Versal Unfoldings of Equivariant Linear Hamiltonian Vector Fields *

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Abstract

We prove an equivariant version of Galin's theorem on versal deformations of infinitesimally symplectic matrices. Matrix families of codimension zero and one are classified, and the results are used to study the movement of eigenvalues in one parameter families.

1 Introduction

In dissipative dynamical systems, an equilibrium (trivial solution) can lose stability when eigenvalues of a linearized vector field cross the imaginary axis as a bifurcation parameter is varied. Generically, such a loss of stability occurs at a steady-state bifurcation where a simple eigenvalue passes through 0, or at a Hopf bifurcation where a pair of simple complex conjugate eigenvalues passes through the imaginary axis away from zero.

Suppose now that the dynamical system is equivariant with respect to a compact Lie group of symmetries Γ, and that the trivial solution is invariant under Γ. Then the eigenvalues passing through the imaginary axis

*Appeared: *Math. Proc. Camb. Phil. Soc.* **114** (1993) 559-573
†Supported in part by NSF Grant DMS-9101836 and by the Texas Advanced Research Program (003652037)

need not be simple, see Golubitsky, Stewart and Schaeffer [**?**]. Indeed, in a steady-state bifurcation generically the multiplicity of the eigenvalue passing through zero has multiplicity equal to the dimension of an absolutely irreducible representation of Γ. (These dimensions are finite but may be arbitrarily large as in the case of $\Gamma = \mathbf{O}(3)$.) The situation for Hopf bifurcation is analogous. The corresponding local dynamics for the steady-state and Hopf bifurcations with symmetry are far richer than in the bifurcations without symmetry. Nevertheless, the expected movement of eigenvalues and the resulting change in stability of the trivial solution is (up to multiplicity) identical in the symmetric and nonsymmetric contexts.

The situation is considerably more complicated for local bifurcations in Hamiltonian systems. One intrinsic difficulty is that it is impossible to prove that a solution is asymptotically stable by a purely linear analysis. A necessary condition is that the solution is spectrally stable, that is the linearization is semisimple and all the eigenvalues lie on the imaginary axis. If there are eigenvalues lying off the imaginary axis, then the solution is both linearly and nonlinearly unstable.

Suppose that we agree to concentrate on the linear aspects of these local bifurcations. Then the simplified question is how can a trivial solution lose spectral stability and become linearly unstable. It is easy to see that this can only happen if eigenvalues moving along the imaginary axis happen to collide. If they collide at zero there is a steady-state bifurcation, otherwise there is a 1-1 resonance. After the bifurcation, the eigenvalues either remain on the imaginary axis or they move into the left and right-half of the complex plane. We say that the eigenvalues *pass* or *split*. It is splitting of eigenvalues that corresponds to loss of spectral stability and is dangerous in the sense of Krein. It is well-known that, in the absence of symmetry, generically the colliding eigenvalues are simple and split. A 1-1 resonance at which splitting occurs is often called a *Hamiltonian Hopf bifurcation*, see [**?**].

Again the presence of a compact Lie group of symmetries Γ changes the expected behaviour at a local bifurcation. However this time the effect of Γ is not restricted to forcing multiplicity of eigenvalues. Indeed, Γ-equivariance strongly alters the expected movement of eigenvalues and has corresponding implications for the spectral stability of the trivial solution. The required computations have been performed in the case of steady-state bifurcation by Golubitsky and Stewart [**?**] and in the case of the 1-1 resonance by Dellnitz, Melbourne and Marsden [**?**]. However these computations are ad hoc and mode interactions, for example, would very likely be intractable by these methods.

Even when there is no symmetry, the computation of movement of eigenvalues is nontrivial using ad hoc methods. Galin [?] (see also Koçak [?], and when there is a time-reversal symmetry, Wan [?], [?])) was able to tackle such problems in a more organized manner by computing normal forms for parametrized families of linear Hamiltonian vector fields. Then the computation can be performed using the simpler normal forms. In this paper, we present the equivariant analogue of Galin's results.

We summarize Galin's results leaving precise definitions for later. Let J be a nonsingular skew-symmetric real $m \times m$ matrix. Necessarily $m = 2n$ is even. Let $\mathbf{sp}_{2n}$ denote the vector space of *infinitesimally symplectic* matrices A, those satisfying $AJ + JA^t = 0$. Two matrices A and A' are equivalent if there is a nonsingular matrix P that is *symplectic* ($PJP^T = J$) such that $PAP^{-1} = A'$. The orbit of A consists of all matrices in $\mathbf{sp}_{2n}$ that are equivalent to A. Define $\operatorname{codim} A$ to be the codimension of the orbit of A in $\mathbf{sp}_{2n}$. It follows from results of Arnold [?] that $\operatorname{codim} A$ is the minimal number of parameters required in a versal unfolding and is equal to the dimension of the centralizer of A. Moreover a versal unfolding of A can be computed by exhibiting a basis for the centralizer. Galin's codimension formula gives $\operatorname{codim} A$ in terms of the sizes of the Jordan blocks of A.

Of course, eigenvalues are invariants of the orbits in $\mathbf{sp}_{2n}$, and for this reason it is not possible to have an orbit of codimension zero. Moreover, the codimension of an orbit has an arbitrarily large lower bound depending on the size of the matrices under consideration. To get around these problems it is convenient to identify orbits that share certain common features if the exact values of the eigenvalues are ignored. The resulting collections of orbits are called bundles and have the property that there are bundles of codimension zero. Moreover the union of bundles of codimension zero forms an open dense subset of $\mathbf{sp}_{2n}$. In addition, the codimension of a bundle is independent of the size of the matrices, and it is possible to isolate those parts of a matrix that have positive codimension.

Galin's results are obtained by computing the centralizers of the normal forms computed by Williamson [?] and applying Arnold's results [?] on parametrized families of matrices. We obtain an equivariant version of Galin's results by computing the centralizers of the equivariant normal forms given in Melbourne and Dellnitz [?], and applying Arnold's results. In stating the results, we shall assume some familiarity with the notation and results in [?], in particular Sections 1 and 2 of that paper.

In Section **??** we state the equivariant problem precisely, and proceed as in [?] to reduce to a nonequivariant problem over a real division ring. Then in Section **??** we present codimension formulas over the three nonisomorphic real division rings $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (the real, complex and quaternionic numbers). The codimension formula over $\mathbb{R}$ is Galin's original formula.

