Asymptotic Stability of Heteroclinic Cycles in Systems with Symmetry *

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Abstract

Systems possessing symmetries often admit heteroclinic cycles that persist under perturbations that respect the symmetry. The asymptotic stability of such cycles has previously been studied on an ad hoc basis by many authors. Sufficient conditions, but usually not necessary conditions, for stability of these cycles have been obtained via a variety of different techniques.

We begin a systematic investigation into the asymptotic stability of such cycles. A general sufficient condition for asymptotic stability is obtained, together with algebraic criteria for deciding when this condition is also necessary. These criteria are always satisfied in $\mathbb{R}^3$ and often satisfied in higher dimensions. We end by applying our results to several higher dimensional examples that occur in mode interactions with $\mathbf{O}(2)$ symmetry.

1 Introduction

Let $\xi_1, \ldots, \xi_m$ be equilibria of a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. If there are trajectories $\{y_1(t), \ldots, y_m(t)\}$ with the property that $y_j(t)$ is backward asymptotic to ξ_j and forward to ξ_{j+1} then it is usual to call the collection of trajectories $\{\xi_j, y_j(t)\}$ a heteroclinic cycle. (Here we use the convention that $\xi_{m+1} = \xi_1$.)

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Typically one does not expect heteroclinic cycles to exist for general vector fields. However, Field [7] has shown that heteroclinic cycles can occur robustly in symmetric systems. More recently, Guckenheimer and Holmes [12] showed that robust heteroclinic cycles occur naturally in low codimension bifurcation theory. Since this paper of [12], several authors have exploited symmetry to compute examples of robust heteroclinic cycles, see [1], [2], [3], [8], [9], [17], [20], [21], [23], [24], [25], [26]. Robust heteroclinic cycles are prevalent also in population dynamics. References may be found in Hofbauer and Sigmund [15].

Many of the heteroclinic cycles in the above references can be asymptotically stable. Then the cycles lead to interesting phenomena such as intermittency and bursting in the dynamics. Early investigations of the asymptotic stability of robust heteroclinic cycles forced by symmetry yielded sufficient conditions based on the relative magnitudes of the real parts of certain eigenvalues at each equilibrium along the cycle. Usually, these conditions were not optimal.

Melbourne, Chossat and Golubitsky [23] required a fairly general setting for a stability theorem and proved a sufficient condition for stability that applies to all their examples. In particular, they included cycles connecting relative equilibria (flow-invariant group orbits). However, their condition for stability fails to be optimal for two reasons. First, the condition involves the so-called 'radial' or 'branching' eigenvalue at each relative equilibrium. We will show that the radial eigenvalues are irrelevant for questions of stability (thus generalizing a result of Melbourne [21] and answering a conjecture of Armbruster [1]). Second, there is an example of Field and Swift [9] (see also Hofbauer and Sigmund [9]) for which the optimal conditions for asymptotic stability are quite different from those in [23] (even neglecting radial eigenvalues).

The example of [9] indicates that the theory of asymptotic stability of a heteroclinic cycle forced by symmetry is unexpectedly rich. A further complicating issue is as follows. It is clear that for asymptotic stability it is necessary that the whole unstable manifold of ξ_j is asymptotic to ξ_{j+1} (or at least the group orbit through ξ_{j+1}. However Melbourne [22] has shown that there are physically meaningful notions of stability even when this requirement is relaxed. (See also recent work of [4], [16] and [19].) It is evident that the theory of asymptotic stability of heteroclinic cycles is only part of a more general stability theory.

This paper represents a first step in a systematic investigation of the stability of heteroclinic cycles. We shall only consider asymptotic stability, and so in our definition of heteroclinic cycle, Definition 2.1, we require that the entire unstable manifold of ξ_j is asymptotic to the group orbit through ξ_{j+1}. Then we say that the heteroclinic cycle consists of the collection of unstable manifolds. Our definitions in the present paper are formulated in such a way as to facilitate a return to the issues raised above in a subsequent paper.

Our results yield necessary and sufficient conditions for asymptotic stability of many of the cycles in the above references. Theorem 2.7 gives a sufficient condition (2.3) for asymptotic stability of heteroclinic cycles forced by symmetry. Our condition is similar to that of [23] but is independent of the radial eigenvalues.

Theorems 2.9 and 3.1 are concerned with necessity of the sufficient condition (2.3). Indeed, we give algebraic criteria under which condition (2.3) is necessary and sufficient. These criteria are automatically satisfied in $\mathbb{R}^3$ and we recover a result of [21].

The remainder of this paper is structured as follows. In §2 we define precisely what we mean by a heteroclinic cycle forced by symmetry, and state our main theorems when the group of symmetries is finite. Then in §3 we generalize the setting to incorporate continuous groups of symmetries.

The fundamentals of asymptotic stability theory are covered in §4 Although the results of this section fall into the folklore variety, it is hard to find rigorous proofs in the literature. Stability properties of the heteroclinic cycle can be understood in terms of stability of an invariant set under a Poincaré map associated with the cycle. Some of the technical details in this section are deferred to the appendix. We prove our main results in §5.

Finally, §6 consists of examples. We compute sufficient conditions for the asymptotic stability of heteroclinic cycles in codimension two mode-interactions with $\mathbf{O}(2)$ symmetry and show that this condition fails to be optimal in only one case. In doing so, we regain, and in many cases substantially improve upon, the conditions of [24], [3] and [23]. This section can be read independently of §§4 and 5.

2 Heteroclinic cycles forced by symmetry, and their geometry

Throughout this section we restrict to the case when the group of symmetries is finite. There are four subsections. In subsection 2.1 we define precisely what we mean by a heteroclinic cycles and its asymptotic stability. Then in subsection 2.2 we introduce the idea that these heteroclinic cycles are robust under certain perturbations. This is formalized in the hypothesis (H1) which is assumed to hold throughout the paper. Symmetry provides a natural setting for hypothesis (H1) to hold, and we review some basic facts about the lattice of isotropy subgroups.

In subsection 2.3 we use the geometry of the heteroclinic cycle to pick out certain eigenvalues along the cycle. It is the relative magnitudes of the real parts of these eigenvalues that drive the stability of the heteroclinic cycle. In subsection 2.4 we state our main results in terms of these eigenvalues.

2.1 The main definitions

Suppose that Γ is a finite Lie group acting linearly on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Γ-equivariant vector field. That is

$$f(\gamma x) = \gamma f(x), \text{ for all } \gamma \in \Gamma.$$

Definition 2.1 Suppose that ξ_j, $j = 1, \ldots, m$ are hyperbolic equilibria with stable and unstable manifolds $W^s(\xi_j)$ and $W^u(\xi_j)$. The set of group orbits of the unstable manifolds

$$X = \{W^u(\gamma\xi_j),\, j = 1, \ldots, m,\, \gamma \in \Gamma\},$$

forms a *heteroclinic cycle* provided $\dim W^u(\xi_j) \geq 1$ and

$$W^u(\xi_j) - \{\xi_j\} \subset \bigcup_{\gamma\in\Gamma} W^s(\gamma\xi_{j+1}).$$

Remark 2.2 (a) The case $m = 1$ is sometimes distinguished, and the cycle called a *homoclinic cycle*. Our methods for determining asymptotic stability are independent of m.
(b) Define the *principal unstable manifold* $W^{pu}(\xi_j)$ to be the invariant manifold tangent to the generalized eigenspace of the unstable eigenvalues with

maximal real part. Then we may speak more generally of a heteroclinic cycle replacing $W^u(\xi_j)$ in Definition 2.1 throughout by $W^{pu}(\xi_j)$. These more general cycles cannot be asymptotically stable. Nevertheless, they may have strong stability properties, see [22], [20].

Definition 2.3 A heteroclinic cycle X is said to be *stable* if for any neighborhood U of X, there exists a smaller neighborhood V such that trajectories starting in V remain in U for all forward time.

The cycle is *asymptotically stable* if V can be chosen so that in addition trajectories starting in V are asymptotic to X.

The cycle is *unstable* if it is not stable.

2.2 Robust heteroclinic cycles and symmetry

For a general vector field without symmetry, a heteroclinic cycle is necessarily structurally unstable. However, symmetry may force the flow-invariance of certain subspaces and this may permit structural stability to occur. We shall make the following standing hypothesis:

(H1) For each j, there is a flow-invariant subspace P_j such that $W^u(\xi_j) \subset P_j$ and ξ_{j+1} is a sink in P_j.

Remark 2.4 (a) Hypothesis (H1) guarantees robustness of the heteroclinic cycle within the class of vector fields that leave the subspaces P_j invariant. That is, a heteroclinic cycle satisfying hypothesis (H1) persists under small perturbations that preserve the invariance of these subspaces, see Proposition 2.5 below.
(b) For optimal results we shall take P_j to be the smallest possible subspace such that hypothesis (H1) is satisfied.
(c) Set $L_j = P_j \cap P_{j-1}$. In many examples, the problem reduces to one where $\dim L_j = 1$ and $\dim P_j = 2$, hence our notation. However this restriction is not necessary for our results to be valid.
(d) Robust heteroclinic cycles occur naturally in systems with symmetry, the flow-invariant subspaces arising as fixed-point spaces for the action of the symmetry group. A second context in which robust heteroclinic cycles occur naturally is in population dynamics, see Hofbauer and Sigmund [15]. Here the invariant subspaces P_j arise in the boundary of inadmissible regions of phase space. See also Brannath [4] and Gaunersdorfer [10].

We conclude this subsection by reviewing some basic group representation theory, (see Golubitsky, Stewart and Schaeffer [11] for more details). Let Γ be a compact Lie group acting on $\mathbb{R}^n$. The *isotropy subgroup* of a point $x \in \mathbb{R}^n$ is defined to be the subgroup of Γ,

$$\Sigma_x = \{\gamma \in \Gamma | \gamma x = x\}.$$

If $\Sigma \subset \Gamma$ is an isotropy subgroup, then there is a corresponding subspace of $\mathbb{R}^n$ called the *fixed-point subspace* of Σ,

$$\mathrm{Fix}(\Sigma) = \{x \in \mathbb{R}^n | \sigma x = x \text{ for all } \sigma \in \Sigma\}.$$

If f is a Γ-equivariant vector field, and Σ is an isotropy subgroup, we have

$$f(\mathrm{Fix}(\Sigma)) \subset \mathrm{Fix}(\Sigma).$$

In particular, fixed-point subspaces of isotropy subgroups are invariant under the flow of an equivariant vector field. Thus we have the following basic result.

