1 - \lambda_0)\gamma_0 u_0 + \lambda_0 v_1.$$

Then $u_1 - \gamma_0 u_0 = \lambda_0(v_1 - \gamma_0 u_0)$ and we compute that

$$\begin{aligned}\|\gamma_0^{-1} u_1 - u_0\|_s = \|u_1 - \gamma_0 u_0\|_s = \|\lambda_0(v_1 - \gamma_0 u_0)\|_s &\leq \|\lambda_0(v_1 - v_0)\|_s + \|\lambda_0(\gamma_0 u_0 - v_0)\|_s \\ &\leq \|v_1 - v_0\|_s + \|\lambda_0(\gamma_0 u_0 - v_0)\|_s < \delta + \delta = 2\delta,\end{aligned}$$

where we have used hypotheses (H1)(a) and (H2)(a), and estimates (2) and (7).

Moreover, $u_1 - v_1 = (1 - \lambda_0)(\gamma_0 u_0 - v_1)$ so $\|u_1 - v_1\|_w < \eta$ by (6). It follows from (5) that

$$\|\Phi_T(u_1) - \Phi_T(v_1)\|_w < \frac{\epsilon}{3}. \tag{9}$$

Writing

$$\Phi_T(u_1) - \Phi_T(\gamma_0 u_0) = [\Phi_T(u_1) - \Phi_T(v_1)] + [\Phi_T(v_1) - \Phi_T(v_0)] + [\Phi_T(v_0) - \Phi_T(\gamma_0 u_0)]$$

we have

$$
\begin{aligned}
\|\Phi_T(\gamma_0^{-1}u_1) - \Phi_T(u_0)\|_s &= \|\Phi_T(u_1) - \Phi_T(\gamma_0 u_0)\|_s \\
&\geq \|\Phi_T(v_1) - \Phi_T(v_0)\|_w - \|\Phi_T(u_1) - \Phi_T(v_1)\|_w - \|\Phi_T(\gamma_0 u_0) - \Phi_T(v_0)\|_w \\
&\geq \epsilon - \frac{\epsilon}{3} - \frac{\epsilon}{3} = \frac{\epsilon}{3},
\end{aligned}
$$

where we have used hypothesis (H1)(a), Γ-equivariance of Φ_T, and estimates (5), (8) and (9).

Summarizing, we have shown that there is an $\epsilon > 0$ such that for all $\delta > 0$ there is a $T > 0$ and a $w = \gamma_0^{-1}u_1$ such that

$$
\|w - u_0\|_s < 2\delta \quad \text{and} \quad \|\Phi_T(w) - \Phi_T(u_0)\|_s \geq \frac{\epsilon}{3}
$$

giving ss-instability of u_0 and the proof is complete. ∎

In certain situations we obtain a more powerful result, namely in the presence of an additional hypothesis:

(H4) For any $\epsilon > 0$, there exists $\delta > 0$ such that for any $u \in \mathcal{B}$

$$
\sup_{\gamma\in\Gamma}\|\gamma u\|_w < \delta \ \text{implies that}\ \|u\|_s < \epsilon.
$$

Note that hypothesis (H4) clearly holds for the setup in Section 2.3 where we have $\|u\|_s = \sup_{\gamma\in\Gamma}\|\gamma u\|_w$.

Lemma 4.3 *Suppose that (H1-H4) hold and that $u_0 \in \mathcal{B}$ has cocompact isotropy Σ. Then u_0 is ss-stable if and only if u_0 is sw-stable.*

Proof We prove the nontrivial direction, namely that sw-stability implies ss-stability. Let $\epsilon > 0$ and choose $\delta > 0$ as in (H4). By Proposition 4.1, γu_0 is sw-stable for all $\gamma \in \Gamma$. Hence, for each γ, there exists $\eta = \eta(\gamma) > 0$ such that $\|u - \gamma u_0\|_s < \eta$ implies that $\|\Phi_t(u) - \Phi_t(\gamma u_0)\|_w < \delta$ for all $t > 0$. By the proof of Proposition 4.1, we can take $\eta(\gamma) = \eta(1_\Gamma)/\|\gamma\|_w$ where 1_Γ is the identity element in Γ. By (H1)(c), $\eta(\gamma)$ depends continuously on γ. Clearly $\eta(\gamma)$ can be chosen to be constant on Σ-cosets. Since Γ/Σ is compact, it follows that $\eta > 0$ can be chosen independent of γ.

Suppose that $\|v - u_0\|_s < \eta$ and let $\gamma \in \Gamma$. By hypothesis (H1)(a), $\|\gamma v - \gamma u_0\|_s < \eta$, so by the above argument with $u = \gamma v$ we have $\|\Phi_t(\gamma v) - \Phi_t(\gamma u_0)\|_w < \delta$ for all $t > 0$. By equivariance, $\|\gamma(\Phi_t(v) - \Phi_t(u_0))\|_w < \delta$ for all $t > 0$ and all $\gamma \in \Gamma$. By (H4), we deduce that $\|\Phi_t(v) - \Phi_t(u_0)\|_s < \epsilon$ for all $t > 0$ and so u_0 is ss-stable. ∎

Combining Theorem 4.2 and Lemma 4.3 we have:

Theorem 4.4 *Suppose that $u_0, v_0 \in \mathcal{B}$ and $v_0 \in \mathrm{clos}_w(\Gamma u_0)$. Suppose further that v_0 has cocompact isotropy. If u_0 is ss-stable then v_0 is also ss-stable.*

The means, for example, that the existence of an ss-stable relative equilibrium of the form in Figure 1(a) implies that both the uniform and the stripe co-solutions are ss-stable. Similarly if the spiral solution in Figure 2 is ss-stable then the 'far field' roll solutions are are ss-stable.

5 Discussion

We give a novel way of trying to understand the qualitative behaviour of dynamics on unbounded domains. There is clearly a great deal more that can be investigated by making use of a strong and a weak norm that satisfy assumptions such as (H1-H4) to a co-solution.

One direction that seems worth pursuing is the generalisation to transients. In particular initial conditions may converge to relative equilibria in a weak sense, and this gives further predictions for the existence of co-solutions (see for example the spiral wind-up discussed in [1]); we note that the results in Section 4 apply equally for solutions and co-solutions that are not relative equilibria.

In another direction, the results above are purely 'topological' in nature and do not attempt to understand the smooth dynamics. This setting may give a way to obtain results that relate smooth dynamical properties such as spectral stability to topological properties such as ss- and sw-stability and indeed to understand what qualitative ingredients a bifurcation theory for such systems should have.

Similarly it would be interesting to discuss asymptotic stability as well as Lyapunov stability in the setting; we observe that similar to the case for Lyapunov stability there will be several inequivalent notions of asymptotic stability depending on choice of norm.

Finally, we remark that there are situations where a flow that is continuous in the weak and strong topology becomes continuous in only the strong topology due to the appearance of mean flow effects [12]. At this point, certain of our hypotheses are violated, and it would be interesting to understand how this impacts on the relationship between local and global dynamics.

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