In Section **??** we define the codimension of a bundle and list versal unfoldings for bundles of codimension zero and one. As an application, in Section **??** we give simplified proofs of results in [?] and [?]. Finally, in Section **??** we verify the codimension formulas presented in Section **??**.

2 The equivariant Galin theorem

We begin by recalling some notation from [?]. Suppose that Γ is a compact Lie group acting on $\mathbb{R}^n$. By a Γ-*equivariant matrix* we mean an $n \times n$ real matrix that commutes with the action of Γ. Let $\mathbf{sk}_\Gamma$ denote the set of nonsingular skew-symmetric Γ-equivariant matrices. If $R \in \mathbf{sk}_\Gamma$, we define $\mathbf{sp}_\Gamma(R)$ to be the vector space of Γ-equivariant matrices satisfying $MR + RM^T = 0$. The pair of matrices (M, R) is called a Γ-symplectic pair. Two Γ-symplectic pairs (M, R) and (M', R') are *equivalent*, $(M, R) \sim (M', R')$, if there is a nonsingular Γ-equivariant matrix P such that

$$PMP^{-1} = M', \quad PRP^T = R'.$$

Suppose that (M, R) is a Γ-symplectic pair. An *unfolding* of M is a parametrized family of matrices $M(\alpha) \in \mathbf{sp}_\Gamma(R)$ where $\alpha \in \mathbb{R}^k$ for some k, $M(\alpha)$ is smooth (C^∞) in α in a neighborhood of 0, and $M(0) = M$. Suppose that $M(\alpha)$ and $N(\beta)$ are two unfoldings of M, with $\alpha \in \mathbb{R}^k$ and $\beta \in \mathbb{R}^\ell$. We say that $N(\beta)$ *factors through* $M(\alpha)$ if there is a mapping $\phi : \mathbb{R}^\ell \to \mathbb{R}^k$ smooth near 0 satisfying $\phi(0) = 0$ such that $(N(\beta), R) \sim (M(\phi(\beta)), R)$, for β near 0, where the family of transformations giving the equivalence is an unfolding of the identity matrix. An unfolding $M(\alpha)$ of M is *versal* if every unfolding of M factors through $M(\alpha)$.

The *orbit* of M consists of all matrices $M' \in \mathbf{sp}_\Gamma(R)$ such that $(M, R) \sim (M', R)$. Finally we define the *centralizer* $C(M, R)$ to consist of those matrices in $\mathbf{sp}_\Gamma(R)$ that commute with M.

Theorem 2.1 *Let (M, R) be a Γ-symplectic pair. Then M has a versal unfolding $M(\alpha)$. Moreover if k is the minimal number of parameters in a versal unfolding, then* *(a) k is equal to the codimension of the orbit of M in $\mathbf{sp}_\Gamma(R)$.*

(b) $k = \dim C(M, R)$.

(c) A versal unfolding of M is given by

$$M(\alpha) = M + \alpha_1 G_1^T + \cdots + \alpha_k G_k^T,$$

where $\{G_1, \ldots, G_k\}$ is a basis for $C(M, R)$.

Proof When $\Gamma = \mathbf{1}$, the result reduces to Lemmas 2 and 3 and Theorem 1 in Galin [**?**] which themselves follow from results of Arnold [**?**]. The proof for a general compact Lie group Γ is completely analogous to this special case. ∎

We define the *codimension of M*, $\operatorname{codim} M$, to be the number k characterized in Theorem **??**. It follows from the theorem that we can calculate $\operatorname{codim} M$ and a versal unfolding of M by computing the centralizer $C(M, R)$.

To simplify the computations, we exploit the isotypic decomposition of $\mathbb{R}^n$ under the action of Γ. Recall from Subsection 2.1 of [**?**] that

$$M \cong M_1 \oplus \cdots \oplus M_\ell, \quad R \cong R_1 \oplus \cdots \oplus R_\ell,$$

where $M_j, R_j \in \operatorname{Hom}(\mathcal{D}_j^{m_j})$ and for each j, $\mathcal{D}_j$ is isomorphic to one of the real division rings $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. In addition, $R_j \in \mathbf{sk}_j$ the set of nonsingular skew-symmetric matrices with entries in $\mathcal{D}_j$. In the obvious notation, $M_j \in \mathbf{sp}_j(R_j)$. We say that (M_j, R_j) is a symplectic pair over $\mathcal{D}_j$.

Define $C(M_j, R_j)$ to consist of those matrices $G \in \mathbf{sp}_j(R_j)$ satisfying $GM_j = M_jG$. Then it follows easily that

$$\dim C(M, R) = \dim C(M_1, R_1) \oplus \cdots \oplus \dim C(M_\ell, R_\ell).$$

In addition, if $M_j(\alpha_j)$ is a versal unfolding of M_j, then a versal unfolding of M is given by

$$M(\alpha) = M_1(\alpha_1) \oplus \cdots \oplus M_\ell(\alpha_\ell).$$

3 Codimension formulas over a real division ring

Suppose that $\mathcal{D}$ is a real division ring and that (M, R) is a symplectic pair over $\mathcal{D}$. Suppose further that (M, R) is in normal form, that is (M, R) is a direct sum of normal form summands from Table 1, 2 or 3 of [**?**]. In this section we give a formula for $\operatorname{codim} M = \dim C(M, R)$. Verification of the formulas is postponed to Section **??**.

Recall that normal form summands are determined uniquely by their modulus μ, size k, and index $\rho = \pm 1$. A summand of modulus μ has a quadruplet of eigenvalues $\pm\mu$, $\pm\bar{\mu}$. Each modulus has nonnegative real part and in addition has nonnegative imaginary part when $\mathcal{D} = \mathbb{R}$ or $\mathbb{H}$.

We begin by grouping together those normal form summands of (M, R) with the same modulus μ to obtain the direct sum

$$(M, R) = \bigoplus_{\mu} (M_\mu, R_\mu).$$

Now matrices that commute with M preserve the generalized eigenspaces of M. Moreover, in the case $\mathcal{D} = \mathbb{C}$, commuting matrices preserve the eigenspaces of μ and $\bar{\mu}$ separately. It follows that the centralizer of M block-diagonalizes into a direct sum of centralizers of the summands M_μ so that

$$\dim C(M, R) = \sum_{\mu} \dim C(M_\mu, R_\mu).$$

Hence it is sufficient to give a formula for $\dim C(M, R)$ where (M, R) is in normal form and all the summands of M have the same modulus μ.

