Validity, Universality and Structure of the Ginzburg-Landau Equation

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Abstract

We present new results on the validity, universality and structure of the Ginzburg-Landau equation on the line.

1 Introduction

Ginzburg-Landau equations [10, 21] are universal modulation equations (also known as amplitude or envelope equations) that govern the local dynamics of spatially-extended systems of partial differential equations (PDEs) undergoing certain bifurcations (see for example [3, 15, 16]). The equations play the same role for spatially-extended systems that Landau equations [9] play for problems with bounded domains.

The simplest form of the Ginzburg-Landau equation is

$$\partial_T A = c_0 \partial_X^2 A + c_1 A + c_2 |A|^2 A, \tag{1.1}$$

where $A = A(X) : \mathbb{R} \to \mathbb{C}$ is a complex amplitude function and $c_0, c_1, c_2 \in \mathbb{R}$. We also consider the complex Ginzburg-Landau equation which has the same form with $c_0, c_2 \in \mathbb{C}$, $c_1 \in \mathbb{R}$. In the physics literature, such equations are derived either phenomenologically or via asymptotic expansion from underlying PDEs.

There are two distinct and complementary rigorous approaches to justifying the Ginzburg-Landau equation (1.1). The first approach is to show that solutions to

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the Ginzburg-Landau equation approximate solutions to the underlying system of PDEs. See Schneider [19], and also [2, 6]. (The analogous results for the complex Ginzburg-Landau equation can be found in Schneider [20].) The second approach due to Melbourne [11, 12] is to derive a universal reduced equation that is a nontruncated version of the Ginzburg-Landau equation, so that

(i) Formally, the reduced equation agrees with the Ginzburg-Landau equation when truncated at leading order.

(ii) There is locally a one-to-one correspondence between *essential solutions* of the nontruncated Ginzburg-Landau equation and the underlying system of PDEs.

Essential solutions were defined in [1] (see also [14]) and roughly speaking are solutions that are small in space and time. Our approach generalizes the results of Iooss *et al.* [8, 7] who studied equilibrium and time-periodic solutions.

We note that the approach in [11, 12] is more in keeping with the well-known approach of *equivariant bifurcation theory* [5], where center manifold reduction leads to equations that, when truncated at leading order, generically coincide with the Landau equations. This point of view is discussed at length in [13]. The main difference from equivariant bifurcation theory is that the group of symmetries is not compact leading to problems with continuous spectra. In particular, the center manifold theorem does not apply and the reduced equations are not finite-dimensional.

In this article, we explore further the formal agreement (i) between the reduced equation truncated at lowest order and equation (1.1), and we prove two rigorous results in this direction. Consider spatially-extended systems of PDEs governing functions $u = u(x, z) : \mathbb{R} \times \Omega \to \mathbb{R}^s$ where $z \in \Omega$ denotes the bounded spatial variables. Let $\lambda \in \mathbb{R}$ denote a bifurcation parameter, and suppose that at $\lambda = 0$ there is a steady-state bifurcation with nonzero critical wavenumber $k_c > 0$. (See [12] or Section 3 for precise definitions of these terms.) Under certain technical assumptions, it is shown in [11, 12] that generically there is a reduction to a nontruncated Ginzburg-Landau equation (see equation (3.2) below). The two new results in this paper are as follows:

Theorem 1.1 *Let $\nu \in (0, \frac{1}{2})$. Generically, there is a semilinear equation of the form*

$$\partial_t A = c_0 \partial_x^2 A + c_1 \lambda A + c_2 |A|^2 A + o(\partial_x^2 A, \lambda A, A^3), \tag{1.2}$$

such that essential solutions satisfying $\|A\| \leq |\lambda|^{2\nu}$ are locally in one-to-one correspondence with such solutions for the underlying system of PDEs.

Since we expect branches of solutions to have amplitude $\lambda^{1/2}$, we obtain effective results by choosing $\nu < 1/4$ in Theorem 1.1.

Theorem 1.2 *Nondegenerate essential solutions A_0 to the (truncated) Ginzburg-Landau equation* (1.1) *correspond to (discontinuous) branches of solutions to the underlying system of PDEs of the form*

$$u(x,z,t) = \sqrt{\lambda}\Big(A_0(\sqrt{\lambda}x,\lambda t)v_0(z)e^{ik_cx} + \overline{A}_0(\sqrt{\lambda}x,\lambda t)\overline{v}_0(z)e^{-ik_cx}\Big) + o(\sqrt{\lambda}). \quad (1.3)$$

for some function $v_0 : \Omega \to \mathbb{C}^s$.

The terms $o(\cdots)$ in equation (1.2) are nonlocal, incorporating derivatives of all orders. Nevertheless, the nonlinear terms are analytic. Our approach is flexible enough to incorporate additional nonlinear terms. For example, Theorem 1.1 remains valid with equation (1.2) replaced by a semilinear equation of the form

$$\partial_t A = c_0\partial_x^2 A + c_1\lambda A + c_2|A|^2A + ic_3|A|^2\partial_x A + ic_4A^2\partial_x\overline{A} + c_5|A|^4A + o(\partial_x^2 A, \lambda A, \partial_x A^3, A^5).$$

The remainder of this paper is organized as follows. In Section 2, we present our results starting from the (generalized) Swift-Hohenberg equation. The extension to a universal theory for systems of PDEs is given in Section 3. The corresponding results for the complex Ginzburg-Landau equation are stated in Section 4.

2 Derivation of the Ginzburg-Landau equation

In this section, we recall the derivation in [11] of the one-dimensional Ginzburg-Landau equation. In addition, we clarify and extend certain aspects of this derivation.

To fix ideas, we begin with a simple example. (The extension to general systems of PDEs is given in Section 3.) Consider the one-dimensional Swift-Hohenberg equation

$$\partial_t u = -(D^2+1)^2u + \lambda u - u^3. \quad (2.1)$$

where u is a real-valued function, and $D = \partial_x$.