Proposition 2.5 *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is Γ-equivariant, and that there is a heteroclinic cycle satisfying hypothesis (H1). If each P_j is the fixed-point subspace of an isotropy subgroup of Γ, then the heteroclinic cycle persists under small Γ-equivariant perturbations of f.*

There is a partial ordering on conjugacy classes of isotropy subgroups of Γ, defined as follows:

$$\Sigma_1 < \Sigma_2 \text{ if } \Sigma_1 \subset \gamma^{-1}\Sigma_2\gamma \text{ for some } \gamma \in \Gamma,$$

that is, Σ_1 is contained in some conjugate of Σ_2. By abuse of terminology, we refer to the partially ordered set of conjugacy classes of isotropy subgroups as the *lattice of isotropy subgroups.*

Let Σ be an isotropy subgroup. Recall that $\mathbb{R}^n$ can be written as a direct sum of Σ-irreducible subspaces

$$\mathbb{R}^n = V_0 \oplus \cdots \oplus V_p. \tag{2.1}$$

Some of the V_i may be Σ-isomorphic, that is they carry isomorphic representations of Σ. Group together the isomorphic representations to obtain

$$\mathbb{R}^n = W_0 \oplus \cdots \oplus W_q, \tag{2.2}$$

where each W_j is a direct sum of irreducible subspaces, and two irreducible subspaces are contained in the same W_j if and only if they are isomorphic. The decomposition in (2.2) is called the *isotypic decomposition*, and the W_j are called the *isotypic components*. We may choose $W_0 = \mathrm{Fix}(\Sigma)$. Unlike the decomposition in (2.1), the isotypic decomposition is unique. Since the isotypic components carry nonisomorphic representations of Σ, any linear map L commuting with the action of Σ satisfies $L(W_j) \subset W_j$. If $\xi_j \in \mathrm{Fix}(\Sigma)$ then the linearization $(df)_{\xi_j}$ commutes with Σ. It follows that each eigenvector of the linearization lies in an isotypic component of Σ. Moreover, generically each generalized eigenspace lies in a single isotypic component.

2.3 Geometry of heteroclinic cycles

Our conditions for asymptotic stability will depend on the magnitudes of the real parts of certain eigenvalues of the linearization of the vector field f at each equilibrium. The geometry of a heteroclinic cycle satisfying hypothesis (H1) allows us to divide the eigenvalues into four classes, as shown schematically in Figure 1. Let $-r_j$ be the maximum real part of eigenvalues of $(df)_{\xi_j}$ restricted to $L_j = P_j \cap P_{j-1}$, and let $-c_j$ be the maximum real part of the remaining eigenvalues in P_{j-1}. Thus r_j, c_j are positive and correspond to the weakest *radial* and *contracting* eigenvalues at ξ_j.

We define $e_j > 0$ to be the maximum real part of an eigenvalue of $(df)_{\xi_j}$, the strongest *expanding* eigenvalue. We refer to all the nonradial eigenvalues in P_j as the expanding eigenvalues even though some of these may have negative real part. (Note that at least one of the eigenvalues has to have positive real part.) Finally, let t_j be the maximum real part of eigenvalues whose eigenvectors are normal to $P_{j-1}+P_j$, the weakest *transverse* eigenvalue. If $\mathbb{R}^n = P_{j-1} + P_j$, then set $t_j = -\infty$. Since all the eigenvalues with positive real part have eigenvectors in P_j, it follows that $t_j < 0$.

Figure 1: Assignment of radial, contracting, expanding and transverse eigenvalues at the relative equilibrium ξ_j.

Remark 2.6 We have defined r_j, c_j and e_j so that they are positive, but t_j is negative. The more general notion of stability in [22], [19] occurs when certain of the transverse eigenvalues have positive real part, and we have made our choice to avoid the use of double negatives in that work.

2.4 Statement of the main results

We begin by stating a sufficient condition for asymptotic stability of heteroclinic cycles. The result depends only implicitly on the presence of symmetry. The explicit dependence is on the flow-invariant subspaces in (H1).

Theorem 2.7 *Suppose that X is a heteroclinic cycle satisfying hypothesis (H1). Then X is asymptotically stable provided the condition*

$$\prod_{j=1}^{m} \min(c_j, e_j - t_j) > \prod_{j=1}^{m} e_j, \tag{2.3}$$

is satisfied.

Remark 2.8 (a) The $e_j - t_j$ terms in condition (2.3) are due to the flow-invariance of the subspaces P_j. In systems without symmetry, there is no distinction between contracting and transverse eigenvalues and condition (2.3) reduces to the standard (and intuitive) condition that $\prod_{j=1}^{m} c_j > \prod_{j=1}^{m} e_j$.
(b) Condition (2.3) does not involve the magnitudes of the radial eigenvalues $-r_j$ and is thus a refinement of the condition in [23, Theorem 5.1]. In particular, we have verified the conjecture of [1]. The proof of Theorem 2.7 shows that this is again a result of the flow-invariance of the subspaces P_j.

In order to discuss necessity of condition (2.3) for asymptotic stability we need to take account of how symmetry enters into the problem. Suppose that in hypothesis (H1), $P_j = \mathrm{Fix}(\Sigma_j)$ for some isotropy subgroup Σ_j. We introduce two further hypotheses.

(H2) the eigenspaces corresponding to c_j, t_j, e_{j+1} and t_{j+1} lie in the same Σ_j-isotypic component.

(H3) $\dim W^u(\xi_j) = 1$.

Theorem 2.9 *Let Γ be a finite group acting on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Γ-equivariant vector field. Suppose that X is a heteroclinic cycle for f satisfying hypotheses (H1)–(H3). Then generically, condition (2.3) is necessary and sufficient for asymptotic stability of X.*

Remark 2.10 (a) In §3 we show that Theorem 2.9 holds when Γ is a compact Lie group. Within this context, hypothesis (H3) can be weakened considerably.
(b) Hypotheses (H2) and (H3) are automatic when $n = 3$, so Theorem 2.9 generalizes a result of [21].
(c) The eigenvalues corresponding to c_j, t_j, e_{j+1} and t_{j+1} lie in $P_j^\perp$. Now $P_j = \mathrm{Fix}(\Sigma_j)$ is always an isotypic component for Σ_j. It follows that if each isotypic decomposition of $\mathbb{R}^n$ under Σ_j is into two isotypic components, then hypothesis (H2) is valid. This is the case in many of the examples in §6.

3 Continuous groups of symmetries

In this section we modify the definitions in §2 to include the case when Γ is any compact group of symmetries. We shall be interested in heteroclinic cycles that connect normally hyperbolic group orbits of equilibria, or more generally, normally hyperbolic relative equilibria.

Recall that a flow-invariant set ξ_j is a *relative equilibrium* if ξ_j is a group orbit under the action of Γ. Krupa [18] shows that if ξ_j is a relative equilibrium, then in a neighborhood of ξ_j the vector field f can be decomposed as $f_N + f_T$, where both the *normal vector field* f_N and the *tangent vector field* f_T are equivariant, f_T is tangential to group orbits. Moreover, the dynamics of f may be understood as the dynamics of f_N coupled with drift along group orbits. It follows from results of Field [7] that the real parts of the eigenvalues of the linearization of f_N at a point $x_j \in \xi_j$ are independent of the choice of the point x_j and independent of the decomposition into normal and tangent vector fields. In particular it makes sense to say that a relative equilibrium ξ_j is *hyperbolic* if $x_j \in \xi_j$ is a hyperbolic equilibrium of f_N.

We generalize Definition 2.1 and speak of heteroclinic cycles connecting hyperbolic relative equilibria ξ_j. The heteroclinic cycle X is defined to be the set of group orbits of the unstable manifolds $W^u(\xi_j)$ of the relative equilibria. The definition of asymptotic stability of X is unchanged.

As before, we use the geometry of the heteroclinic cycle to define the radial, contracting, expanding and transverse directions. The only difference is that we work with the linearization of the normal vector field f_N at each relative equilibrium ξ_j. More precisely, for each j we choose an $x_j \in \xi_j$ and consider the linearization $(df_N)_{x_j}$. Associate with each relative equilibrium ξ_j the eigenvalue data r_j, c_j, $e_j > 0$ and $t_j < 0$ defined in terms of the real parts of the eigenvalues of $(df_N)_{x_j}$. This is independent of the choice of $x_j \in \xi_j$ by the aforementioned results of [7].

Suppose that X is a heteroclinic cycle between relative equilibria and satisfying hypothesis (H1) with $P_j = \mathrm{Fix}(\Sigma_j)$ for some isotropy subgroup Σ_j. Then the statement and proof of Theorem 2.7 go through without any changes and X is asymptotically stable provided condition (2.3) holds. In addition, Theorem 2.9 is still valid, that is if hypotheses (H1)–(H3) are satisfied, then generically condition (2.3) is both necessary and sufficient for asymptotic stability. In fact we can weaken hypothesis (H3) as follows. Let $N(\Sigma_j)$ denote the normalizer of Σ_j in Γ.

(H3)$'$ $\dim W^u(\xi_j) = \dim (N(\Sigma_j)/\Sigma_j) + 1$.

Of course (H3)$'$ reduces to (H3) when Γ is finite.

Theorem 3.1 *Suppose that Γ is a compact Lie group acting on $\mathbb{R}^n$ and that X is a heteroclinic cycle satisfying hypothesis (H1), (H2) and (H3)$'$. Then generically X is asymptotically stable if and only if condition (2.3) is satisfied.*

Remark 3.2 (a) In many examples,

$$\dim P_j = \dim (N(\Sigma_j)/\Sigma_j) + 2, \tag{3.1}$$

this being required in order to establish existence of a heteroclinic connection in P_j by the Poincaré-Bendixson theorem (see for example [23]). When equation (3.1) holds hypothesis (H3)$'$ is automatically satisfied.