It is convenient to divide the complex plane into four regions,

region I μ not real or purely imaginary.

region II μ real and nonzero.

region III μ purely imaginary and nonzero.

region IV $\mu = 0$.

We shall write $\mu \in \mathrm{I}$ if μ is in region I, and so on. Suppose that (M, R) consists of r normal form summands each with modulus μ. Let $k_1 \geq \cdots \geq k_r$

be the sizes of the normal form summands. Define the *weight* $w(\mu)$ of μ to be $w(\mu) = 1$ if $\mu \in \mathrm{III}$ and $w(\mu) = 2$ if $\mu \in \mathrm{I}$. If $\mu \in \mathrm{II}$ then $w(\mu) = \dim_{\mathbb{R}} \mathcal{D}$ $(= 1, 2$ or $4)$. In these cases, the dimension of the centralizer of M is given by

$$\dim C(M,R) = w(\mu) \sum_{i=1}^{r} (2i-1)k_i. \tag{3.1}$$

Next suppose that $\mu = 0$. If $\mathcal{D} = \mathbb{C}$ then $\dim C(M,R)$ is still given by equation (**??**) with $w(0) = 1$. If $\mathcal{D} = \mathbb{H}$ then the formula is modified slightly:

$$\dim C(M,R) = \sum_{i=1}^{r} [2(2i-1)k_i - \delta_i],$$

where $\delta_i = 1$ if k_i is odd and $\delta_i = 0$ if k_i is even. Finally we consider the case $\mathcal{D} = \mathbb{R}$ and $\mu = 0$. Suppose that there are r summands from row 5 of Table 1, [**?**] of size $k_1 \geq \cdots \geq k_r$, (k_i even) and s summands from row 6 of size $\ell_1 \geq \cdots \geq \ell_s$ (ℓ_j odd). Then

$$\dim C(M,R) = \tfrac{1}{2} \sum_{i=1}^{r} (2i-1)k_i \; + \; \sum_{j=1}^{s} [2(2j-1)\ell_j + 1] \; + \; 2 \sum_{i=1}^{r} \sum_{j=1}^{s} \min(k_i, \ell_j).$$

Observe that when $\mathcal{D} = \mathbb{R}$, the codimension formulas coincide with those of Galin.

Remark 3.1 Suppose that a symplectic pair (M,R) over $\mathcal{D}$ has several summands with the same modulus μ. Then it follows from the formulas in this section that $\dim C(M,R)$ is rather large. In particular, if $r \geq 2$ in the above codimension formulas, then $\dim C(M,R) \geq 4$. If $r \geq 2$ and $\mu \in \mathrm{I}$, then $\dim C(M,R) \geq 8$. The consequence for Γ-symplectic pairs of small enough codimension is that normal form summands with common modulus lie in distinct isotypic components. We shall exploit this fact in Section **??**.

4 Codimension of bundles

In this section we consider bundles of Γ-equivariant infinitesimally symplectic matrices and enumerate the bundles of low codimension. The definitions and basic properties are presented in Subsection **??**. In Subsection **??** we list the normal form summands whose bundles have low codimension. Using this information, we enumerate in Subsection **??** the bundles of infinitesimally symplectic matrices with codimension ≤ 1.

4.1 Bundles of infinitesimally symplectic matrices

Our aim in this section is to define the notions of bundle and bundle codimension. The definition of bundle is somewhat longwinded and we proceed in the following stages. In the first and most tedious stage we consider symplectic pairs over a real division ring $\mathcal{D}$ and define bundle equivalence of those symplectic pairs possessing a single quadruplet of eigenvalues. Then we define bundle equivalence for Γ-symplectic pairs possessing single quadruplets of eigenvalues. Finally, we define bundle equivalence for arbitrary Γ-symplectic pairs.

Suppose that $\mathcal{D}$ is a real division ring and that (M, R) is a symplectic pair over $\mathcal{D}$ with a single quadruplet of eigenvalues $\pm\mu$, $\pm\bar{\mu}$. By Theorem 2.4 in [**?**], (M, R) can be written uniquely as a direct sum of normal form summands (M_i, R_i), $i = 1, \ldots, r$, where (M_i, R_i) has modulus μ_i, size k_i, $k_1 \geq \cdots \geq k_r$, and index ρ_i. Let q denote the number of moduli with negative imaginary part (so $q = 0$ except possibly when $\mathcal{D} = \mathbb{C}$ and $\mu \in \mathrm{I} \cup \mathrm{III}$). Define $\mathrm{ind}_k = \sum_{k_i = k} \rho_i$. We associate with the symplectic pair (M, R) the set of invariants

$$I = \{q;\ k_1, \ldots, k_r;\ \mathrm{ind}_{k_1}, \ldots, \mathrm{ind}_{k_r}\}.$$

It follows from [**?**] that I forms a complete set of invariants for equivalence classes of symplectic pairs over $\mathcal{D}$ with this single quadruplet of eigenvalues.

Now suppose that (M_1, R_1) and (M_2, R_2) are two symplectic pairs over $\mathcal{D}$ each with a single quadruplet of eigenvalues (not necessarily the same quadruplet). Let I_1 and I_2 denote the corresponding sets of invariants. We say that the pairs (M_1, R_1) and (M_2, R_2) are *bundle equivalent* provided the eigenvalues lie in the same region (I–IV) of the complex plane and $I_1 = I_2$.

Next suppose that Γ acts on $\mathbb{R}^n$ and that (M, R) is a Γ-symplectic pair with a single quadruplet of eigenvalues. As in Section **??** we can write (M, R) as a direct sum of symplectic pairs (M_i, R_i) over real division rings $\mathcal{D}_j$ corresponding to the isotypic decomposition of $\mathbb{R}^n$. Two such Γ-symplectic pairs are *bundle equivalent* if the corresponding summands on each isotypic component are bundle equivalent.

Finally, suppose that (M, R) is a general Γ-symplectic pair. Write (M, R) as a direct sum of summands (M_μ, R_μ) that are a direct sum of all those normal from summands with eigenvalue μ. (This is analogous to the decomposition in Section **??** but we now decompose over quadruplets of eigenvalues rather than moduli). Two symplectic pairs (M_1, R_1) and (M_2, R_2) are *bundle equivalent* if there is a one-to-one correspondence between summands $(M_{1,\mu_1}, R_{1,\mu_1})$ and $(M_{2,\mu_2}, R_{2,\mu_2})$ that have a single quadruplet of eigenvalues and are bundle equivalent.