Viewing $u : \mathbb{R}^2 \to \mathbb{R}$ as a function of x and t, we rewrite the equation in the form

$$0 = \Phi(u,\lambda) = -\partial_t u - (D^2+1)^2u + \lambda u - u^3. \quad (2.2)$$

The linearization around the trivial solution is given by $L_\lambda = -\partial_t - (D^2+1)^2 + \lambda$. The kernel of $L = L_0$ consists of Fourier modes $e^{ikx}e^{i\ell t}$ with $(k,\ell) = (\pm 1, 0)$. Modes with $(k,\ell) \sim (\pm 1, 0)$ are also considered to be critical since they contribute eigenvalues $-(k^2-1)^2 - i\ell$ that are arbitrarily close to 0.

The primary aim is to reduce equation (2.2) to an equation of the form

$$0 = \Psi(A,\lambda) = -\partial_t A + 4D^2A + \lambda A - 3|A|^2A + \cdots \quad (2.3)$$

We use the implicit function to establish a one-to-one correspondence locally (near $(u, \lambda) = (0, 0)$ and $(A, \lambda) = (0, 0)$) between solutions of $0 = \Phi$ and $0 = \Psi$.

As shown in [11, 12], there is a Banach space $\mathcal{X}$ of bounded continuous functions $u : \mathbb{R}^2 \to \mathbb{R}$ that has certain convenient properties. We will describe these properties as they are needed. For the moment, we note that $e^{ikx}e^{i\ell t} \in \mathcal{X}$ for all k, ℓ and that the spectrum of a linear operator on $\mathcal{X}$ is the closure of the set of eigenvalues arising from $e^{ikx}e^{i\ell t}$.

As usual, we define the graph norm $\|u\|_L = \|u\| + \|Lu\|$ and denote by $\mathcal{X}_L$ the subspace of $\mathcal{X}$ with $\|u\|_L < \infty$. Then $\mathcal{X}_L$ is a dense subspace of $\mathcal{X}$ and $\Phi : \mathcal{X}_L \times \mathbb{R} \to \mathcal{X}$ is an analytic nonlinear operator.

Fix $\delta > 0$, and let $\mathcal{X}^\delta$ consist of functions comprising 'critical' Fourier modes with $||k| - 1| < \delta$ and $|\ell| < \delta^2$. Let $\mathcal{X}^c$ be the complementary space (so either $||k| - 1| \geq \delta$ or $|\ell| \geq \delta^2$). An important property of the space $\mathcal{X}$ is that $\mathcal{X}^\delta$ and $\mathcal{X}^c$ are *closed* subspaces and we have the closed splitting $\mathcal{X} = \mathcal{X}^\delta \oplus \mathcal{X}^c$. Similarly, we have the closed splitting $\mathcal{X}_L = \mathcal{X}^\delta \oplus \mathcal{X}_L^c$ (noting that $\mathcal{X}_L^\delta \cong \mathcal{X}^\delta$). The linear operators L_λ preserve the splittings:

$$L_\lambda : \mathcal{X}^\delta \to \mathcal{X}^\delta, \qquad L_\lambda : \mathcal{X}_L^c \to \mathcal{X}^c.$$

Since the eigenvalues of L restricted to $\mathcal{X}_L^c$ are bounded away from 0, the linear operator $L : \mathcal{X}_L^c \to \mathcal{X}^c$ is an isomorphism with bounded inverse L^{-1}. Hence, we can use the implicit function theorem to solve locally for noncritical Fourier modes in terms of critical Fourier modes. More precisely, we use the method of Liapunov-Schmidt reduction (see for example [4]). Define complementary projections

$$E : \mathcal{X} \to \mathcal{X}^c, \qquad I - E : \mathcal{X} \to \mathcal{X}^\delta.$$

Equation (2.2) is equivalent to the equations

$$0 = E\Phi(v + w, \lambda), \qquad 0 = (I - E)\Phi(v + w, \lambda),$$

where $v \in \mathcal{X}^\delta$, $w \in \mathcal{X}_L^c$. By the implicit function theorem, there is a unique analytic function $W : \mathcal{X}^\delta \times \mathbb{R} \to \mathcal{X}_L^c$ defined locally (for (u, λ) near $(0, 0)$) such that

$$E(\Phi(v + W(v, \lambda), \lambda) \equiv 0, \qquad W(0, 0) = 0.$$

Hence, there is locally a one-to-one correspondence between solutions of equation (2.2) and the reduced equation

$$0 = \phi(v, \lambda) = (I - E)\Phi(v + W(v, \lambda), \lambda),$$

where $\phi : \mathcal{X}^\delta \times \mathbb{R} \to \mathcal{X}^\delta$ is an analytic nonlinear operator. A computation (see Appendix B) shows that the reduced equation is

$$0 = \phi(v,\lambda) = (I - E)\big\{L_\lambda v - v^3 - 3v^2 L_\lambda^{-1} E v^3 + O(v^7)\big\}.$$

The next step is to make the substitution

$$v(x,t) = B(x,t)e^{ix} + \overline{B}(x,t)e^{-ix}, \tag{2.4}$$

where B is a complex amplitude function. This is a well-defined substitution: since v incorporates Fourier modes with $(k,\ell) \sim (\pm 1, 0)$, we require that B incorporates Fourier modes with $(k,\ell) \sim (0,0)$. Accordingly, we define $\mathcal{Y}$ (in the same manner as $\mathcal{X}$) to a function space of *complex valued* continuous functions $B : \mathbb{R}^2 \to \mathbb{C}$, and $\mathcal{Y}^\delta$ is the subspace of functions that incorporate only those Fourier modes with $|k| < \delta$ and $|\ell| < \delta^2$. For any fixed $\delta \in (0,1)$, equation (2.4) defines a one-to-one correspondence between $v \in \mathcal{X}^\delta$ and $B \in \mathcal{Y}^\delta$.