(b) Examples of heteroclinic cycles where equation (3.1) fails are given by Armbruster and Chossat [2] and Swift and Barany [26]. The existence of certain connections in these cases is established by construction of a Liapunov function and numerical simulation respectively. We note that hypothesis (H3)$'$ is valid in the examples of [2] but fails for some of the examples in [26].

(c) The standing hypothesis (H1) is sufficient to incorporate all examples of robust heteroclinic cycles that we know of. However, it is possible to generalize hypothesis (H1) to take account of the continuous symmetries. Let x_j be a point in ξ_j and let Δ_j be the isotropy subgroup of x_j. Let Δ_j^0 be the connected component of the identity in Δ_j.

($\tilde{\mathrm{H}}$1) For each j there is an isotropy subgroup Σ_j with

$$P_j = \bigcup_{\delta \in \Delta_j^0} \delta \mathrm{Fix}(\Sigma_j),$$

such that $W^u(\xi_j) \subset P_j$ and ξ_{j+1} is a sink in P_j.

As in Proposition 2.5, heteroclinic cycles satisfying ($\tilde{\text{H1}}$) persist under Γ-equivariant perturbations. In addition, Theorem 2.7 holds for heteroclinic cycles satisfying hypothesis ($\tilde{\text{H1}}$). Moreover, Theorem 3.1 is valid under a similar modification to hypothesis (H3)$'$:

$(\tilde{\text{H3}})'$ $\dim(W^u(\xi_j) \cap \text{Fix}(\Sigma_j)) = \dim(N(\Sigma_j)/\Sigma_j) + 1.$

4 Poincaré maps and asymptotic stability

In this section, we set up the foundations for our analysis of the stability of heteroclinic cycles satisfying hypothesis (H1). Our main results are not surprizing, and probably fall into the 'folklore' category. However, it is hard to find proofs elsewhere. Moreover, many of the results in this paper are proved using the methods developed in this section.

The main tool in our analysis is the Poincaré map. In subsection 4.1 we construct the Poincaré map as a composition of 'first hit maps' defined in a neighborhood of each relative equilibrium ξ_j and 'connecting diffeomorphisms'. In subsection 4.2 we obtain standard estimates on the first hit maps and show that the Poincaré map is well-defined. Then in subsection 4.3 we relate asymptotic stability of the cycle with the stability of a certain invariant set for the Poincaré map. In subsection 4.4 we consider the issue of genericity of the connecting diffeomorphisms.

4.1 Construction of the Poincaré map

We begin by linearizing the normal vector field f_N in a neighborhood of each relative equilibrium. In §2(c) we used the geometry of the heteroclinic cycle to partition the eigenvalues of the linearization at each relative equilibrium into four groups: radial, contracting, expanding and transverse. Now, in the region of linearized flow, we introduce local coordinates (u, v, w, z) around ξ_j corresponding to the radial, contracting, expanding and transverse directions. Recall that at least one of the expanding eigenvalues has positive real part, but there might in addition be expanding eigenvalues with negative real part. Write $w = (w^+, w^-)$ corresponding to the partition of the expanding eigenvalues into those with positive and negative real part.

Let $||$ denote the euclidean norm in the local coordinates. Scaling the local coordinates if necessary, we may assume that the unit 'cube'

P_{j-1}

$H_j(in)$

L_j

P_j

/users/krupa/pictures/cycles2.eps

$H_j(out)$

Figure 2: The cross-sections $H_j^{(in)}$ and $H_j^{(out)}$

$\{|u|, |v|, |w^+|, |w^-|, |z| \leq 1\}$ lies within the region of linearized flow. We shall define various cross-sections to the flow near the heteroclinic cycle. In the linearized flow, the connection leaving ξ_j must lie in the subspace $\{u = v = w^- = z = 0\}$ and so we define the cross-section

$$H_j^{(out)} = \{(u, v, w, z) \mid |u| \leq 1,\ |v| \leq 1,\ |w^+| = 1,\ |w^-| \leq 1,\ |z| \leq 1\}.$$

Define the generalized origin

$$O = W^u(\xi_j) \cap H_j^{(out)} = \{u = v = w^- = z = 0,\ |w^+| = 1\}.$$

The connection approaching ξ_j lies in the subspace P_{j-1} which is coordinatized locally by u and v. We define the cross-section

$$H_j^{(in)} = \{(u, v, w, z) \mid |u|^2 + |v|^2 = 1,\ |w| \leq 1,\ |z| \leq 1\}.$$

Again we define the generalized origin $O = W^u(\xi_{j-1}) \cap H_j^{(in)} \subset \{w = z = 0\}$. See Figure 2 for a diagram of the cross-sections $H_j^{(in)}$ and $H_j^{(out)}$.

We now define the *first hit maps*

$$\phi_j : H_j^{(in)} \to H_j^{(out)},$$

and the *connecting diffeomorphisms*

$$\psi_j : H_j^{(out)} \to H_{j+1}^{(in)}.$$

Then define $g_j : H_j^{(in)} \to H_{j+1}^{(in)}$ to be the composition $g_j = \psi_j \circ \phi_j$. Finally set $g = g_k \circ \cdots \circ g_1$. This is our Poincaré map $g : H_1^{(in)} \to H_1^{(in)}$.

Remark 4.1 (a) At present g is not well-defined. In particular, the first hit maps ϕ_j are not defined when $w^+ = 0$ so g is not defined at O. However, Corollary 4.4 below shows that g is well-defined and continuous on a certain subset of $H_1^{(in)}$.
(b) The connecting diffeomorphism ψ_j maps neighborhoods of $O \subset H_j^{(out)}$ homeomorphically onto neighborhoods of $O \subset H_{j+1}^{(in)}$.
(c) We began this section by linearizing the vector field in a neighborhood of each relative equilibrium. In general, this change of coordinates is not C^1. Nevertheless we refer to the maps ψ_j as connecting diffeomorphisms, and indeed our techniques in subsequent sections proceed as if they are C^1. One way around this is to assume finitely many nondegeneracy conditions on the linearization of $(df)_{\xi_j}$ for each j, so that there is indeed a C^1 change of coordinates. Even if these conditions fail the results on stability remain valid and can be proved by combining our methods with the results of Deng [6].

4.2 Estimates on the first hit maps

The first hit map ϕ_j may be computed explicitly, where defined, using the linear flow near ξ_j. Recall that we have the eigenvalue data $-r_j, -c_j, e_j, t_j$ corresponding to the (u, v, w, z) directions. In addition we have $w = (w^+, w^-)$. The eigenvalues in the w^+ directions have positive real part and e_j is the largest real part. Let $-\hat{e}_j$ denote the smallest real part of the eigenvalues in the w^- directions, so $\hat{e}_j > 0$.

Proposition 4.2 *Let $y = (u, v, w^+.w^-, z) \in H_j^{(in)}$, $w^+ \neq 0$, and let $\epsilon > 0$. There is a constant K such that*

$$\begin{aligned}
|\phi_j^u(y)| &\leq K|w^+|^{r_j/e_j-\epsilon}, \\
|\phi_j^v(y)| &\leq K|w^+|^{c_j/e_j-\epsilon}, \\
|\phi_j^{w^-}(y)| &\leq K|w^+|^{\hat{e}_j/e_j-\epsilon}, \\
|\phi_j^z(y)| &\leq K|w^+|^{-t_j/e_j-\epsilon}|z|.
\end{aligned}$$

Proof The linear flow $\phi_j : H_j^{(in)} \to H_j^{(out)}$ takes the form

$$\phi_j(u, v, w^+, w^-, z) = (e^{-R_j t}u, e^{-C_j t}v, e^{E_j^+ t}w^+, e^{-E_j^- t}w^-, e^{T_j t}z),$$

where t is time and $-R_j$, $-C_j$, E_j^+, $-E_j^-$, T_j are matrices. A trajectory hits $H_j^{(out)}$ when $|e^{E_j^+ t}w^+| = 1$ and we may estimate the time of flight t using this equation. Suppose that $\delta > 0$. It follows from the proof of [13, Theorem 1, page 145], that there is a positive constant k such that

$$|e^{E_j^+ t}w^+| \leq k e^{(e_j+\delta)t}|w^+|.$$

From this inequality follows the estimate

$$-t \leq \frac{1}{e_j+\delta}\ln(k|w^+|).$$

We can now estimate $e^{-R_j t}u$ say. Again, there is a positive constant ℓ such that

$$|e^{-R_j t}u| \leq \ell e^{-(r_j-\delta)t}|u|.$$

Substituting in the estimate for $-t$ yields

$$|e^{-R_j t}u| \leq K|w^+|^{(r_j-\delta)/(e_j+\delta)}|u|,$$

where $K > 0$. In $H_j^{(in)}$, $|u| \leq 1$ and so we obtain

$$|\phi_j^u(y)| \leq K|w^+|^{(r_j-\delta)/(e_j+\delta)}.$$

Now choose δ so that $(r_j-\delta)/(e_j+\delta) > r_j/e_j - \epsilon$ in order to obtain the required estimate for $|\phi_j^u|$. The estimates for the remaining components are similar. Note that in the case of ϕ_j^z we can remove the factor of $|z|$ but choose not to. ■

Remark 4.3 The occurrence of an $\epsilon > 0$ in the proposition is due to the fact that linearization at ξ_j may be nonsemisimple. If the linearization is semisimple, then we may take $\epsilon = 0$ (again see [13]).