Definition 4.1 If $R \in \mathbf{sk}_\Gamma$, then $M_1, M_2 \in \mathbf{sp}_\Gamma$ are *bundle equivalent* if (M_1, R) and (M_2, R) are bundle equivalent. An equivalence class of bundle equivalent orbits in $\mathbf{sp}_\Gamma(R)$ is called a *bundle*. The *bundle codimension* $\operatorname{codim}_{\mathrm{b}} M$ of $M \in \mathbf{sp}_\Gamma(R)$, is defined to be the codimension of the bundle of orbits of M in $\mathbf{sp}_\Gamma(R)$.

Remark 4.2 There are certain seemingly arbitrary choices that we have made in the definition of bundle equivalence. For example, it is not clear in general that it is important to preserve indices. However, when $\mathcal{D} = \mathbb{C}$ there are nonisomorphic symplectic forms (see [**?**], [**?**], [**?**]) and corresponding to these are noncongruent matrices $R \in \mathbf{sk}$. Preservation of indices is crucial to distinguish the nonisomorphic symplectic forms.

In addition, the treatment of eigenvalues and moduli is designed to be consistent with the results in [**?**], see Section **??** of this paper.

Suppose that (M, R) is a Γ-symplectic pair in normal form. Let $\mu \in \mathbb{C}$ and define the *deficit* $d(\mu) = 1/2$ if $\mu \neq 0$ and $d(0) = 0$. Then

$$\operatorname{codim}_{\mathrm{b}} M = \operatorname{codim} M - \sum d(\mu),$$

where the sum is over all distinct eigenvalues μ of M. (Note that we work with eigenvalues rather than moduli here.) Thus a quadruplet of eigenvalues $\pm\mu$, $\pm\bar{\mu}$ corresponds to a deficit of 2, 1 or 0 depending on whether $\mu \in \mathrm{I}, \mathrm{II} \cup \mathrm{III}$ or IV.

The computation of the bundle codimension is facilitated by the following observations. First, the bundle codimension is superadditive, that is if (M_1, R_1) and (M_2, R_2) are normal forms and $M = M_1 \oplus M_2$, $R = R_1 \oplus R_2$, then

$$\operatorname{codim}_{\mathrm{b}} M \geq \operatorname{codim}_{\mathrm{b}} M_1 + \operatorname{codim}_{\mathrm{b}} M_2. \tag{4.1}$$

We have equality in equation (**??**) if and only if M_1 and M_2 have no common nonzero eigenvalues, and zero eigenvalues of M_1 and M_2 lie in distinct isotypic components.

Second, if M_1 and M_2 have the property that common moduli lie in distinct isotypic components, then

$$\operatorname{codim}_{\mathrm{b}} M = \operatorname{codim}_{\mathrm{b}} M_1 + \operatorname{codim}_{\mathrm{b}} M_2 + \sum d(\mu),$$

where the sum is over all common eigenvalues μ.

4.2 Normal form summands with low bundle codimension

Tables **??** and **??** contain those normal form summands (M, R) with bundle codimension 0 and 1 respectively, together with their versal unfoldings. The tables are obtained as follows. First set $r = 1$ in the codimension formulas in Section **??** (since we are working with single normal form summands). (For the case $\mu = 0$, $\mathcal{D} = \mathbb{R}$, take $r + s = 1$.) Recall that

$$\operatorname{codim}_{\mathrm{b}} M = \operatorname{codim} M - \sum d(\mu), \tag{4.2}$$

where the sum is over the quadruplet of eigenvalues corresponding to the modulus μ. Given a bundle codimension, we can choose one of the four regions for the modulus μ and solve equation (**??**) for the possible division rings $\mathcal{D}$ and sizes k. For example, to find the sizes of the normal form summands with $\mu \in \mathrm{II}$ of bundle codimension 1, we solve the equation

$$w(\mu)k = 1 + \sum d(\mu) = 2.$$

Hence $k = 2/w(\mu)$ yielding a summand of size 2 if $\mathcal{D} = \mathbb{R}$, size 1 if $\mathcal{D} = \mathbb{C}$, and no solution if $\mathcal{D} = \mathbb{H}$. The region for μ, size k, and division ring $\mathcal{D}$, specify uniquely (up to index) a normal form summand (M, R) in one of Tables 1, 2 and 3 in [?].

Finally, the centralizer $C(M, R)$ may be computed either directly, or using the results in Section **??**. We use the transpose of a set basis elements for $C(M, R)$ to construct a versal unfolding of M suppressing those basis elements that correspond to merely perturbing the moduli within their region.

μ	$\mathcal{D}$	k	M	R
I	$\mathbb{R}$, $\mathbb{C}$	1	$\begin{pmatrix} \alpha + i\beta & 0 \\ 0 & -\alpha + i\beta \end{pmatrix}_{\mathbb{C}}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\mathbb{C}}$
I	$\mathbb{H}$	1	$\begin{pmatrix} \alpha + i\beta & 0 \\ 0 & -\alpha + i\beta \end{pmatrix}_{\mathbb{H}}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\mathbb{H}}$
II	$\mathbb{R}$	1	$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
III	$\mathbb{R}$, $\mathbb{C}$	1	$(i\beta)_{\mathbb{C}}$	$\rho i_{\mathbb{C}}$
III	$\mathbb{H}$	1	$(i\beta)_{\mathbb{H}}$	$\rho i_{\mathbb{H}}$

Table 1: Normal form summands (M, R) with $\operatorname{codim}_{\mathrm{b}} M = 0$. The first three columns show the region of the modulus μ, the underlying division ring $\mathcal{D}$, and the size k of each summand; $\alpha > 0$, $\beta \neq 0$, $\beta > 0$ unless $\mathcal{D} = \mathbb{C}$, $\rho = \pm 1$

μ	$\mathcal{D}$	k	M	R
II	$\mathbb{R}$	2	$\begin{pmatrix} \begin{matrix} \alpha & 1 \\ \lambda & \alpha \end{matrix} & 0 \\ 0 & \begin{matrix} -\alpha & -1 \\ \lambda & -\alpha \end{matrix} \end{pmatrix}$	$\begin{pmatrix} 0 & \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \end{pmatrix}$
II	$\mathbb{C}$	1	$\begin{pmatrix} \alpha + i\lambda & 0 \\ 0 & -\alpha + i\lambda \end{pmatrix}_{\mathbb{C}}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\mathbb{C}}$
III	$\mathbb{R}$, $\mathbb{C}$	2	$\begin{pmatrix} i\beta & 1 \\ \lambda & i\beta \end{pmatrix}_{\mathbb{C}}$	$\rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\mathbb{C}}$
III	$\mathbb{H}$	2	$\begin{pmatrix} i\beta & 1 \\ \lambda & i\beta \end{pmatrix}_{\mathbb{H}}$	$\rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\mathbb{H}}$
IV	$\mathbb{R}$	2	$\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$	$\rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
IV	$\mathbb{C}$	1	$i\lambda_{\mathbb{C}}$	$\rho i_{\mathbb{C}}$
IV	$\mathbb{H}$	1	$i\lambda_{\mathbb{H}}$	$i_{\mathbb{H}}$