After the substitution, we obtain a complex amplitude equation $0 = \psi(B,\lambda)$ where $B : \mathcal{Y}^\delta \times \mathbb{R} \to \mathcal{Y}^\delta$ is analytic. To be more explicit, we introduce the following notation. Let $\mathcal{P} : \mathcal{Y} \to \mathcal{Y}^\delta$ be projection, and define $\mathcal{Q} = I - \mathcal{P}$. Define $L_\lambda[j](B) = e^{-ijx} L_\lambda(Be^{ijx})$. Then we have the equation

$$0 = \psi(B,\lambda) = \mathcal{P}\big\{L_\lambda[1]B - 3|B|^2 B - 3\overline{B}^2 L_\lambda[3]^{-1} B^3 - 18|B|^2 Z - 9B^2\overline{Z} + O(B^7)\big\},$$

where $Z = L_\lambda[1]^{-1}\mathcal{Q}(|B|^2 B)$. Moreover, $L_\lambda[1] = -\partial_t + 4D^2(1-(i/2)D)^2 + \lambda$. (Strictly speaking, we should speak of an operator $\psi : \mathcal{Y}^\delta \times \mathcal{Y}^\delta \times \mathbb{R} \to \mathcal{Y}^\delta$ that is analytic in B, $\overline{B}$ and λ, but our abuse of notation should not lead to any confusion.)

Now we work backwards from an equation of the form (2.3). Keeping in mind the desired unbounded linear terms in the Ginzburg-Landau equation, we define $\mathcal{Y}_2 \subset \mathcal{Y}$ using the graph norm $\|A\|_2 = \|A\| + \|(-\partial_t + D^2)A)\|$. In the obvious notation, we have the closed splittings $\mathcal{Y} = \mathcal{Y}^\delta \oplus \mathcal{Y}^c$, $\mathcal{Y}_2 = \mathcal{Y}^\delta \oplus \mathcal{Y}_2^c$. Consider an equation of the form

$$0 = \Psi(A,\lambda) = M_\lambda A - 3|A|^2 A - 3\overline{A}^2 N_\lambda A^3 + H(A,\lambda), \tag{2.5}$$

where $\Psi : \mathcal{Y}_2 \times \mathbb{R} \to \mathcal{Y}$ is an analytic nonlinear operator, and

(i) $M_\lambda : \mathcal{Y}_2 \to \mathcal{Y}$ is a bounded linear operator and $M = M_0 : \mathcal{Y}_2^c \to \mathcal{Y}^c$ is an isomorphism (so the graph norms corresponding to M and $-\partial_t + D^2$ are equivalent.)

(ii) $N_\lambda : \mathcal{Y} \to \mathcal{Y}$ is a bounded linear operator.

(iii) $H : \mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$ is analytic and satisfies $H = O(A^7)$.

Condition (i) ensures that the critical Fourier modes at $\lambda = 0$ lie in $\mathcal{Y}^\delta$. By the Liapunov-Schmidt method described above, we can locally reduce the equation $0 = \Psi(A, \lambda)$ to an equation $0 = \widetilde{\psi}(B, \lambda)$ where $\widetilde{\psi} : \mathcal{Y}^\delta \times \mathbb{R} \to \mathcal{Y}^\delta$. A calculation shows that

$$\widetilde{\psi}(B, \lambda) = \mathcal{P}\{M_\lambda B - 3|B|^2 B - 3\overline{B}^2 N_\lambda B^3 - 18|B|^2 Z' - 9B^2 \overline{Z}' + O(B^7)\},$$

where $Z' = M_\lambda^{-1} \mathcal{Q}(|B|^2 B)$.

Next, we match up the terms in $\widetilde{\psi}$ and ψ. The operators $\widetilde{\psi}$ and ψ agree at linear order if and only if $\mathcal{P}M_\lambda = \mathcal{P}L_\lambda[1] = -\partial_t + 4D^2(1 - (i/2)D)^2 + \lambda$. It is sufficient to choose $M_\lambda = -\partial_t + 4D^2 M' + \lambda$ where M' is a linear isomorphism such that $\mathcal{P}M' = \mathcal{P}(1 - (i/2)D)^2$. (That is, we choose M' to have symbol η, where η is a C^∞ function, bounded away from zero and infinity, satisfying $\eta(k) = (1 + k/2)^2$ for $|k| < \delta$.) This can be accomplished if $\delta < 1/2$ say.

The cubic terms in $\widetilde{\psi}$ and ψ already coincide, so we turn to the fifth order terms. First, we modify the definition of M_λ so that M_λ and $L_\lambda[1]$ coincide for all Fourier modes with $|k| < 3\delta$. To maintain the requirements in (i) it may be necessary to shrink δ ($\delta < 1/6$ suffices). We then have that $Z' = Z$. In addition, we choose N_λ to be a bounded linear operator coinciding with $L_\lambda[3]^{-1}$ for $|k| < 3\delta$, $|\ell| < 3\delta^2$. Again, we can choose N_λ to have a C^∞ symbol.

Finally, it follows by [11, Proof of Theorem 3.6] that there is an analytic choice of higher order terms H such that the reduced operators $\widetilde{\psi}$ and ψ are identical. In particular, we have proved that there is locally a one-to-one correspondence between solutions of equations (2.2) and (2.5). In other words, there is a one-to-one correspondence between *essential* solutions of the Swift-Hohenberg equation (2.1) and the (nontruncated) Ginzburg-Landau type equation

$$\partial_t u = 4M' D^2 A + \lambda A - 3|A|^2 A + O(A^5) \tag{2.6}$$

We now describe in more detail the structure of equations (2.5) and (2.6).

Normal form symmetry Equations (2.5) and (2.6) have a normal form symmetry $A \mapsto Ae^{i\theta}$ for all θ. By choosing δ small enough, we can arrange that this normal form symmetry occurs to arbitrarily high order in the Taylor expansion of Ψ. More precisely, to arbitrarily high order, the general term in Ψ (neglecting derivatives and multiplication by λ) is of the form $|A|^{2k} A$. In particular, such terms are odd.