Corollary 4.4 *Let U be a neighborhood of $O \subset H_j^{(out)}$. There is a neighborhood V of $O \subset H_j^{(in)}$ such that $\phi_j : V - \{w^+ = 0\} \to U$ is well-defined and continuous.*

Proof The estimates in Proposition 4.2 show that provided $y \in V$ has w^+-component small enough but nonzero, the u, v, w^-, z components of $\phi_j(y)$ are small so that $\phi_j(y)$ is close to O. (We have used the fact that $r_j, c_j, \hat{e}_j$ and e_j are positive, and t_j is negative. Also $|z| \leq 1$.) ■

Let W denote the union of the stable manifolds of the relative equilibria in the cycle

$$W = \bigcup_{j=1}^{k} \bigcup_{\gamma \in \Gamma} W^s(\gamma\xi_j). \tag{4.1}$$

Corollary 4.5 *There is a neighborhood S of $O \in H_1^{(in)}$ such that*

$$g : S - W \to H_1^{(in)} - W$$

is well-defined and continuous.

Proof Observe that the subspace $\{w^+ = 0\}$ in $H_j^{(in)}$ is a subset of W. Now W is forward and backward invariant under the flow and hence the complement of W is forward and backward invariant under the maps ϕ_j and ψ_j. It follows from Remark 4.1(b) and Corollary 4.4 that we may choose a neighborhood S of O in $H_1^{(in)}$, so that $g : S - W \to H_1^{(in)}$ is well-defined and continuous. ■

4.3 Asymptotic stability

Our main result in this subsection is to show that asymptotic stability of the heteroclinic cycle is determined by the w^+-component of the Poincaré map g. This is to be expected intuitively since all the other components lie in the stable manifold of ξ_1. To make this precise, we introduce the notion of transverse stability. Let E denote the stable manifold/subspace of ξ_1 within the region of linearized flow intersected with $H_1^{(in)}$. Recall that the sets W and S were defined in equation (4.1) and Corollary 4.5.

Definition 4.6 The origin O in $H_1^{(in)}$ is *transversely stable* under the map g if for any neighbourhood U of E there is a neighborhood V of O satisfying the condition

(a) If $y \in V - W$ and $g^i(y) \in S$ for $i = 0, \ldots, r-1$ then $g^i(y) \in U$ for $i = 0, \ldots, r$.

The origin O is *transversely asymptotically stable* if in addition V can be chosen so that

(b) If $y \in V - W$ and $g^i(y) \subset S$ for all $i \geq 0$ then $\mathrm{dist}(g^r(y), E) \to 0$.

The origin O is *transversely unstable* if it is not transversely stable.

The interrelation of the various sets (apart from W) in Definition 4.6 are shown in Figure 3.

Figure 3: The sets $H_1^{(in)}$, S, U, V and E in the definition of transverse stability

Theorem 4.7 *Let X be a heteroclinic cycle satisfying hypothesis (H1).*

(a) *X is (asymptotically) stable if and only if the origin $O \subset H_1^{(in)}$ is transversely (asymptotically) stable under the Poincaré map g.*

(b) *X is unstable if and only if O is transversely unstable under g.*

Proof The proof is given in the appendix. ■

A consequence of Theorem 4.7 is the intuitively obvious statement that if a heteroclinic cycle is contained entirely in a proper flow-invariant subspace, then stability of the cycle is governed by stability within that subspace.

Corollary 4.8 *Suppose that X is a heteroclinic cycle in $\mathbb{R}^n$ satisfying hypothesis (H1), and that $Q \subset \mathbb{R}^n$ is a flow-invariant subspace containing X. Then*

(a) *X is (asymptotically) stable in $\mathbb{R}^n$ if and only if X is (asymptotically) stable in Q.*

(b) *X is unstable in $\mathbb{R}^n$ if and only if X is unstable in Q.*

4.4 Genericity of connecting diffeomorphisms

Many of the forthcoming stability theorems will require hypotheses stating that certain nondegeneracy conditions on the linear coefficients of the connecting diffeomorphisms ψ_j are valid. It seems plausible that such conditions are valid for an open dense set of Γ-equivariant vector fields f. The results of this subsection ensure that this is indeed the case.

The connecting diffeomorphism $\psi_j : H_j^{(in)} \to H_j^{(out)}$ is Σ_j-equivariant with respect to the (isomorphic) actions of Σ_j induced on $H_j^{(in)}$ and $H_j^{(out)}$. Identifying these cross-sections with $\mathbb{R}^k$ (where $k = n - \dim P_j$) we can write $\psi_j : \mathbb{R}^k \to \mathbb{R}^k$.

Since the heteroclinic cycle is robust, it makes sense to talk about the connecting diffeomorphism ψ_j corresponding to each vector field close to the original vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. With this in mind, the following proposition can be stated at least roughly. The technicalities related to making the statement completely precise are deferred to the appendix.

Proposition 4.9 *Suppose that there is a property P that holds for an open and dense set of Σ_j-equivariant diffeomorphisms $\psi : \mathbb{R}^k \to \mathbb{R}^k$. Then for an open and dense set of vector fields on $\mathbb{R}^n$ near to the vector field f, the corresponding connecting diffeomorphism ψ_j satisfies property P.*

Remark 4.10 The open and dense sets in Proposition 4.9 will be interpreted as subsets of spaces of C^r vector fields/diffeomorphisms (with the C^r topology) defined by finitely many transversality conditions, see the appendix. In our examples, the heteroclinic cycles arise in the context of local bifurcation theory. By standard arguments, there is a residual set in the space of families of C^r vector fields for which the corresponding connecting diffeomorphisms satisfy property P. This residual set is defined by countably many transversality conditions.

It follows from Proposition 4.9 that generically the only restrictions on the connecting diffeomorphisms are the symmetry restrictions on mappings from $H_j^{(out)}$ to $H_j^{(in)}$. In particular, such mappings are forced to vanish at linear order in directions tangent to the group orbit. The next result which is proved in the appendix shows that some directions are automatically exempt from such restrictions.

Lemma 4.11 *Let O and O' denote the generalized origins in $H_j^{(out)}$ and $H_{j+1}^{(in)}$ respectively. Then*

(a) the eigenspaces of c_j and t_j have trivial intersection with $T_y\Gamma y$ for all $y \in O$.

(b) the eigenspaces of e_{j+1} and t_{j+1} have trivial intersection with $T_y\Gamma y$ for all $y \in O'$.

5 Proof of the main results

In this section we prove our main results, Theorems 2.7 and Theorems 3.1. Both rely on a basic result about stability of the origin for mappings of the plane. This stability result is proved in subsection 5.1. Then subsections 5.2 and 5.3 contain the proofs of the theorems.

5.1 A basic stability result

Lemma 5.1 *Suppose that $g = g_m \circ \cdots \circ g_1$ where $g_j : \mathbb{R}^2 \to \mathbb{R}^2$ has at lowest order the form*

$$g_j(w, z) = (A_j w^{a_j} z^{b_j} + B_j w^{c_j} z^{d_j}, C_j w^{a_j} z^{b_j} + D_j w^{c_j} z^{d_j})$$

with $a_j, b_j, c_j, d_j \geq 0$ and $a_i + b_i$, $c_i + d_i > 0$. Let $\rho_j = \min(a_j + b_j, c_j + d_j)$ and $\rho = \rho_1 \cdots \rho_m$. Then 0 is an asymptotically stable fixed point of g if $\rho > 1$. Suppose further that $A_j, B_j, C_j, D_j \neq 0$ and that $a_j + b_j \neq c_j + d_j$. Then 0 is unstable if $\rho < 1$.

Proof The statement about asymptotic stability is easily verified by working in polar coordinates. We prove instability when $\rho < 1$. First choose constants $\beta > \alpha > 0$ such that

$$4\alpha|A_j| \leq |C_j| \leq \frac{1}{4}\beta|A_j|, \quad 4\alpha|B_j| \leq |D_j| \leq \frac{1}{4}\beta|B_j|,$$

for $j = 1, \ldots, m$. We may define a cone in $\mathbb{R}^2$,

$$\mathcal{C} = \{\alpha|w| \leq |z| \leq \beta|w|\}.$$

Then, set $M = \frac{1}{2}\min(\alpha^{b_j}|A_j|, \alpha^{d_j}|B_j|)$.

We claim that if $(w, z) \in \mathcal{C}$ is close enough to zero then $g_j(w, z) \in \mathcal{C}$ and $|g_j^w(w, z)| \geq M|w|^{\rho_j}$. It follows that $|g^w(w, z)| \geq M^m|w|^\rho$ and the w coordinate is expanding since $\rho < 1$.

It remains to verify the claim. Let $(w, z) \in \mathcal{C}$ and suppose that $a_j + b_j < c_j + d_j$. Then $|w^{a_j} z^{b_j}| \geq \alpha^{b_j}|w|^{a_j+b_j}$. But

$$|w^{c_j} z^{d_j}| \leq \beta^{d_j}|w|^{c_j+d_j} = o(|w^{a_j} z^{b_j}|).$$

It follows that at lowest order

$$g_j^w(w, z) = A_j w^{a_j} z^{b_j}, \quad g_j^z(w, z) = C_j w^{a_j} z^{b_j}.$$

Hence if (w, z) is close enough to 0 we have the estimates,

$$\tfrac{1}{2}|A_j w^{a_j} z^{b_j}| \leq |g_j^w(w, z)| \leq 2|A_j w^{a_j} z^{b_j}|,$$

$$\tfrac{1}{2}|C_j w^{a_j} z^{b_j}| \leq |g_j^z(w, z)| \leq 2|C_j w^{a_j} z^{b_j}|.$$

We compute that

$$\begin{aligned}\alpha|g_j^w(w,z)| &\leq 2\alpha|A_j w^{a_j} z^{b_j}| \\ &\leq \frac{1}{2}|C_j w^{a_j} z^{b_j}| \\ &\leq |g_j^z(w,z)|.\end{aligned}$$

Similarly, $|g_j^z(w,z)| \leq \beta|g_j^w(w,z)|$, and so $g_j(w,z) \in \mathcal{C}$ as required. In addition,

$$\begin{aligned}|g_j^w(w,z)| &\geq \frac{1}{2}|A_j w_j^a z_j^b| \\ &\geq \frac{1}{2}\alpha^{b_j}|A_j w^{a_j+b_j}| \\ &\geq M|w|^{\rho_j}.\end{aligned}$$

The argument in the case $a_j + b_j > c_j + d_j$ is almost identical. ■

5.2 Sufficiency

Proof of Theorem 2.7 Since condition (2.3) is assumed to be valid, we may find an $\epsilon > 0$ small enough such that

$$\prod_{j=1}^{m} \min\left\{\frac{c_j}{e_j} - \epsilon, 1 - \frac{t_j}{e_j} - \epsilon\right\} > 1.$$

Recall that the Poincaré map $g : S - W \to H_1^{(in)}$ is the composition of the first hit maps $g_1, \ldots, g_m$. Write g_j in components $g_j = (g_j^u, g_j^v, g_j^w, g_j^z)$. Let $y = (u,v,w,z) \in H_j^{(in)}$ lie in the domain of g_j. We claim that there is a constant M such that at lowest order

$$\begin{aligned}|g_j^w(y)| &\leq M(|w|^{c_j/e_j-\epsilon} + |w|^{-t_j/e_j-\epsilon}|z|), \\ |g_j^z(y)| &\leq M(|w|^{c_j/e_j-\epsilon} + |w|^{-t_j/e_j-\epsilon}|z|).\end{aligned}$$

It then follows from Lemma 5.1 that the origin $O \subset H_1^{(in)}$ is transversely asymptotically stable under g. By Theorem 4.7 the heteroclinic cycle is asymptotically stable.