Table 2: Versal unfoldings of normal form summands (M, R) with $\operatorname{codim}_{\mathrm{b}} M = 1$. The first three columns show the region of the modulus μ, the underlying division ring $\mathcal{D}$, and the size k of each summand. The (real) unfolding parameter is denoted by λ. The trivial unfolding parameters that adjust the values of α and β are not shown; $\alpha > 0$, $\beta \neq 0$, $\beta > 0$ unless $\mathcal{D} = \mathbb{C}$, $\rho = \pm 1$

4.3 Enumeration of bundles with low codimension

Now suppose that Γ acts on $\mathbb{R}^n$ and that (M, R) is a Γ-symplectic pair. In Theorems **??** and **??** we give necessary and sufficient conditions for M to have bundle codimension 0 and 1.

Theorem 4.3 *Let (M, R) be a Γ-symplectic pair. Then $\operatorname{codim}_{\mathrm{b}} M = 0$ if and only if (M, R) is a direct sum of normal form summands (M_j, R_j) such that*

(a) $\operatorname{codim}_{\mathrm{b}} M_j = 0$ *for each j, and*

(b) If $\mu \in \mathbb{C}$ there is at most one j such that M_j has eigenvalue μ.

Theorem 4.4 *Let (M, R) be a Γ-symplectic pair. Then $\operatorname{codim}_{\mathrm{b}} M = 1$ if and only if (M, R) is a direct sum of a normal form (M_0, R_0) with* $\operatorname{codim}_{\mathrm{b}} M_0 = 0$ *and a normal form summand (M_1, R_1) with modulus μ such that either*

(a) $\operatorname{codim}_{\mathrm{b}} M_1 = 1$ *and μ is not an eigenvalue of M_0, or*

(b) $\operatorname{codim}_{\mathrm{b}} M_1 = 0$*, μ is an eigenvalue of M_0, and lies in the region* $\mathrm{II} \cup \mathrm{III}$*, and either*

(i) the summand of (M_0, R_0) with eigenvalue μ belongs to a distinct isotypic component of $\mathbb{R}^n$, or

(ii) μ is not a modulus of M_0 (necessarily $\mu \in \mathrm{III}$ and $\mathcal{D} = \mathbb{C}$).

The proof of Theorems **??** and **??** is almost immediate from the definitions. Note that by Remark **??** there is at most one summand from each isotypic component with a given modulus μ. (This is true also for bundles of codimension two.) It is this fact that allows us to work in terms of normal form summands.

5 Application to bifurcation theory

In this section, we apply the results of the previous section to the problems considered by Golubitsky and Stewart [**?**] and by Dellnitz, Melbourne and Marsden [**?**]. In particular, we recover the results in those papers without resorting to any ad hoc computations.

We begin with the steady-state bifurcation [?]. Looking for eigenvalues in region IV in Tables **??** and **??** we find that zero eigenvalues occur generically in a one-parameter family. If $\mathcal{D} = \mathbb{R}$ then there is a single normal form summand of size 2 with zero eigenvalues and a versal unfolding is given by

$$\begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}.$$

The eigenvalues are given by $\mu = \pm\sqrt{\lambda}$ and the eigenvalues split.

On the other hand, if $\mathcal{D} = \mathbb{C}$ or $\mathbb{H}$, then there is a single summand of size 1 with zero eigenvalues and a versal unfolding is given by $i\lambda_{\mathcal{D}}$. Over $\mathcal{D}$ the eigenvalue is $i\lambda$ corresponding to eigenvalues $\pm i\lambda$ when $\mathcal{D} = \mathbb{C}$ and $\pm i\lambda$ with multiplicity 2 when $\mathcal{D} = \mathbb{H}$. In both cases, the eigenvalues pass through zero along the imaginary axis.

Next we turn to the 1-1 resonance [?]. This time we look for eigenvalues in region III. According to Table **??**, generically for each $\beta > 0$ there is at most one summand with eigenvalues $\pm i\beta$. Moreover such summands have size 1. In a one-parameter family, this picture can change in one of three ways corresponding to parts (a), (b)(i), and (b)(ii) of Theorem **??**.

(a) There is a summand of size 2 with eigenvalues $\pm i\beta$.

(b)(i) There are two summands of size 1 from distinct isotypic components with eigenvalues $\pm i\beta$.

(b)(ii) There are two summands of size 1 from the same isotypic component with eigenvalues $\pm i\beta$, but with distinct moduli.

Case (a) can occur with $\mathcal{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and the versal unfolding is

$$\begin{pmatrix} i\beta & 1 \\ \lambda & i\beta \end{pmatrix}_{\mathcal{R}},$$

where $\mathcal{R} = \mathbb{C}$ if $\mathcal{D} = \mathbb{R}$, and $\mathcal{R} = \mathcal{D}$ otherwise. The eigenvalues are given by $\mu = i\beta \pm \sqrt{\lambda}$ indicating that the eigenvalues split.

Case (b)(i) corresponds to the 'independent passing' case in [?]. The unfolding parameter moves the eigenvalues apart, but they remain on the imaginary axis.

Finally, case (b)(ii) only occurs when $\mathcal{D} = \mathbb{C}$ and corresponds to the 'mysterious' cases in [?] which were apparently of a different nature to the independent passing cases. Here they are quite analogous. Moreover with hindsight, we can see that this was forced from the outset by the fact that real-complex commuting matrices preserve the eigenspaces of $i\beta$ and $-i\beta$ separately. (We note also that this phenomenon is independent of the indices ρ and hence occurs for each of the nonisomorphic symplectic forms on $\mathbb{C}^2$, [**?**] (both *complex of the same type* and *complex duals* in the terminology of [**?**])).