We refer to [11] for a proof that this symmetry occurs to arbitrarily high orders. In the tail, there are terms of the form $A^p \overline{A}^q e^{i(p-q-1)x}$. For a proof that such terms are unavoidable, and a discussion of the repercussions of this fact, we again to refer to [11].

Pseudodifferential structure Equations (2.5) and (2.6) are pseudodifferential equations, incorporating terms and derivatives of all orders. Note that the fifth order term $-3\overline{A}^2 N_\lambda A^3$ involves the linear operator N_λ which coincides with $L_\lambda[3]^{-1}$ for small wavenumbers. But $L_\lambda[3] = -\partial_t - ((D+3i)^2+1)^2 + \lambda$, so inversion leads to derivatives of all orders in x and t.

Even at linear order, there are x-derivatives of all orders thanks to the linear isomorphism M'. The symbol of M' is C^∞ but not analytic.

The cubic terms in (2.6) happen to have a polynomial symbol, but this is an artifact of the simplicity of the Swift-Hohenberg equation. If there are quadratic terms, or if there are derivatives in the cubic terms, in equation (2.2), then the situation is more complicated. However, as shown in [11], the cubic term in equation (2.6) can always be chosen to have a C^∞ symbol (see also Appendix B).

In Melbourne [11], it was suggested that the fifth order terms in (2.6) might have discontinuous symbols due to the presence of the projection operator $\mathcal{Q}$ in ψ. This fear turns out not to be realized, since exactly the same terms occur in $\widetilde{\psi}$. In Appendix B, we prove that this 'coincidence' at fifth order occurs for all operators Φ regardless of the nature of the quadratic, cubic and quartic terms.

Proposition 2.1 *Consider the generalized Swift-Hohenberg equation*

$$0 = -\partial_t u - (D^2+1)^2 u + \lambda u + N(u,\lambda), \tag{2.7}$$

where $N : \mathcal{X}_L \times \mathbb{R} \to \mathcal{X}$ is any constant coefficient analytic nonlinear differential operator satisfying $N(0,\lambda) = 0$, $(dN)_{0,\lambda} = 0$. Then locally, solutions of (2.7) are in one-to-one correspondence with solutions of an equation of the form

$$0 = \Psi(A,\lambda) = -\partial_t A + 4M'D^2 A + \lambda A + R(A,\lambda), \tag{2.8}$$

where $\Psi : \mathcal{Y}_2 \times \mathbb{R} \to \mathcal{Y}$ is analytic, $M' : \mathcal{Y} \to \mathcal{Y}$ is a linear isomorphism (on the whole of $\mathcal{Y}$), $R : \mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$ is analytic, and $(dR)_{0,\lambda} = 0$. Moreover,

$$R(A,\lambda) = C_3(A,\overline{A};A,\lambda) + C_5(A,\overline{A};A,\overline{A};A,\lambda) + O(A^7),$$

where $C_3(\cdot,\lambda)$ and $C_5(\cdot,\lambda)$ are bounded multilinear operators with C^∞ symbols.

Conjecture 2.2 In Proposition 2.1, we can arrange that the nonlinear terms through arbitrarily high order have smooth symbols.

Relation to the Ginzburg-Landau equation. I As described in [13], a lesson to be learnt from Landau theory is that reduced equations cannot necessarily be truncated at lowest order. A two-step approach is preferable: I. Reduce to a nontruncated version of the desired equation, and II. Determine to what extent truncation is legitimate.

Consider the Swift-Hohenberg equation (2.1). Comparing the reduced equation (2.6) with the Ginzburg-Landau equation (1.1), we see that step I has been accomplished except for the factor M' in the linear terms. In fact, the linear isomorphism M' can be chosen arbitrarily close to the identity in the following sense. As far as the linear terms are concerned, it suffices that M' coincides with $(1-(i/2)D)^2$ on $\mathcal{X}^\delta$. But the latter operator has norm $(1+\delta/2)^2 < 1+2\delta$ for δ small. Hence we can choose M' such that $\|M'-I\| < 2\delta$.

(If we require also that the fifth order terms have a smooth symbol, then we require that M' coincides with $(1-(i/2)D)^2$ on $\mathcal{X}^{3\delta}$. For δ small enough we can arrange that $\|M'-I\| < 4\delta$.)

For the generalized Swift-Hohenberg equation (2.7), we similarly choose C_3 in equation (2.8) in such a way that $C_3(A,\overline{A};A,\lambda) = c_2|A|^2A + O(\delta A^3, \lambda A^3)$ for some $c_2 \in \mathbb{R}$. In this way, we obtain the following result.

Proposition 2.3 *There exists a $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$, there is locally a one-to-one correspondence between solutions of the generalized Swift-Hohenberg equation* (2.7) *and solutions of an equation of the form*

$$0 = \Psi_\delta(A,\lambda) = -\partial_t A + 4D^2A + \lambda A + c_2|A|^2A + H_\delta(A,\lambda), \tag{2.9}$$

where $\Psi_\delta : \mathcal{Y}_2 \times \mathbb{R} \to \mathcal{Y}$ and $H_\delta : \mathcal{Y} \times \mathbb{R} \to \mathcal{Y}$ are analytic for each δ, and $H_\delta(A,\lambda) = O(\delta D^2 A, \delta A^3, \lambda A^3, A^5)$.

For each fixed δ in Proposition 2.3, the correspondence holds locally on a full neighborhood of $(0,0) \in \mathcal{Y}_2 \times \mathbb{R}$. We note however that the neighborhood shrinks to zero as $\delta \to 0$. To overcome this difficulty, we set $\delta = \lambda^{\nu'}$ where $0 < \nu' < 1/2$. For λ fixed, the implicit function theorem holds on neighborhoods of order $\|(L|_{\mathcal{X}_L^c})^{-1}\|^{-1}$ and this is of order $\delta^2 = \lambda^{2\nu'}$. (The estimate in $\mathcal{Y}_2$ is identical.) Choosing $\nu \in (0,\nu')$, we obtain the following version of Theorem 1.1.