It remains to verify the claim. We have the estimates

$$\begin{aligned} |\phi_j^v(y)| &\leq K|w|^{c_j/e_j-\epsilon}, \\ |\phi_j^z(y)| &\leq K|w|^{-t_j/e_j-\epsilon}|z|, \end{aligned} \tag{5.1}$$

from Proposition 4.2. We shall show that in a neighborhood U of O in $H_j^{(out)}$, there is a positive constant L such that

$$|\psi_j^w(p)|,\ |\psi_j^z(p)| \leq L(|v|+|z|) + o(|v|,|z|), \tag{5.2}$$

for $p=(u,v,w,z)\in U$. Inequalities (5.1) and (5.2) combine to produce the required estimates with $M=KL$.

It is convenient to momentarily introduce global coordinates

$$(s,t)\in\mathbb{R}^n = P_j \oplus P_j^{\perp}.$$

Near $H_j^{(out)}$, s and t represent (u,w) and (v,z). However near $H_{j+1}^{(in)}$ they represent (u,v) and (w,z). The important observation is that

$$\psi_j^t(s,0)=0,$$

this following from flow-invariance of P_j. Transferring back to the local coordinates, we have that

$$\psi_j^w(y)=\psi_j^z(y)=0, \text{ whenever } v=z=0.$$

This implies that

$$\psi_j^w(p)=A(p)v+B(p)z+o(v,z),$$

and a similar expression for ψ_j^z. Thus we have obtained the estimates in (5.2) with

$$L=\max_{p\in O}\max\{|A(p)|,|B(p)|\}.$$

■

5.3 Necessity

In this subsection, we prove Theorem 3.1. Then Theorem 2.9 is a special case. First, refine the local coordinates (u, v, w, z) of §2(d) as follows. Write $v = (v_1, v_2)$, $w = (w_1, w_2)$, $z = (z_1, z_2)$ where v_1, w_1 and z_1 are coordinates on the eigenspaces of the eigenvalues with real part $-c_j$, e_j and t_j respectively. Note that the subspace corresponding to w_2 is a sum of eigenspaces with negative real part. Moreover, by (H3) (w_1, w_2) correspond to (w^+, w^-) in §4(a).

Now we shall make some genericity assumptions. (Some of these assumptions can be removed but we shall not distinguish between the essential and inessential assumptions.) Recall that $\rho_j = \min(c_j/e_j, 1 - t_j/e_j)$.

(G1) $c_j/e_j \neq 1 - t_j/e_j$.

(G2) The linearization $(df_N)_{x_j}$, $x_j \in \xi_j$, is semisimple. Moreover each eigenspace lies inside one Σ_j-isotypic component.

By hypothesis (H3), $W^u(\xi_{j-1})$ intersects $H_j^{(in)}$ in finitely many group orbits lying in $\{w = z = 0\}$ (the intersection consists of two points if Γ is finite). If one point on such a group orbit has nonzero v_1 and v_2 coordinates, then so do all points on that group orbit and the norm of these coordinates is constant.

(G3) The v_1 and v_2 coordinates of the points in $W^u(\xi_{j-1}) \cap H_j^{(in)}$ are nonzero.

Proposition 5.2 *Suppose that (G2) and (G3) hold. Let $\mathcal{C} \in H_j^{(in)}$ be the cone*

$$\mathcal{C} = \{y = (u, v, w, z) \in H_j^{(in)},\ |z_2| \leq \gamma |z_1|\},$$

where γ is a positive constant. If $y \in \mathcal{C}$ is close to $W^u(\xi_j)$, then

$$\phi_j^{v_1}(y) = |w_1|^{c_j/e_j} v_1, \quad \phi_j^{z_1}(y) = |w_1|^{-t_j/e_j} z_1, \tag{5.3}$$

$$|\phi_j^{v_2}(y)| = o(|\phi_j^{v_1}(y)|), \quad |\phi_j^{z_2}(y)| = o(|\phi_j^{z_1}(y)|). \tag{5.4}$$

Proof The expressions for $\phi_j^{v_1}$ and $\phi_j^{z_1}$ are obtained by following the proof of Proposition 4.2 but exploiting hypothesis (H3) and genericity assumption (G2). Hypothesis (H3) implies that all the eigenvalues with positive real part are real and equal. Hence the expression $e^{E_j^+ t} w$ that appears in the proof of Proposition 4.2 is replaced by $e^{e_j t} w_1$. It follows from assumption (G2) that no ϵ is required and we can solve for the time of flight exactly: $t = -\frac{1}{e_j} \ln |w_1|$. The equations in (5.3) easily follow.

Now let $-c_j'$ denote the real part of the weakest contracting eigenvalues other than those with real part $-c_j$. Then $c_j' > c_j$. Similarly define $t_j' < t_j$. Arguing as in Proposition 4.2 there is a constant K such that

$$|\phi_j^{v_2}(y)| \leq K |w_1|^{c_j'/e_j} |v_2|, \quad |\phi_j^{z_2}(y)| \leq K |w_1|^{-t_j'/e_j} |z_2|.$$

Now $|v_1|$ and $|v_2|$ are nonzero by (G3) and we have the first estimate in (5.4). The second estimate follows from the fact that y lies in the cone $\mathcal{C}$. ■

In order to compute the lowest order terms of the Poincaré map g we must expand each connecting diffeomorphism ψ_j about each of the group orbits in (G3). The fact that there are finitely many such orbits rather than one does not complicate the analysis in any way other than leading to cumbersome notation. In addition, the fact that these group orbits may be continuous is not a complication since the connecting diffeomorphisms based at each point on the group orbit are conjugated by the group elements. Hence, we shall proceed as if there were only one point in the intersection of $W^u(\xi_{j-1}) \cap H_j^{(in)}$. Let p_j denote the v_1-coordinate of this point.

It follows from Proposition 5.2 that at lowest order the w_1, z_1 and z_2 components of g_j have the form

$$\begin{aligned} g_j^{w_1}(y) &= |w_1|^{c_j/e_j} A_j v_1 + |w_1|^{-t_j/e_j} B_j z_1, \\ g_j^{z_1}(y) &= |w_1|^{c_j/e_j} C_j v_1 + |w_1|^{-t_j/e_j} D_j z_1, \\ g_j^{z_2}(y) &= |w_1|^{c_j/e_j} E_j v_1 + |w_1|^{-t_j/e_j} F_j z_1, \end{aligned}$$

where A_j—F_j are constant matrices. We note that if there are no transverse eigenvalues at ξ_j or ξ_{j+1}, or all of the transverse eigenvalues have the same real part, then not all of the matrices will be present.

At this point it is convenient to make two additional assumptions for ease of exposition. Having proved the theorem under these assumptions, we shall then sketch the proof when these assumptions are relaxed. Our additional assumptions are

(A1) There is at most one transverse eigenvalue with real part t_j at each ξ_j.

(A2) c_j corresponds to real eigenvalues (possibly with multiplicity).

Finally we shall require some further genericity assumptions on the matrices A_j—D_j. In fact, the matrices B_j and D_j, if they occur, have one column as a consequence of (A1). By Proposition A.2 and Lemma 4.11 we may assume that the entries are all nonzero or equivalently

(G4) $\ker B_j = 0, \quad \ker D_j = 0.$

The remaining matrices may have nontrivial kernels. However, generically these kernels do not contain the points p_j.

(G5) $A_j p_j \neq 0, \quad C_j p_j \neq 0.$

We can use these genericity assumptions to define cones in analogy with the proof of lemma 5.1. There are constants $\beta > \alpha > 0$ such that

$$4\alpha|A_j p_j| \leq |C_j p_j| \leq \frac{1}{4}\beta|A_j p_j|, \quad 4\alpha|B_j z_1| \leq |D_j z_1| \leq \frac{1}{4}\beta|B_j z_1|,$$

for each j and all z_1. In addition, there is a constant $\gamma > 0$ such that

$$|E_j p_j| \leq \frac{1}{4}\gamma|C_j p_j|, \quad |F_j p_j| \leq \frac{1}{4}\gamma|D_j p_j|.$$

Define a cone $\mathcal{C}_j$ inside $H_j^{(in)}$,

$$\mathcal{C}_j = \{\alpha|w_1| \leq |z_1| \leq \beta|w_1|, \ |z_2| \leq \gamma|z_1|\} \subset H_j^{(in)}.$$

Finally, let $M = \frac{1}{2}\min(|A_j p_j|, \alpha|B_j|)$.