We illustrate this viewpoint further by making a prediction for dissipative systems. Suppose that Γ is a group with complex irreducibles present and consider Γ-equivariant Hopf/Hopf mode-interactions. Such an interaction is at least codimension two. Moreover if there is a 1-1 resonance, then the codimension is at least three and generically the linearization is nonsemisimple. We claim that semi-simple 1-1 resonances can occur generically in a three-parameter family. This is no more or less surprizing than the corresponding phenomenon in one-parameter families for Hamiltonian systems. However the significance for dissipative systems is much less due to the high codimension.

6 Derivation of the codimension formulas

In this section, we obtain the codimension formulas listed in Section **??**. We divide the section into three subsections. In Subsection **??** we work within the axiomatic framework set up in [**?**] and prove an abstract version of Galin's theorem. As in [**?**], it is only the case of normal form summands with modulus 0 over $\mathbb{R}$ that does not fit into this abstract setting. This is considered as a special case in Subsection **??**. Finally, in Subsection **??** we use these results to derive the required formulas.

6.1 Axiomatic framework

Let $\mathcal{D}$ be one of the real division rings $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and let $\mathcal{M}$ denote the space of $p \times p$ matrices with entries in $\mathcal{D}$. We denote by $1_{\mathcal{M}}$ the identity matrix in $\mathcal{M}$. Suppose that L and Q are matrices with entries in $\mathcal{M}$. Define $Z(L)$ to be the set of matrices with entries in $\mathcal{M}$ that commute with L. Our aim is to compute the centralizer

$$C(L,Q) = \{G \in Z(L);\, GQ + QG^T = 0\}.$$

In [?], the emphasis was on the complementary space

$$Z(L,Q) = \{G \in Z(L);\ GQ - QG^T = 0\}.$$

We shall exploit the following relationship between $C(L,Q)$ and $Z(L,Q)$.

Proposition 6.1 *Suppose that Q is nonsingular, $Q^T = \pm Q$ and L satisfies $LQ + QL^T = 0$. Then $Z(L) = C(L,Q) \oplus Z(L,Q)$.*

Proof If $G \in C(L,Q) \cap Z(L,Q)$ then $GQ = QG^T = -GQ$. Since Q is nonsingular, we have that $C(L,Q) \cap Z(L,Q) = 0$. Now write $G = (C+Z)/2$ where

$$C = G - QG^TQ^{-1}, \quad Z = G + QG^TQ^{-1}.$$

An easy computation shows that if $G \in Z(L)$, then $C \in C(L,Q)$ and $Z \in Z(L,Q)$ as required. ■

Recall from [?] that a pair of matrices (L,Q) is a *W-summand (of size k)* if $L = I_k \otimes \pi + N_k \otimes \phi$ and $Q = Y_k \otimes \tau$, where $\pi, \phi, \tau \in \mathcal{M}$ and Y_k is a real nonsingular $k \times k$ matrix such that certain hypotheses are satisfied including the following for some choice of $\sigma_1, \sigma_2 = \pm 1$.

(H1) ϕ, τ, Y_k are nonsingular.

(H2) π is semisimple, $\quad \tau^T = \tau^{-1} = \sigma_1 \tau, \quad Y_k^T = Y_k^{-1} = -\sigma_1 Y_k$.

(H3) $\pi\tau = -\tau\pi^T, \quad \phi\tau = -\sigma_2 \tau \phi^T, \quad N_k Y_k = \sigma_2 Y_k N_k^T$.

(H4) $Z(\pi) \subset Z(\phi)$.

Suppose that (L_i, Q_i) is a W-summand of size k_i for $i = 1, \ldots, r$, and that $k_1 \geq k_2 \geq \cdots \geq k_r$. Let $L = L_1 \oplus \cdots \oplus L_r$ and $Q = Q_1 \oplus \cdots \oplus Q_r$. Then (L,Q) is a *W-sum* if

(a) $L_i = I_{k_i} \otimes \pi + N_{k_i} \otimes \phi$ where π and ϕ are independent of i.

(b) If $k_i = k_j$ then $L_i = L_j$ and $Q_i = Q_j$.

(c) If $k_i > k_j$ and A is a $k_j \times k_j$ matrix, then $Y_{k_i} \begin{pmatrix} 0 \\ A \end{pmatrix} = \begin{pmatrix} Y_{k_j} A \\ 0 \end{pmatrix}$.

If (L,Q) is a W-sum, define $Z(L,Q)^0$ to consist of those matrices $H \in Z(L,Q)$ that have the form $H = H_1 \oplus \cdots \oplus H_r$ where $H_i = \rho_i I_{k_i} \otimes 1_{\mathcal{M}}$ and $\rho_i = \pm 1$. (This is slightly different from the definition of $Z(L,Q)^0$ in [**?**].)

Theorem 6.2 (Abstract Galin Theorem) *Suppose that (L,Q) is a W-sum with summands of size $k_1 \geq \cdots \geq k_r$ and let $H \in Z(L,Q)^0$.*

(a) If $\sigma_2 = 1$, then

$$\dim C(L,HQ) = \sum_{i=1}^{r} k_i[i \dim Z(\pi) - \dim Z(\pi,\tau_i)].$$

(b) If $\sigma_2 = -1$, then

$$\dim C(L,HQ) = \sum_{i=1}^{r}[(i-1/2)k_i \dim Z(\pi) + \delta_i d_i],$$

where $d_i = \dim Z(\pi)/2 - \dim Z(\pi,\tau_i)$ *and* $\delta_i = \begin{cases} 1; & k_i \text{ odd} \\ 0; & k_i \text{ even} \end{cases}$.

Remark 6.3 In many cases we find that $\dim Z(\pi,\tau_i) = \dim Z(\pi)/2$ for each i. Then we have the uniform simplified formula

$$\dim C(L,HQ) = \frac{\dim Z(\pi)}{2} \sum_{i=1}^{r}(2i-1)k_i.$$

The remainder of this section is devoted to proving Theorem **??**. We begin by considering the structure of matrices in $C(L,HQ)$. We can partition such a matrix P into blocks P_{ij}, $1 \leq i,j \leq r$ where P_{ij} is a $k_i \times k_j$ matrix with entries in $\mathcal{M}$. The conditions $PL = LP$, $PHQ + HQP^T = 0$ become

$$P_{ij}L_j = L_i P_{ij}, \quad P_{ij}Q_j + \rho_i\rho_j Q_i P_{ji}^T = 0.$$

Let $k_{ij} = \min(k_i,k_j)$.