Theorem 2.4 *There is a $\lambda_0 > 0$ such that for each λ, ν with $|\lambda| < \lambda_0$, $0 < \nu < 1/2$, there is locally a one-to-one correspondence between solutions of the generalized Swift-Hohenberg equation* (2.7) *and solutions of an equation of the form*

$$0 = \Psi_{\lambda,\nu}(A) = -\partial_t A + 4D^2A + \lambda A + c_2|A|^2A + H_{\lambda,\nu}(A), \tag{2.10}$$

where $\Psi_{\lambda,\nu} : \mathcal{Y}_2 \to \mathcal{Y}$ and $H_{\lambda,\nu} : \mathcal{Y} \to \mathcal{Y}$ are analytic for each λ, and $H_{\lambda,\nu}(A) = O(\lambda^\nu D^2 A, \lambda^\nu A^3, A^5)$. Moreover, the neighborhood of validity includes solutions satisfying $\|u\|_L, \|A\|_2 \le |\lambda|^{2\nu}$.

Relation to the Ginzburg-Landau equation. II A standard method for passing from nontruncated reduced equations to truncated (Landau) equations is to scale and apply the implicit function theorem; see for example Sattinger [18]. Then all nondegenerate solutions of the truncated equation correspond to smooth branches of solutions to the reduced equations and hence to smooth branches of solutions to the underlying equations.

The same method applies here, except that the scaling yields an equation that is everywhere discontinuous in λ except at $\lambda = 0$. Fortunately, this is sufficient regularity to apply the implicit function theorem (though the resulting branches of solutions are again everywhere discontinuous in λ except at $\lambda = 0$).

Here are the details. Starting from equation (2.8), we make the standard scaling

$$A(x,t) = \epsilon A_0(X,T), \quad X = \epsilon x, \quad T = \epsilon^2 t, \quad \lambda = \epsilon^2.$$

Since M' is bounded with smooth symbol $\eta(k) = 1 + O(k)$, we can write

$$M' = \eta(-i\partial_x) = \eta(-i\epsilon\partial_X) = 1 + O(\epsilon).$$

Similarly, $C(A, \overline{A}; A, \lambda) = c_2|A|^2 A + O(\epsilon A^3)$. Substituting into (2.6) and dividing throughout by ϵ^3, we obtain the equation $0 = G(A_0, \epsilon)$ where $G : \mathcal{Y}_2 \times \mathbb{R} \to \mathcal{Y}$ is analytic in A_0 and is given by

$$G(A_0, \epsilon) = -\partial_T A_0 + 4\partial_X^2 A_0 + A_0 + c_2|A_0|^2 A_0 + O(\epsilon).$$

We caution that this equation has no regularity in ϵ (scaling x and t in terms that involve the projection $\mathcal{Q}$ is a discontinuous operation). However, G is analytic in A_0 for each fixed ϵ and is jointly continuous at $(A_0, 0)$ for each A_0. As shown in Appendix A, the implicit function theorem is valid, and we conclude the following special case of Theorem 1.2.

Theorem 2.5 *Suppose that A_0 is a nondegenerate solution to the equation $0 = G(A_0, 0)$ (by nondegenerate, we mean that $(dG)_{A_0,0} : \mathcal{Y}_2 \to \mathcal{Y}$ has a bounded inverse). Then locally, there is a unique family of solutions $A_0(\epsilon)$ to the equation $G(A_0, \epsilon) = 0$. This family of solutions corresponds to a (discontinuous) branch of solutions to the generalized Swift-Hohenberg equation (2.7) of the form (1.3) (with $k_c = 1$, $v_0 = 1$).*

3 Systems of PDEs and Universality

So far, we have stated our results for the (generalized) Swift-Hohenberg equation (2.7). We now show that these results hold universally for systems of PDEs.

The general setting is systems of PDEs involving functions $u = u(x, z, t) : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}^s$ where there is a single unbounded domain variable $x \in \mathbb{R}$ and arbitrarily many bounded domain variables $z \in \Omega$. (So the spatial domain $\mathbb{R} \times \Omega$ is cylindrical.) For ease of exposition, we shall restrict to the case when there are no bounded domain variables ($\Omega = \{0\}$) and refer to Melbourne [12] for the general case.

We also assume Euclidean $\mathbf{E}(1)$ symmetry, where $\mathbf{E}(1)$ consists of translations $x \mapsto x + b$ and reflections $x \mapsto -x$. We suppose that $\mathbf{E}(1)$ acts on functions $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^s$ as

$$u(x,t) \mapsto u(x-b,t), \qquad u(-x,t) \mapsto Au(x,t), \tag{3.1}$$

where A is an $s \times s$ orthogonal matrix with $A^2 = I$. Such actions are called *physical* in [12].

A suitable function space is given by $\mathcal{X}^s = (\mathcal{X})^s$. We consider nonlinear partial differential operators $\Phi^s : \mathcal{X}^s \times \mathbb{R} \to \mathcal{X}^s$ satisfying the following properties:

(H1) Φ^s is $\mathbf{E}(1)$-equivariant: $\Phi^s(\gamma u, \lambda) = \gamma \Phi^s(u, \lambda)$ for all $\gamma \in \mathbf{E}(1)$, $\lambda \in \mathbb{R}$, and u in the domain of Φ^s.

We note that the (generalized) Swift-Hohenberg equation is equivariant under the *scalar* action of $\mathbf{E}(1)$, where $s = 1$ and $A = 1$. Equivariance with respect to translations $u(x,t) \mapsto u(x-b,t)$ implies that the partial differential operators are constant coefficient.

(H2) $\Phi^s(0, \lambda) \equiv 0$.

(H3) There is a linear operator $L^s : \mathcal{X}^s \to \mathcal{X}^s$ such that $\Phi^s : \mathcal{X}^s_{L^s} \times \mathbb{R} \to \mathcal{X}^s$ is analytic, and $(d\Phi^s)_{0,0} = L^s$.