Lemma 5.3 *Suppose that the genericity assumptions (G1–5) and the additional assumptions (A1,2) are valid. Let $y = (u, v, w, z) \in \mathcal{C}_j$. If y is close enough to $W^u(\xi_j)$, then $g_j(y) \in \mathcal{C}_{j+1}$ and $|g_j^{w_1}(y)| \geq M|w_1|^{\rho_j}$.*

Proof In the case when there is no z_2 component,the proof is completely analogous to the proof of Lemma 5.1. If there is a z_2 component, then the definition of γ and the corresponding adaptation of the cone guarantees that $\mathcal{C}_j$ is still mapped into $\mathcal{C}_{j+1}$. Then Proposition 5.2 ensures that the v_2 and z_2

components of $\phi_j(y)$ remain higher order terms throughout. The proof now proceeds as in Lemma 5.1. ■

It is clear that Theorem 2.9 follows from the lemma. It remains to relax assumptions (A1) and (A2). First consider the possibility that the eigenvalues corresponding to t_j are not simple though still real. Observe that if $1 - t_j/e_j < c_j/e_j$ for each j, then the heteroclinic cycle is asymptotically stable by Theorem 2.7. Hence we may assume without loss of generality that $c_1/e_1 < 1 - t_1/e_1$. Then the conclusion of Lemma 5.3 still holds for g_1. Moreover $g_1^{z_1}(y) = w_1^{c_j/e_j} C_1 v_1$. Suppose that $1 - t_2/e_2 < c_2/e_2$. In order to obtain the required estimate for g_2 we need the genericity assumptions

$$B_2 C_1 p_1 \neq 0, \quad D_2 C_1 p_1 \neq 0.$$

Also we must modify the definition of α, β and γ in the obvious way.

Finally we must address the possibility that the eigenvalues corresponding to c_j and t_j may be complex. We shall sketch the argument in the case that $c_j/e_j < 1 - t_j/e_j$ and the c_j eigenvalues are a complex conjugate pair $-(c_j + i\omega_j)$ (with multiplicity). Suppose for simplicity that there is no z_2 component. Then at lowest order

$$g_j^{w_1}(y) = |w_1|^{c_j/e_j} A_j R_{\theta_j(w_1)} v_1, \quad g_j^{z_1}(y) = |w_1|^{c_j/e_j} C_j R_{\theta_j(w_1)} v_1,$$

where R_θ is a rotation matrix and

$$\theta_j(w_1) = \frac{\omega_j}{e_j} \ln w_1.$$

As w_1 varies close to 0, $A_j R_{\theta_j(w_1)} p_j$ vanishes arbitrarily often. Nevertheless we may use this oscillation to argue that often this quantity is large in magnitude. The idea is to demonstrate expansion of intervals rather than individual points. Formally, fix $\hat{w}_1 > 0$ and define a slice

$$I(\hat{w}_1) = \{y \in H_j^{(in)} | w_1 \in [0, \hat{w}_1]\}.$$

Lemma 5.4 *There is a $\delta > 0$ such that for any $y' \in [0, \delta \hat{w}_1^{c_j/e_j}]$ there exists $y \in I(\hat{w}_1)$ with $|g_j^{w_1}(y)| = y'$. Moreover, if $y \in \mathcal{C}_j$ then $g_j(y) \in \mathcal{C}_{j+1}$.*

Proof Choose $\delta_0 < 1$ so that $\theta_j(\hat{w}_1) - \theta_j(\delta_0 \hat{w}_1) \geq 2\pi$. Then there is a $w_1 \in [\delta_0 \hat{w}_1, \hat{w}_1]$ such that

$$\frac{1}{2}|A_j p_j| \leq |A_j R_{\theta_j(w_1)} p_j| \leq |A_j p_j|, \quad \frac{1}{2}|C_j p_j| \leq |C_j R_{\theta_j(w_1)} p_j| \leq |C_j p_j|.$$

It follows that there is an $y = (u, v, w, z) \in I(\hat{w}_1)$ such that $|g_j^{w_1}(y)| = \delta \hat{w}_1^{c_j/e_j}$ where $\delta \geq \frac{2}{3}|A_j p_j| \delta_0^{c_j/e_j}$. There is also an $y = (u, v, w, z) \in I(\hat{w}_1)$ with $w_1 \in [\delta_0 \hat{w}_1, \hat{w}_1]$ such that $g_j^{w_1}(y) = 0$. The first statement of the lemma follows by continuity. The proof of the statement about the cones is straightforward. ■

6 Examples in mode interactions with $\mathbf{O}(2)$ symmetry

In two-parameter families of vector fields, one may expect to find points where a steady-state loses stability by having eigenvalues of the linearized equation simultaneously at

(a) $0,\ 0$, steady-state/steady-state,

(b) $0,\ \pm\omega i$, steady-state/Hopf,

(c) $\pm\omega_1 i,\ \pm\omega_2 i,\ \omega_1/\omega_2$ irrational, Hopf/Hopf.

Generically in a two-parameter family, these are the only eigenvalues on the imaginary axis and one may reduce to center manifolds of dimension two, three and four respectively.

When there is a symmetry group present, eigenvalues may be forced to be multiple, the multiplicities corresponding (roughly speaking) to the dimensions of irreducible representations of the group. In the case of $\mathbf{O}(2)$-symmetry, the situation can be described as follows. Irreducible representations of $\mathbf{O}(2)$ are either one or two-dimensional, and the eigenvalues of the linearized equation may generically have multiplicity one or two.

It follows, that, when there is $\mathbf{O}(2)$-symmetry present, case (a) may lead to a center-manifold of dimension two, three or four. The corresponding dimensions for case (b) are 3, 4, 5 and 6, and for case (c) are 4, 6 and 8. It turns out that structurally stable heteroclinic cycles only occur when all eigenvalues are double, and the center manifold has the highest dimension available.

The steady-state/steady-state mode interaction has been analyzed by [3] and [24], and the steady-state/Hopf and Hopf/Hopf interactions by [23]. In this section, we shall assume that coefficients in the Taylor expansions of the various vector fields are in the regimes of existence of the heteroclinic cycles. In particular hypothesis (H1) is assumed to hold. For these heteroclinic cycles, each heteroclinic connection is shown to exist via application of the Poincaré-Bendixson theorem after phase-amplitude reduction. By Remark 3.2(a) hypothesis (H3)$'$ is valid for all the cycles in this section. It turns out that hypothesis (H2) fails only in one case of the Hopf/Hopf interaction.

6.1 Steady-state/steady-state

Coordinates $z = (z_1, z_2)$ can be chosen on the four-dimensional center manifold $\mathbb{C}^2$ so that the action of $\mathbf{O}(2)$ is given by

$$\begin{aligned} \theta \cdot z &= (e^{li\theta} z_1, e^{mi\theta} z_2), \\ \kappa \cdot z &= (\bar{z}_1, \bar{z}_2), \end{aligned}$$

where l and m are coprime positive integers $1 \leq l \leq m$.

It turns out that structurally stable heteroclinic cycles occur only in the case $l = 1$, $m = 2$. (According to conventions other than that adopted in this paper, the cycles are *homoclinic* cycles.) There is a single equilibrium on the cycle with isotropy $\mathbf{D}_2$ generated by rotation through π and the reflection κ. The heteroclinic connections lie in $\mathrm{Fix}(\kappa)$ or conjugate copies of this fixed-point subspace.

Even though the cycle lies in a four-dimensional space, one of the eigenvalues is forced to be zero since the equilibria lie on a continuous group orbit. Hence the normal vector field at each equilibrium has only three eigenvalues. We label these (real) eigenvalues $-r$, $-c$ and e. In particular, hypothesis (H2) (and (H3)$'$) is trivially valid, and by Theorem 3.1 we have that generically the heteroclinic cycle is asymptotically stable if and only if $c > e$.

This result is the same is that obtained in [24] and is an improvement on [3, Proposition 5.1]. In the latter reference it is shown that (in our notation) the condition $\min\{r, c\} > e$ is sufficient for asymptotic stability.

6.2 Steady-state/Hopf

In this case we have a six-dimensional center manifold $\mathbb{C}^3$. We can choose coordinates $z = (z_0, z_1, z_2)$ so that the $\mathbf{O}(2)$-action has the form

$$\begin{aligned} \phi \cdot (z_0, z_1, z_2) &= (e^{ki\phi} z_0, e^{li\phi} z_1, e^{-li\phi} z_2) \quad \phi \in \mathbf{SO}(2), \\ \kappa \cdot (z_0, z_1, z_2) &= (\bar{z}_0, z_2, z_1), \end{aligned}$$

where k and l are positive coprime integers.

In addition to the $\mathbf{O}(2)$-symmetry, there is an approximate phase-shift symmetry $\mathbf{SO}(2)$ arising from the Hopf bifurcation. We may assume that the vector field is in Birkhoff normal form up to any required order in the Taylor expansion. Then $\mathbf{SO}(2)$ acts by

$$\theta \cdot (z_0, z_1, z_2) = (z_0, e^{i\theta} z_1, e^{i\theta} z_2).$$

It turns out that structurally stable heteroclinic cycles occur only when $k = l = 1$. We now run quickly through the structure of the lattice of isotropy subgroups that is relevant to the existence of heteroclinic cycles. Define the following subgroups:

$$\begin{aligned}
\mathbb{Z}_2(\kappa) &= \{1, \kappa\}, \\
\mathbb{Z}_2(\kappa \cdot (\pi, \pi)) &= \{1, \kappa \cdot (\pi, \pi)\}, \\
\mathbb{Z}_2^c &= \{1, (\pi, \pi)\}.
\end{aligned}$$

The relevant portion of the lattice of isotropy subgroups is shown in Figure 4. The isotropy subgroups (1)–(4) are given together with their fixed-point subspaces in Table 1.

$$\begin{tikzcd}
(1) & (2) \\
(3) \arrow[u] \arrow[ur] & (4) \arrow[u] \arrow[ul]
\end{tikzcd}$$

Figure 4: Lattice connections for the cycle between equilibria and periodic solutions in the steady-state/Hopf interaction

Theorem 3.1 of [23] guarantees (under certain open conditions) the existence of a heteroclinic cycle between equilibria with isotropy (1) and periodic solutions with isotropy (2). The heteroclinic connections lie in Fix(3) and Fix(4). The equilibrium has one radial eigenvalue $-r_1$, a zero eigenvalue and contracting and expanding eigenvalues of multiplicity two with real parts $-c_1$ and e_1 respectively. The periodic solution has a radial eigenvalue $-r_2$, two zero eigenvalues, and simple real contracting, expanding and transverse eigenvalues $-c_2$, e_2 and t_2 respectively. Each of the isotropy subgroups (3) and (4)

	Isotropy subgroup	Fixed-point subspace
(1)	$\mathbb{Z}_2(\kappa)\times\mathbf{SO}(2)$	$(x,0,0)$
(2)	$\mathbb{Z}_2(\kappa)\oplus\mathbb{Z}_2^c$	$(0,z_1,z_1)$
(3)	$\mathbb{Z}_2(\kappa)$	(x,z_1,z_1)
(4)	$\mathbb{Z}_2(\kappa\cdot(\pi,\pi))$	(iy,z_1,z_1)

Table 1: Isotropy subgroups and fixed-point subspaces for the cycle between equilibria and periodic solutions in the steady-state/Hopf interaction

is a two element group and hence has precisely two distinct irreducible representations. The isotypic decomposition under each group consists of two components and (H2) is valid by Remark 2.10(b).