Proposition 6.4 *A matrix $P = \{P_{ij}\}$ lies in $Z(L)$ if and only if the following is true.*

(a) $P_{ij} = \begin{pmatrix} 0 & F_{ij} \end{pmatrix}$ *if* $k_i \le k_j$ *or* $P_{ij} = \begin{pmatrix} F_{ij} \\ 0 \end{pmatrix}$ *if* $k_i \ge k_j$, *where* F_{ij} *is a* $k_{ij} \times k_{ij}$ *matrix with entries in* $\mathcal{M}$.

(b) $F_{ij} = \sum_{s=0}^{k_{ij}-1} N_{k_{ij}}^s \otimes f_{ij,s}$ *where* $f_{ij,s} \in Z(\pi)$.

Proof See [**?**]. ∎

We shall refer to the blocks P_{ii} as diagonal blocks and P_{ij}, $i \neq j$, as off-diagonal blocks. Note that $F_{ii} = P_{ii}$ for each i.

Proposition 6.5 *Suppose that $P \in Z(L)$. Then $P \in C(L, HQ)$ if and only if*

$$f_{ij,s}\tau_j + \rho_i\rho_j\sigma_2^s\tau_i f_{ji,s}^T = 0,$$

for all $1 \le i \le j \le r$.

Proof Observe that $P \in C(L, HQ)$ if and only if the equation $P_{ij}Q_j + \rho_i\rho_j Q_i P_{ji}^T$ holds for all i, j. We shall compute the restrictions that these conditions impose for $i \le j$. By taking transposes it can be seen that the remaining conditions impose no further restrictions.

By [**?**] we have for $i \le j$,

$$P_{ij}(Y_{k_j} \otimes 1_{\mathcal{M}}) = (Y_{k_i} \otimes 1_{\mathcal{M}})\tilde{P}_{ij},$$

where $\tilde{P}_{ij} = \begin{pmatrix} 0 \\ \tilde{F}_{ij} \end{pmatrix}$ and $\tilde{F}_{ij} = \sum_{s=0}^{k_{ij}-1} \sigma_2^s (N_{k_{ij}}^s)^T \otimes f_{ij,s}$. Hence

$$\begin{aligned} Q_i P_{ji}^T &= -\rho_i\rho_j P_{ij} Q_j \\ &= -\rho_i\rho_j P_{ij}(Y_{k_j} \otimes 1_{\mathcal{M}})(I_{k_j} \otimes \tau_j) \\ &= -\rho_i\rho_j (Y_{k_i} \otimes 1_{\mathcal{M}})\tilde{P}_{ij}(I_{k_j} \otimes \tau_j) \\ &= -\rho_i\rho_j Q_i (I_{k_i} \otimes \tau_i^{-1})\tilde{P}_{ij}(I_{k_j} \otimes \tau_j). \end{aligned}$$

This yields the restriction

$$(I_{k_i} \otimes \tau_i) P_{ji}^T = -\rho_i \rho_j \tilde{P}_{ij} (I_{k_j} \otimes \tau_j),$$

or

$$\sum_{s=0}^{k_j - 1} (N_{k_j}^s)^T \otimes \tau_i f_{ji,s}^T = -\rho_i \rho_j \sum_{s=0}^{k_j - 1} \sigma_2^s (N_{k_j}^s)^T \otimes f_{ij,s} \tau_j.$$

The result follows from the linear independence of the matrices $N_{k_j}^s$. ■

Proof of Theorem ??
First suppose that $P \in Z(L)$. It follows from Proposition **??** that each block P_{ij} is determined by k_{ij} elements $f_{ij,0}, \dots, f_{ij,k_{ij}-1} \in Z(\pi)$. For example, block P_{11} is determined by k_1 such elements, blocks P_{12}, P_{22} and P_{21} by k_2 elements and so on. It follows that

$$\dim Z(L) = \sum_{i=1}^{r} (2i - 1) k_i \dim Z(\pi).$$

Moreover the off-diagonal blocks contribute $\sum_{i=1}^r (2i-2) k_i \dim Z(\pi)$.

Now suppose that $P \in C(L, HQ)$. If $i < j$, then it follows from Proposition **??** that we may consider $f_{ji,s}$ as being an arbitrary element of $Z(\pi)$ but then $f_{ij,s}$ is determined. Hence the contribution to $\dim C(L, HQ)$ from the off-diagonal terms is half the contribution to $\dim Z(L)$, namely

$$\sum_{i=1}^{r} (i - 1) k_i \dim Z(\pi). \tag{6.1}$$

Next we compute the contribution to $\dim C(L, HQ)$ of the diagonal blocks. By Proposition **??**,

$$f_{ii,s} \tau_i = -(\sigma_2)^s \tau_i f_{ii,s}^T.$$

If $\sigma_2 = 1$ then this implies that $f_{ii,s} \in C(\pi, \tau_i)$. Thus the diagonal blocks contribute $\sum_{i=1}^r k_i \dim C(\pi, \tau_i)$ or by Proposition **??**,

$$\sum_{i=1}^{r} k_i (\dim Z(\pi) - \dim Z(\pi, \tau_i)). \tag{6.2}$$

Part (a) of the theorem is obtained by adding (??) and (??).

Finally suppose that $\sigma_2 = -1$. Then $f_{ii,s} \in Z(\pi,\tau_i)$ or $C(\pi,\tau_i)$ depending on whether s is odd or even. Since $\dim Z(\pi) = \dim Z(\pi,\tau_i) \oplus \dim C(\pi,\tau_i)$ we may pair off terms so that if k_i is even, the block P_{ii} contributes $\dim Z(\pi) k_i/2$. If k_i is odd, then there is an additional term

$$\dim C(\pi,\tau_i) = \dim Z(\pi) - \dim Z(\pi,\tau_i).$$

Putting all of this together yields part (b) of the theorem. ■

6.2 The zero eigenvalue case

In this subsection we consider the zero eigenvalue case when $\mathcal{D} = \mathbb{R}$.