(H4) The kernel of L^s consists of Fourier modes $v_0 e^{ikx} e^{i\ell t}$ with $(k, \ell) = (\pm k_c, 0)$, where $k_c > 0$ and $v_0 \in \mathbb{C}^s$.

Hypothesis (H2) states that there is a fully symmetric trivial solution $u \equiv 0$. By (H4), the trivial solution undergoes a bifurcation at $\lambda = 0$ and this is a steady-state bifurcation (since the critical modes have $\ell = 0$) with nonzero critical wavenumber $k_c > 0$. Under certain technical assumptions, Melbourne [12] proves that generically $\dim \ker L^s = 2$ in (H4). When these technical assumptions hold and Hypotheses (H1)–(H4) are valid, we say that the $\mathbf{E}(1)$-equivariant system of PDEs $0 = \Phi^s(u, \lambda)$ undergoes a *steady-state bifurcation with nonzero critical wavenumber.*

Lemma 3.1 ([12]) *Suppose that an $\mathbf{E}(1)$-equivariant system of PDEs $0 = \Phi^s(u, \lambda)$ undergoes a steady-state bifurcation with nonzero critical wavenumber k_c. Generically, there exists a nonlinear (pseudodifferential) operator $\Phi : \mathcal{X} \times \mathbb{R} \to \mathcal{X}$ (with $s = 1$) such that*

(1) Φ *is equivariant with respect to the scalar action of* $\mathbf{E}(1)$ *(*$s = 1$ *and* $A = 1$ *in* (3.1)*).*

(2) $\Phi(0, \lambda) \equiv 0$.

(3) There is a linear operator $L : \mathcal{X} \to \mathcal{X}$ *such that* $\Phi : \mathcal{X}_L \times \mathbb{R} \to \mathcal{X}$ *is analytic, and* $(d\Phi)_{0,0} = L$.

(4) The kernel of L *consists of Fourier modes* $e^{ikx}e^{i\ell t}$ *with* $(k, \ell) = (\pm k_c, 0)$*. Moreover, the symbol of* L *is* C^∞ *and has the form* $-i\ell - \alpha(k^2 - k_c^2)^2 + O(k^2 - k_c^2)^3$*, where* $\alpha \in \mathbb{R}$.

(5) Locally there is a one-to-one correspondence between solutions of $0 = \Phi^s$ *and* $0 = \Phi$.

Combining this lemma with the techniques in Section 2, we obtain (with $c_0 = 4\alpha$):

Theorem 3.2 *Suppose that an* $\mathbf{E}(1)$*-equivariant system of PDEs* $0 = \Phi^s(u, \lambda)$ *undergoes a steady-state bifurcation with nonzero critical wavenumber. Generically,*

(a) *There is locally a one-to-one correspondence between solutions of* $0 = \Phi^s(u, \lambda)$ *and solutions of an equation of the form*

$$0 = \Psi(A, \lambda) = -\partial_t A + c_0 M' D^2 A + c_1 M'' \lambda A + C(A, \overline{A}, A, \lambda) + H(A, \lambda), \tag{3.2}$$

where $\Psi : \mathcal{Y}_2 \times \mathbb{R} \to \mathcal{Y}$ *is an analytic operator,* $c_0, c_1, c_2 \in \mathbb{R}$*,* M'*,* M'' *are linear isomorphisms on* $\mathcal{Y}$*,* $C(\cdot, \lambda)$ *is a bounded trilinear operator on* $\mathcal{Y}$*, and* $H = O(A^5)$ *is analytic on* $\mathcal{Y}$.

(b) *For any* $q \geq 1$*, there exists* $\delta_0 > 0$ *such that for any* $\delta \in (0, \delta_0)$*, equation* (3.2) *can be written as*

$$0 = \Psi_\delta(A, \lambda) = -\partial_t A + c_0 D^2 A + c_1 \lambda A + c_2 |A|^2 A + O(\delta D^2 A, \delta \lambda A, \delta A^3, A^5), \tag{3.3}$$

where

(i) The Taylor expansion of Ψ_δ *truncated at order* A^q *is equivariant with respect to the* S^1 *action* $A \mapsto e^{i\theta} A$*. In particular* Ψ_δ *is odd and constant coefficient through order* q.

(ii) The symbols of terms in Ψ_δ *through fifth order (and conjecturally through order* q*) are* C^∞.

(c) *Let* $0 < \nu < \nu' < 1/2$ *and set* $\delta = \lambda^{\nu'}$ *in part (b). Then equation* (3.3) *becomes*

$$0 = \Psi_{\lambda,\nu}(A) = -\partial_t A + c_0 D^2 A + c_1 \lambda A + c_2 |A|^2 A + O(\lambda^\nu D^2 A, \lambda^{\nu+1} A, \lambda^\nu A^3, A^5). \tag{3.4}$$

For each $\lambda > 0$*, there is locally a one-to-one correspondence between solutions* u *of* $0 = \Phi^s(\cdot, \lambda)$ *and solutions* A *of* $0 = \Psi_{\lambda,\nu}$ *and the neighborhood of validity includes solutions satisfying* $\|u\|_{L^s}, \|A\|_2 \leq |\lambda|^{2\nu}$*.*

(d) *Nondegenerate solutions of*

$$0 = -\partial_T A_0 + c_0 \partial_X^2 A_0 + c_1 A_0 + c_2 |A_0|^2 A_0$$

correspond to branches of solutions to the underlying system of PDEs $0 = \Phi^s(u, \lambda)$ *of the form* (1.3) *(with* $v_0(z) = v_0 \in \mathbb{C}^s$*).*

4 Complex Ginzburg-Landau Equation

The methods for steady-state bifurcation with nonzero critical wavenumber apply equally to Hopf bifurcation with zero critical wavenumber. (See [13] for a nontechnical account of the results.) The set up is as in Section 3 with Hypothesis (H4) replaced by

(H4′) The kernel of L^s consists of Fourier modes $v_0 e^{ikx} e^{i\ell t}$ with $(k, \ell) = (0, \pm\omega)$, where $\omega > 0$ and $v_0 \in \mathbb{C}^s$.