In our notation, [23, Theorem 3.3] states that the heteroclinic cycle is asymptotically stable provided

$$\min(r_1,c_1)\min(r_2,c_2,e_2-t_2)>e_1e_2.$$

However it follows from Theorem 3.1 that generically the cycle is asymptotically stable if and only if

$$c_1\min(c_2,e_2-t_2)>e_1e_2.$$

6.3 Hopf/Hopf

This time we have an eight-dimensional center manifold $\mathbb{C}^4$. Effectively, the symmetry group is $\mathbf{O}(2)\times T^2$, the T^2-symmetry being present in the normal form and arising from the simultaneous Hopf bifurcations. We can choose coordinates $z=(z_1,z_2,z_3,z_4)$ so that the action of $\mathbf{O}(2)\times T^2$ is as follows:

$$\begin{aligned}
\phi\cdot z &= (e^{il\phi}z_1,e^{-il\phi}z_2,e^{im\phi}z_3,e^{-im\phi}z_4), \quad \phi\in\mathbf{SO}(2),\\
(\psi_1,\psi_2)\cdot z &= (e^{i\psi_1}z_1,e^{i\psi_1}z_2,e^{i\psi_2}z_3,e^{i\psi_2}z_4), \quad (\psi_1,\psi_2)\in T^2,\\
\kappa\cdot z &= (z_2,z_1,z_4,z_3)
\end{aligned}$$

where l and m are positive coprime integers and $l\le m$.

There are several possibilities for heteroclinic cycles and it turns out that the cases $l=m=1$ and $l<m$ are quite different. In order to describe these possibilities it is necessary to reproduce the group-theoretic information in [23].

	Isotropy subgroup	Fixed-point subspace
(1)	$S(0,0,1) \times S(1,-l,0)$	$(z_1,0,0,0)$
(2)	$S(0,0,1) \times \mathbb{Z}_\kappa \times \mathbb{Z}(\pi/l,\pi,0)$	$(z_1,z_1,0,0)$
(3)	$S(0,1,0) \times \mathbb{Z}_\kappa \times \mathbb{Z}(\pi/m,0,\pi)$	$(0,0,z_3,z_3)$
(4)	$S(0,1,0) \times S(1,0,m)$	$(0,0,0,z_4)$
(5)	$S(0,0,1) \times \mathbb{Z}(\pi/l,\pi,0)$	$(z_1,z_2,0,0)$
(6)	$S(0,1,0) \times \mathbb{Z}(\pi/m,0,\pi)$	$(0,0,z_3,z_4)$
(7)	$S(1,l,m)$	$(0,z_2,0,z_4)$
(8)	$S(1,l,-m)$	$(0,z_2,z_3,0)$
(9)	$\mathbb{Z}_\kappa \times \mathbb{Z}(\pi,l\pi,m\pi)$	(z_1,z_1,z_3,z_3)
(10)	$\mathbb{Z}_\kappa(0,\pi,0) \times \mathbb{Z}(\pi,l\pi,m\pi)$	$(z_1,-z_1,z_3,z_3)$
(11)	$\mathbb{Z}_\kappa(0,0,\pi) \times \mathbb{Z}(\pi,l\pi,m\pi)$	$(z_1,z_1,z_3,-z_3)$

Table 2: Isotropy subgroups in the Hopf-Hopf interaction

Define the following subgroups

$$\begin{aligned}
\mathbb{Z}(\phi,\psi_1,\psi_2) &= \text{group generated by } (\phi,\psi_1,\psi_2) \in \mathbf{SO}(2) \times T^2, \\
\mathbb{Z}_\kappa(\phi,\psi_1,\psi_2) &= \text{group generated by } \kappa \cdot (\phi,\psi_1,\psi_2), \\
S(k,l,m) &= \{(k\theta,l\theta,m\theta) \in \mathbf{SO}(2) \times T^2,\ \theta \in S^1\}
\end{aligned}$$

The upper part of the lattice of isotropy subgroups is given in Figure 5. The isotropy subgroups are listed together with their fixed-point subspaces in Table 2.

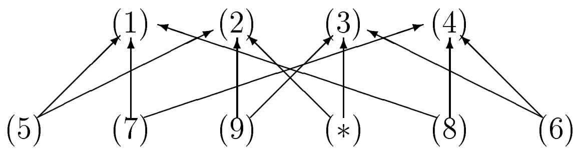

Figure 5: Upper part of lattice of isotropy subgroups in the Hopf/Hopf interaction. Isotropy $(*)$ is (10) if m is odd and (11) if m is even.

The isotropy subgroups (1) and (4) correspond to rotating waves, and the isotropy subgroups (2) and (3) to standing waves. It turns out that for $l = m = 1$ there are three structurally stable heteroclinic cycles between these periodic solutions. One cycle connects the rotating waves and a second cycle connects the standing waves. The third cycle connects all four periodic solutions. When $l < m$, only the cycle connecting the rotating waves can occur. The portions of the lattice of isotropy subgroups corresponding to the three heteroclinic cycles are illustrated in Figure 6.

(a)

(b)

(c)

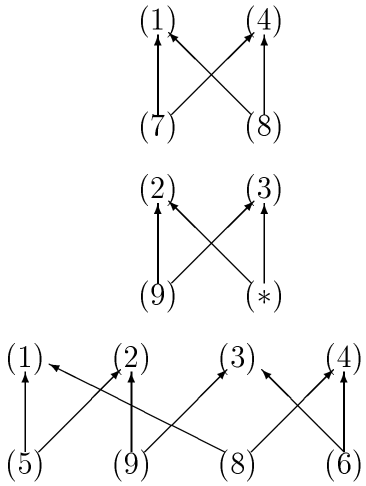

Figure 6: Isotropy connections for the heteroclinic cycles in the Hopf/Hopf interaction: (a) cycle of rotating waves, (b) cycle of standing waves, and (c) cycle of rotating and standing waves.

The case $l = m = 1$

Each of the three heteroclinic cycles is realized in this case. We show that for each cycle, the necessary and sufficient condition of Theorem 3.1 applies. Up to multiplicity forced by the group action, there are precisely four nonzero eigenvalues corresponding to the modes (j), $j = 1, \cdots, 4$ and we label their real parts $-r_j$, $-c_j$, e_j and t_j in the usual way.

Generically the necessary and sufficient conditions for asymptotic stability have the form

$$\prod_{j=1}^{m} \min(c_j, e_j - t_j) > \prod_{j=1}^{m} e_j, \tag{6.1}$$

where $m = 2$ for the cycles (a) and (b) and $m = 4$ for the cycle (c).

In order to establish condition (6.1) we verify hypothesis (H2). By Remark 2.10, it is sufficient to show that the isotypic decompositions of the isotropy subgroups (5)–(11) consist of two components. We give the details for (9), (7) and (5). The arguments for the remaining isotropy subgroups are similar. First, notice that $\mathbb{Z}(\pi, \pi, \pi)$ acts trivially on the whole of $\mathbb{C}^4$ so that, when $l = m = 1$, the isotropy subgroup (9) reduces essentially to a group generated by an element of order two, and we can apply Remark 2.10(b).

Isotropy subgroup (7) reduces to $S(1, 1, 1)$ which acts as

$$z \to (e^{2i\theta} z_1, z_2, e^{2i\theta} z_3, z_4).$$

Again there are two isotypic components $(0, z_2, 0, z_4)$ and $(z_1, 0, z_3, 0)$ corresponding to the 0 and 2 representations of S^1. Finally isotropy subgroup (5) reduces to $S(0, 0, 1) \times \mathbb{Z}(\pi, \pi, 0)$ which acts as

$$z \to (z_1, z_2, e^{i\theta} z_3, e^{i\theta} z_4), \quad z \to (z_1, z_2, -z_3, -z_4).$$

In particular, the action of $\mathbb{Z}(\pi, \pi, 0)$ is subsumed into the action of $S(0, 0, 1)$ and there are two isotypic components corresponding to the 0 and 1 representations of S^1.

The case $l < m$

In this case only the cycle (a) between rotating waves can exist. Although, cycles (b) and (c) are still suggested by the lattice of isotropy subgroups their existence as heteroclinic cycles is ruled out by the structure of the low order equivariant mappings.

Theorem 2.7 guarantees that condition (6.1) remains sufficient for asymptotic stability of the cycle between rotating waves. However this condition is not optimal and a necessary and sufficient condition is given by

$$C_1 + C_2 + T_1 T_2 > \min(2, 1 + C_1 C_2), \tag{6.2}$$

where $C_j = c_j / e_j$ and $T_j = t_j / e_j$.

The derivation of condition (6.2) relies on the transition matrix method of [9] and will appear in future work. Here we show only that hypothesis (H2) is violated at each of the rotating waves. Recall that $(7) = S(1, l, m)$ which acts as

$$\theta \cdot z = (e^{2il\theta} z_1, z_2, e^{2im\theta} z_3, z_4).$$

We claim that the isotypic decomposition under (7) is

$$\mathbb{C}^4 = \{z_2, z_4\} \oplus \{z_1\} \oplus \{z_3\}.$$

Indeed these subspaces correspond to the 0, l and m representations of S^1 which are nonisomorphic since $0 < l < m$. Similarly, the isotypic decomposition under (8) is

$$\mathbb{C}^4 = \{z_2, z_3\} \oplus \{z_1\} \oplus \{z_4\}.$$

It follows that hypothesis (H2) fails at each relative equilibrium.