Theorem 6.6 *Suppose that $R \in \mathbf{sk}$, $M \in \mathbf{sp}(R)$, and that M has only zero eigenvalues. Then the dimension of $C(M,R)$ is given by*

$$\tfrac{1}{2}\sum_{i=0}^{r}(2i-1)k_i + \sum_{j=1}^{s}[2(2j-1)\ell_j + 1] + 2\sum_{i=1}^{r}\sum_{j=1}^{s}\min(k_i,\ell_j).$$

Proof It follows from results in [?] that $(M,R) \sim (L,HQ)$ where $L = L^1 \oplus L^2$, $Q = Q^1 \oplus Q^2$, (L^1,Q^1), (L^2,Q^2) are W-sums with summands of even and odd size respectively and $H = H^1 \oplus H^2$ where $H^j \in Z(L^j,Q^j)^0$. Moreover we can take $H^2 = I$. We have

$$\begin{aligned} L^1 &= N_{k_1} \oplus \cdots \oplus N_{k_r} & L^2 &= (N_{\ell_1} \otimes 1_{\mathbb{C}}) \oplus \cdots \oplus (N_{\ell_s} \otimes 1_{\mathbb{C}}) \\ H^1Q^1 &= \rho_1 X_{k_1} \oplus \cdots \oplus \rho_r X_{k_r} & H^2Q^2 &= (X_{\ell_1} \otimes i_{\mathbb{C}}) \oplus \cdots \oplus (X_{\ell_s} \otimes i_{\mathbb{C}}) \end{aligned}$$

where $\rho_i = \pm 1$. Suppose that $G \in C(L,HQ)$ and write $G = \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix}$. Observe that $G^{11} \in C(L^1,H^1Q^1)$ and $G^{22} \in C(L^2,H^2Q^2)$. Now (L^1,Q^1) and (L^2,Q^2) are W-sums with $\sigma_2 = -1$, so we apply Theorem **??**(b). In the case of (L^1,Q^1) each k_i is even and $\dim Z(\pi) = 1$ so that

$$\dim C(L^1,H^1Q^1) = \sum_{i=1}^{r}(i-1/2)k_i. \qquad (6.3)$$

For (L^2, Q^2) each ℓ_j is odd and $\dim Z(\pi) = 4$, $\dim Z(\pi, \tau) = 1$. Hence

$$\dim C(L^2, H^2 Q^2) = \sum_{j=1}^{s} [4(j - 1/2)\ell_j + 1]. \tag{6.4}$$

It remains to compute the contribution of the blocks G^{12} and G^{21}. These satisfy the equations

$$\begin{aligned} G^{12} L^2 &= L^1 G^{12} && (6.5)\\ G^{21} L^1 &= L^2 G^{21} && (6.6)\\ G^{12} Q^2 &= -H^1 Q^1 (G^{21})^T && (6.7)\\ G^{21} H^1 Q^1 &= -Q^2 (G^{12})^T && (6.8) \end{aligned}$$

We claim that equations (**??**) and (**??**) are redundant. Suppose that G^{12} and G^{21} satisfy equations (**??**) and (**??**). Taking the transpose of equation (**??**) yields equation (**??**). Next solve for G^{12} in terms of G^{21} in equation (**??**) and substitute into equation (**??**) to obtain

$$H^1 Q^1 (G^{21})^T (Q^2)^{-1} L^2 = L^1 H^1 Q^1 (G^{21})^T (Q^2)^{-1}.$$

Using the relations $L^j Q^j + Q^j (L^j)^T = 0$ this reduces to equation (**??**) verifying the claim.

Since equation (**??**) determines G^{21} once G^{12} is given, we have only to compute the number of matrices G^{12} satisfying equation (**??**). Write $G^{12} = \{G^{12}_{ij}\}_{\substack{1 \le i \le r \\ 1 \le j \le s}}$. Then G^{12}_{ij} satisfies the equation

$$N_{k_i} G^{12}_{ij} = G^{12}_{ij} (N_{\ell_j} \otimes I_2).$$

For each i, j, set $G^{12}_{ij} = \{g_{\alpha\beta}\}_{\substack{1 \le \alpha \le k_i \\ 1 \le \beta \le \ell_j}}$ where $g_{\alpha\beta} = (g_{\alpha\beta,1}, g_{\alpha\beta,2})$, $g_{\alpha\beta,t} \in \mathbb{R}$ for $t = 1, 2$. Let $E_t = \{g_{\alpha\beta,t}\}$, $t = 1, 2$. Then it is easily seen that

$$N_{k_i} E_t = E_t N_{\ell_j}.$$

It follows from Proposition **??** that for each $1 \le i \le r$, $1 \le j \le s$, $t = 1, 2$, E_t has the form

$$\begin{pmatrix} 0 & F_t \end{pmatrix}, \text{ or } \begin{pmatrix} F_t \\ 0 \end{pmatrix},$$

where

$$F_t = \sum_{s=0}^{\min(k_i,\ell_j)} f_{ij,t,s} N^s_{\min(k_i,\ell_j)}$$

for $f_{ij,t,s} \in \mathbb{R}$. In particular there are $\min(k_i, \ell_j)$ degrees of freedom in F_t. Summing over i, j and t we see that the number of matrices G^{12} satisfying equation (**??**) is given by

$$2\sum_{i=1}^{r}\sum_{j=1}^{s} \min(k_i, \ell_j). \tag{6.9}$$

The dimension of $C(L, HQ)$ is given by the sum of the contributions in equations (**??**), (**??**) and (**??**). ∎

6.3 Codimension formulas

In this subsection, we use the results of Subsections **??** and **??** to derive the codimension formulas in Section **??**. Suppose that (M, R) is a symplectic pair over $\mathcal{D}$ with a single quadruplet of eigenvalues $\pm\mu$, $\pm\bar{\mu}$. If $\mathcal{D} = \mathbb{R}$ and $\mu = 0$, then the codimension formula follows from Theorem **??**. In all other cases, $(M, R) \sim (L, HQ)$ where (L, Q) is a W-sum, $H \in Z(L, Q)^0$, and the matrices μ, ϕ, τ_i lie in one of the rows (i)–(ix) of Table 4 in [**?**]. The corresponding spaces $Z(\pi)$ and $Z(\pi, \tau)$ are given in Table 5 in [**?**]. For entries other than those in row (ix), we have $\dim Z(\pi) = 2 \dim Z(\pi, \tau)$ and we are in the situation of Remark **??**. Defining the weight $w(\mu)$ to be $\dim Z(\pi)/2$ yields the required result.

Row (ix) corresponds to the case $\mathcal{D} = \mathbb{H}$ and $\mu = 0$. We have $\dim Z(\pi) = 4$ and $\dim Z(\pi, \tau) = 3$ or 1 depending in whether the size k of the summand is odd or even. In addition the number σ_2 in the definition of W-summand has the value -1. The required formula follows from Theorem **??**(b).

Acknowledgment
I am grateful to Henk Broer and in particular Michael Dellnitz for helpful discussions.

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