The conclusions in Theorem 3.2 are unchanged except that now $c_0, c_2 \in \mathbb{C}$, and in part (b)(i) Ψ_δ is constant coefficient to all orders but autonomous only through arbitrary finite order q (see [13]).

A Implicit function theorem

The standard implicit function theorem is stated for C^k functions, $k \geq 1$, and leads to implicit functions that are C^k. If we require less regularity for the implicit function, then the hypotheses may be weakened and the standard proof goes through with no change. Renardy [17, Theorem 2.1]) pointed this out under hypotheses that guaranteed a continuous implicit function, and we generalize further to obtain an implicit function that need not be continuous except at one point. In particular, our version of the implicit function theorem is sufficiently general to prove Theorem 2.5. Our exposition follows Renardy [17].

Theorem A.1 *Let X, Y, Z be Banach spaces and $F : X \times Y \to Z$ a mapping such that*

(i) $F(x_0, 0) = 0$,

(ii) $F(\cdot, y)$ *is* C^1 *for each fixed* $y \in Y$,

(iii) F *is continuous at* $(x_0, 0)$,

(iv) $D_x F$ *is continuous at* $(x_0, 0)$,

(v) $(D_x F)_{x_0,0} : X \to Z$ *is an isomorphism.*

Then in a small enough neighborhood of $(x_0, 0)$ in $X \times Y$, there is a unique solution $x = f(y)$ to the equation $F(x, y) = 0$. The function f satisfies $f(0) = x_0$ and is continuous at 0.

Proof Let $L = (D_x F)_{x_0,0}$ and define $G(x, y) = x - L^{-1}F(x, y)$. Let $H = LG$. By assumption (ii),

$$\begin{aligned} H(x, y) - H(x', y) &= \int_0^1 (d_x H)_{x'+t(x-x'),y}(x - x')dt \\ &= \int_0^1 [(D_x F)_{x_0,0} - (D_x F)_{x'+t(x-x'),y}]dt\,(x - x'). \end{aligned}$$

Choose $\epsilon > 0$ such that $\|\epsilon L^{-1}\| < 1/2$. By (iv), we can choose r' such that the norm of the integrand is smaller than ϵ for $\|x - x_0\|, \|x' - x_0\|, \|y\| \leq r'$. Consequently, $\|H(x, y) - H(x', y)\| \leq \epsilon\|x - x'\|$, and hence

$$\|G(x, y) - G(x', y)\| < \tfrac{1}{2}\|x - x'\|. \tag{A.1}$$

Now let $B \subset X$ denote the closed ball of radius r' centered at x_0. Then, for $x \in B$, we have $\|G(x, y) - G(x_0, y)\| < \frac{1}{2}\|x - x_0\| \leq r'/2$. Hence

$$\|G(x, y) - x_0\| \leq \|G(x_0, y) - x_0\| + r'/2.$$

By (i), $G(x_0, 0) - x_0 = 0$. Hence, by (iii), there is an $r'' > 0$ such that $\|G(x_0, y) - x_0\| < r'/2$, for $\|y\| \leq r''$. Let $r = \min(r', r'')$. Then

$$G(x, y) \in B \quad \text{for all } x, y \text{ with } x \in B \text{ and } \|y\| \leq r. \tag{A.2}$$

Equations (A.1) and (A.2) show that $G(\cdot, y)$ is a contraction mapping on the complete metric space B for each y with $\|y\| < r$. By the contraction mapping theorem, G has a unique fixed point $x = f(y) \in B$ for each y with $\|y\| < r$. In particular, $F(f(y), y) \equiv 0$. ∎

B Smoothness of symbols

In our derivation of the reduced equation $0 = \Psi(A, \lambda)$, it is generally the case that the terms in Ψ have smooth symbols through order five (Proposition 2.1). The proof below is a tedious and unenlightening calculation. We suspect that a more sensible approach would prove smoothness of symbols through arbitrarily high order, but we have not found this approach and Conjecture 2.2 remains a conjecture.

We can disregard λ, so we consider a general analytic nonlinear operator $\Phi : \mathcal{X}_L \to \mathcal{X}$ satisfying $\Phi(0) = 0$, $(d\Phi)_0 = L$. We assume that the linear and nonlinear terms of Φ through fifth order have smooth symbols.

Liapunov-Schmidt reduction leads to the analytic operator $\phi : \mathcal{X}^\delta \to \mathcal{X}$ given by $\phi(v) = (I-E)\Phi(v+W(v))$ where $E\Phi(v+W(v)) \equiv 0$. The Taylor expansion of Φ can be computed by implicit differentiation; see for example [4, Chapter I, Section 3(e)]. Extending the calculations in [4] through fifth order, we find that

$$
\begin{aligned}
&(d\phi)v = Lv, \qquad (d^2\phi)(v,v) = (I-E)(d^2\Phi)(v,v),\\
&(d^3\phi)(v,v,v) = (I-E)\big\{(d^3\Phi)(v,v,v) + 3(d^2\Phi)(v,(d^2W)(v,v))\big\},\\
&(d^4\phi)(v,v,v,v) = (I-E)\big\{(d^4\Phi)(v,v,v,v) + 6(d^3\Phi)(v,v,(d^2W)(v,v))\\
&\qquad\qquad + 4(d^2\Phi)(v,(d^3W)(v,v,v)) + 3(d^2\Phi)((d^2W)(v,v),(d^2W)(v,v))\big\},\\
&(d^5\phi)(v,v,v,v,v) = (I-E)\big\{(d^5\Phi)(v,v,v,v,v) + 10(d^4\Phi)(v,v,v,(d^2W)(v,v))\\
&\qquad\qquad + 15(d^3\Phi)(v,(d^2W)(v,v),(d^2W)(v,v)) + 10(d^3\Phi)(v,v,(d^3W)(v,v,v))\\
&\qquad\qquad + 10(d^2\Phi)((d^2W)(v,v),(d^3W)(v,v,v)) + 5(d^2\Phi)(v,(d^4W)(v,v,v,v))\big\},
\end{aligned}
$$