A Appendix

In this appendix, we give the technical details omitted in §§ 4.3 and 4.4. In particular, we prove Theorem 4.7 and Lemma 4.11 and give a precise version of Proposition 4.9.

We begin by showing that transverse (asymptotic) stability (resp. transverse instability), defined in Definition 4.6, of O under g corresponds roughly to (asymptotic) stability (resp. instability) of O under g. A more precise statement is required since g is not defined at O.

Lemma A.1 *Suppose that $O \subset H_1^{(in)}$ is transversely stable under g. Then for any neighborhood U of O there is a smaller neighborhood V so that $g^i(V - W) \subset U$ for all $i \geq 0$. If O is transversely asymptotically stable then V can be chosen so that in addition* $\mathrm{dist}(g^i(y), O) \to 0$ *for $y \in V - W$.*

Proof Let U be a neighborhood of O. It follows from the estimates in Proposition 4.2 that for $y = (u, v, w^+, w^-, z) \in H_1^{(in)}$, $\phi_1(y) \to O$ as $w^+ \to 0$. Moreover $\phi_1(y) \to O$ uniformly in u, v, w^- and z (recall that $|z| \leq 1$). By continuity of ϕ_j and ψ_j where defined, we have that $g(y) \to O$ uniformly in u, v, w^-, z as $w^+ \to 0$. Hence there is a neighborhood U' of E, $U' \subset H_1^{(in)}$ such that $g(U' - W) \subset S \cap U$.

Now suppose that O is transversely stable under g. Let V be a neighborhood of O satisfying condition (a) of Definition 4.6 with respect to U'. Shrink V if necessary so that $V \subset S \cap U \cap U'$. Observe that $g^i(V - W) \subset S \cap U \cap U'$ for all $i \geq 0$. (This is easily shown by induction on the integer i.) In particular $g^i(V - W) \subset U$ for all $i \geq 0$ as required.

Finally suppose that O is transversely asymptotically stable under g. Shrink V if necessary so that V also satisfies condition (b) of the definition with respect to U'.

Let $y \in V - W$ and $\epsilon > 0$. Using Proposition 4.2 as before, there is a neighborhood U'' of E so that $g(U'' - W)$ is contained in an ϵ-neighborhood of O. By condition (b) of Definition 4.6, eventually $g^i(y) \in U''$. It follows that eventually g^{i+1} is within distance ϵ of O. ∎

Proof of Theorem 4.7 We shall prove part (a). Then part (b) follows immediately from the definitions of instability. Suppose that O is transversely stable under g and let $\hat{U}$ be a neighborhood of the heteroclinic cycle X. Suppose that $U \subset S$ is a neighborhood of O in $H_1^{(in)}$. Let $F_t(x)$ denote the trajectory of a point x under the flow. If $x \in U - W$ there is a $\tau(x) > 0$, the 'first return time', so that $F_t(x) \notin H_1^{(in)}$ for $0 < t < \tau(x)$ and $F_{\tau(x)} = g(x) \in H_1^{(in)}$.

We claim that U can be chosen so that $F_t(x) \in \hat{U}$ for all $x \in U - W$ and $t < \tau(x)$. Indeed, it follows from Remark 4.1(b) and Corollary 4.4 that whenever $F_t(x) \in H_j^{(in)}$ or $H_j^{(out)}$ and $0 \leq t \leq \tau(x)$, we have $F_t(x) \in \hat{U}$. Now we can use compactness of the heteroclinic cycle between $H_j^{(out)}$ and $H_{j+1}^{(in)}$ to show that between these cross-sections $F_t(x) \in \hat{U}$ provided U is chosen small enough. On the other hand, the linearity of the flow between $H_j^{(in)}$ and $H_j^{(out)}$ implies that between these cross-sections $F_t(x) \in \hat{U}$ for U small enough. Thus we have verified the claim.

Let V be a neighborhood as guaranteed by Lemma A.1. For $y \in V - W$ we have that $g^i(y) \in U$ for all $i \geq 0$. It follows that for $i \geq 0$,

$$F_t(y) \in \hat{U} \text{ for } t \in [0, \tau(y) + \tau(g(y)) + \cdots + \tau(g^i(y))].$$

Observe that the time taken to flow from $H_1^{(out)}$ to $H_2^{(in)}$ say is bounded away from zero and so $\tau(y) \geq \tau_0 > 0$ for $y \in V - W$. It follows that $\sum_{i=0}^{i} \tau(g^i(y)) \to \infty$ as $i \to \infty$. Hence $F_t(V - W) \subset \hat{U}$ for all $t \geq 0$.

If O is transversely asymptotically stable, then we can choose V to satisfy the additional property described in Lemma A.1 Let $\hat{U}'$ be a neighborhood of X. Then as above there is a neighborhood V' of O in $H_1^{(in)}$ such that $F_t(V' - W) \subset \hat{U}'$ for all $t \geq 0$. If $y \in V - W$ then $g^i(y) \to O$ so eventually $g^i(y) \in V'$. It follows that eventually $F_t(y) \in \hat{U}'$ and since $\hat{U}'$ is arbitrary, $\mathrm{dist}(F_t(y), X) \to 0$.

Finally, using similar arguments to those used in the construction of U, we can construct a neighborhood $\hat{V}$ of X so that the forward trajectory through any point in $\hat{V} - W$ intersects V and does not leave $\hat{U}$ during the time it takes to reach V. It follows that trajectories starting in $\hat{V} - W$ remain in $\hat{U}$ for all forward time (and are asymptotic to X), and we have proved that the heteroclinic cycle is (asymptotically) stable. ■

Proof of Lemma 4.11 We prove (a) and omit the analogous proof of (b). Choose $x_j \in \xi_j$ with isotropy subgroup Δ_j and let Ω be a closed submanifold of Γ transverse to Δ_j at the identity element $\mathbf{1}$ and such that $\Delta_j \cap \Omega = \{\mathbf{1}\}$. The eigenspaces must have trivial intersection with the tangent space $T_{x_j}\Omega x_j$ and, by continuity, also with $T_y\Omega y$. To complete the proof of (a) we need to show that their intersection with $T_y\Delta_j y$ is also trivial. Let $L = (df_N)_{x_j}$ and suppose that η is an eigenvector of L with eigenvalue λ. Let $\gamma : [0,1] \to \Delta_j$ be a smooth curve with $\gamma(0) = \mathbf{1}$.

Then we compute that

$$\begin{aligned} \lambda\dot{\gamma}\eta &= \lambda\frac{d}{dt}\gamma(t)\eta|_{t=0} = \frac{d}{dt}\gamma(t)\lambda\eta|_{t=0} \\ &= \frac{d}{dt}\gamma(t)L\eta|_{t=0} = L\frac{d}{dt}\gamma(t)\eta|_{t=0} = L\dot{\gamma}\eta. \end{aligned}$$

Hence $\dot{\gamma}\eta$ is also an eigenvector of L with eigenvalue λ. It now follows that $T_y\Delta_j y$ is contained in the span of eigenspaces having nontrivial intersections with P_j. Generically the intersection of the eigenspaces corresponding to c_j and t_j with P_j is trivial and (a) follows. ■

Finally, we consider the problem of making Proposition 4.9 completely precise. Field [7] has studied the corresponding problem for equivariant Poincaré maps around a periodic orbit. Here we follow closely the approach in [7].

If U, V are Γ-invariant open sets in $\mathbb{R}^k$, let $C^r_\Gamma(U,V)$ denote the space of C^r Γ-equivariant maps of U into V. Let C^r_Γ denote the space of C^r equivariant vector fields on $\mathbb{R}^n$. Consider $f \in C^r_\Gamma$ having a heteroclinic cycle with relative equilibria $\{\xi_1, ..., \xi_m\}$. Let $\Phi_t(x)$ denote the flow corresponding to the vector field f.

For $y \in H_j^{(out)}$ let $\rho(y)$ be the least positive time such that $\Phi_{\rho(y)}(y) \in H_{j+1}^{(in)}$. Let O and O' denote the generalized origins in $H_j^{(out)}$ and $H_{j+1}^{(in)}$ respectively, and define

$$\Lambda_0 = \bigcup_{y \in O} \{\Phi_t(y), t \in [0, \rho(y)]\}, \quad \Lambda = \Gamma\Lambda_0.$$

Consider a normal bundle $N(\Lambda)$ (for background on normal bundles see [5]), and let D (resp. D') be the restriction of $N(\Lambda)$ to ΓO (resp. $\Gamma O'$). For $\epsilon > 0$ let D_ϵ be the bundle with the same base space as D and whose fibers are ϵ-balls in the fibers of D. Define D'_ϵ similarly. We choose $N(\Lambda)$ in such a way that

$$D_1 = \Gamma H_j^{(out)}, \quad D'_1 = \Gamma H_{j+1}^{(in)}.$$

Let U be a Γ-invariant neighborhood of Λ with the property that

$$\overline{U} \subset \bigcup_{y \in D_1} \{\Phi_t(y), t \in [0, \rho(y)]\}.$$

Given $\epsilon > 0$ let X_ϵ^r be the subset of $C_\Gamma^r(D_1, D'_1)$ consisting of maps equal to ψ_j outside of D_ϵ.

Proposition A.2 *There exists $\epsilon > 0$, an open neighborhood Q of ψ_j in X_ϵ^{r+1} in the C^{r+1} topology and a continuous map $\chi : Q \to C_\Gamma^r$ with the following properties:*

(a) For $\psi \in Q$, $\chi(\psi) = f$ on $\mathbb{R}^n - U$.

(b) For $\psi \in Q$, $\chi(\psi)$ has connecting diffeomorphism ψ.

(c) $\chi(\psi_j) = f$.

Proof Analogous to the proof of Lemma C, p. 198 in [7]. ∎

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