where

$$
\begin{aligned}
&(d^2W)(v,v) = -L^{-1}E(d^2\Phi)(v,v),\\
&(d^3W)(v,v,v) = -L^{-1}E\big\{(d^3\Phi)(v,v,v) + 3(d^2\Phi)(v,(d^2W)(v,v))\big\},\\
&(d^4W)(v,v,v,v) = -L^{-1}E\big\{(d^4\Phi)(v,v,v,v) + 6(d^3\Phi)(v,v,(d^2W)(v,v))\\
&\qquad\qquad + 4(d^2\Phi)(v,(d^3W)(v,v,v)) + 3(d^2\Phi)((d^2W)(v,v),(d^2W)(v,v))\big\}.
\end{aligned}
$$

Now, quadratic interactions involving $v \in \mathcal{X}^\delta$ leads to wavenumbers (k,ℓ) where k is within distance 2δ of 0 or ± 2. Choosing $\delta < 1/3$ ensures that such terms lie in the kernel of $I-E$ so that ϕ contains no quadratic terms. Similarly, choosing $\delta < 1/5$ ensures that there are no quartic terms. Thus, for $\delta < 1/5$, we have simplified expressions where $(d^2\phi) = 0$, $(d^4\phi) = 0$. Moreover, (d^2W) and (d^4W) no longer explicitly include the projection E Note that the projection E occurs explicitly only in (d^3W) and hence occurs implicitly in the third term in the expression for (d^4W). It follows that only the final three terms in $(d^5\phi)$ contain the projection E and that the remaining terms through fifth order have the form $(I-E)\{$expression with smooth symbols$\}$. Writing $F(v,v,v) = (d^3\Phi)(v,v,v) - 3(d^2\Phi)(v, L^{-1}(d^2\Phi)(v,v))$, we have

$$
\begin{aligned}
&(d\phi)v = Lv, \qquad (d^3\phi)(v,v,v) = (I-E)F(v,v,v),\\
&(d^5\phi)(v,v,v,v,v) = 5(I-E)\big\{2(d^3\Phi)(v,v,(d^3W)(v,v,v))\\
&-2(d^2\Phi)(L^{-1}(d^2\Phi)(v,v),(d^3W)(v,v,v)) + (d^2\Phi)(v,(d^4W)(v,v,v,v)) + \cdots\big\},
\end{aligned}
$$

where

$$
\begin{aligned}
(d^3W)(v,v,v) &= -L^{-1}EF(v,v,v),\\
(d^4W)(v,v,v,v) &= 4L^{-1}(d^2\Phi)(v, L^{-1}EF(v,v,v)) + \cdots
\end{aligned}
$$

(Here, $\cdots$ denotes terms with smooth symbols.)

Observe that

$$F(v,v,w) = (d^3\Phi)(v,v,w) - (d^2\Phi)(L^{-1}d^2\Phi(v,v),w) - 2(d^2\Phi)(v,L^{-1}d^2\Phi(v,w)).$$

It follows that

$$(d^5\phi)(v,v,v,v,v) = -10(I-E)F(v,v,L^{-1}EF(v,v,v)) + \cdots$$

Next we make the substitution $v \mapsto Be^{ix} + \overline{B}e^{-ix}$, If N is an s-linear operator, we define

$$N[j_1,\dots,j_s](B_1,\dots,B_s) = N(B_1e^{ij_1x},\dots,B_se^{ij_sx})e^{-i(j+1+\cdots+j_s)x}.$$

In this notation, we have

$$(d\phi)v \mapsto e^{ix}L[1]B + \text{c.c.}, \qquad (d^3\phi)(v,v,v) \mapsto 3e^{ix}\mathcal{P}F[1,1,-1](B,B,\overline{B}) + \text{c.c.}$$

The fifth order terms are more complicated, but we are only concerned with the terms that involve E. Hence we can write

$$L^{-1}EF(v,v,v) \mapsto 3e^{ix}L[1]^{-1}\mathcal{Q}F[1,1,-1](B,B,\overline{B}) + \text{c.c.} + \cdots$$

so that

$$(d^5\phi)(v,v,v,v,v) \mapsto -30e^{ix}\mathcal{P}\{F[1,1,-1](B,B,\overline{Y}) + 2F[1,-1,1](B,\overline{B},Y) + \cdots\} + \text{c.c.}$$

where $Y = L[1]^{-1}\mathcal{Q}F[1,1,-1](B,B,\overline{B})$.

From the other direction, $\Psi(A)$ reduces to $\psi(B)$ with

$$(d\psi)B = (d\Psi)B, \qquad (d^3\psi)(B,B,\overline{B}) = \mathcal{P}(d^3\Psi)(B,B,\overline{B}),$$
$$(d^5\psi) = \mathcal{P}\big\{(d^5\Psi) - 10(d^3\Psi)(\cdot,\cdot,(d\Psi)^{-1}\mathcal{Q}(d^3\Psi))\big\}.$$

Formally, we choose $(d\Psi) = L[1]$ and $(d^3\Psi) = 3F[1,1,-1]$ ensuring that the linear and cubic terms match. Then the second term in $(d^5\psi)$ is

$$-30\mathcal{P}\{F[1,1,-1](\cdot,\cdot,\overline{Y}) + 2F[1,-1,1](\cdot,Y,\bar{\cdot})\}$$

where $Y = L[1]^{-1}\mathcal{Q}F[1,1,-1]$. Fortunately(?), this matches up with the fifth order terms that we wrote down starting from Φ. Now we simply choose $(d^5\Psi)$ to match up with the fifth order terms that we did not write down (the ones with smooth symbols).

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