Explicit Coupling Argument for Nonuniformly Hyperbolic Transformations

A. Korepanov Z. Kosloff* I. Melbourne

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

26 September 2016 (updated 12 June 2018)

Abstract

The transfer operator corresponding to a uniformly expanding map enjoys good spectral properties. Here it is verified that coupling yields explicit estimates that depend continuously on the expansion and distortion constants of the map.

For nonuniformly expanding maps with a uniformly expanding induced map, we obtain explicit estimates for mixing rates (exponential, stretched exponential, polynomial) that again depend continuously on the constants for the induced map together with data associated to the inducing time.

Finally, for nonuniformly hyperbolic transformations, we obtain the corresponding estimates for rates of decay of correlations.

1 Introduction

It is well-known that the transfer operator associated to a uniformly expanding map enjoys good spectral properties. In particular, there are numerous methods for proving exponential decay of correlations for uniformly expanding maps, see for example [1, 9, 25, 26, 28].

Often, statistical properties of nonuniformly expanding systems are studied by inducing to a uniformly expanding one. Young [31, 32] obtained results on decay of correlations for large classes of such nonuniformly expanding maps, as well as nonuniformly hyperbolic transformations. The rate of decay is related to the tails of the inducing time, with special emphasis placed on exponential tails and polynomial tails. Stretched exponential decay rates (amongst others) were obtained in Maume-Deschamps [22]. The resulting decay rates have the form $O(e^{-cn^\gamma})$ or $O(n^{-\beta})$ where

*Permanent address: Einstein Institute of Mathematics, Hebrew University of Jerusalem, Givat Ram. Jerusalem, 9190401, Israel

$\gamma \in (0,1]$ and $\beta > 0$ are given explicitly, but the implied constants are not and nor is c in the exponential case $\gamma = 1$. An improved estimate of Gouëzel [17] gives sharp decay rates in the stretched exponential case $\gamma \in (0,1)$ but the implied constant remains nonexplicit (as does the constant c in the exponential case).

In this paper, we use an explicit coupling argument to obtain mixing rates with uniform control on the various constants. The main novelty in our results lies in the nonuniformly expanding/hyperbolic setting. However, even for uniformly expanding maps, we expect that our results have numerous applications, see for example [19, 20].

Related results using the coupling method for uniformly expanding maps can be found in both simpler and more complicated situations (usually in low dimensions) in recent papers, for example [14, 29]. See also [21] for an approach using Birkhoff cones for one-dimensional maps. None of these results are formulated in such a way that they can be cited in [19, 20]. In this paper, we work in a general metric space and present a much shorter and more elementary proof than was previously written down. The results then feed into the more complicated argument required in the nonuniformly expanding/hyperbolic setting.

Remark 1.1 After circulating a first version of this paper, we were made aware by Oliver Butterley and Jean-René Chazottes of previous work of Zweimüller [33] which handles the uniformly expanding case. Using a coupling argument for uniform expanding Markov maps defined on a general compact metric space, [33] shows how to obtain exponential decay of correlations with explicit control on the various constants, just as is shown in this paper. Moreover, the setting in [33] (within the uniformly expanding setting) is more general than the one considered here since we assume full branches whereas [33] assumes a "finite images" condition. Assuming full branches simplifies matters considerably but suffices for our purposes in [19, 20].

The compactness assumption in [33] is used only to to prove existence of an invariant density via an Arzelà-Ascoli argument. The proof below of Proposition 2.5 shows how to bypass this, so that compactness of the metric space is not required. For an alternative argument to prove existence of an invariant density without using compactness, see [2] or [1, Lemma 4.4.1].

Hence our results for uniformly expanding maps in Section 2.1 are not new. We include the results for a number of reasons: (a) completeness, especially as they feed into our results for nonuniformly expanding/hyperbolic systems (Sections 2.2 and 2.3) which are new; (b) The arguments are very short and direct; (c) The explicit nature of the constants is stated in a way that is convenient for easy reference (in [32] it is necessary to read the entire proof to see that it gives explicit uniform bounds for the constants).

Remark 1.2 Keller & Liverani [18] considered continuous families of uniformly expanding maps and developed a perturbative theory that gives uniform estimates on the spectra of the associated transfer operators. This idea was used by [13] in the situation of dispersing billiards. However, inducing from continuous families of nonuniformly expanding maps to families of uniformly expanding maps may fail to preserve any useful notion of continuous dependence. In particular, the examples in [19, Section 5] and in [20] do not satisfy the hypotheses of [13, 18].

In this paper, we do not assume any continuous dependence on parameters. Instead, we work with a fixed uniformly expanding map F, and give explicit estimates on the associated transfer operator that depend continuously on the expansion and distortion estimates of F.

Even for nonuniformly expanding/hyperbolic dynamical systems, none of the results in this paper are particularly surprising. Nevertheless, the results go far beyond those previously available. Some examples are listed at the end of Section 2.2. In the case of smooth unimodal maps there are previous results [8, Theorem 1.3] showing exponential decay of correlations up to a finite period with uniform exponent (uniformity of the implied constant is not claimed in [8]). Here we obtain a similar result with uniform exponent and uniform implied constant. In the case of families of Viana maps [30] which are known to have stretched exponential decay of correlations [16], we obtain for the first time uniform estimates on the constants C, c, γ in the stretched exponential decay rate Ce^{-cn^γ}.

Our main results are stated in Section 2 and proved for uniformly expanding, nonuniformly expanding, and nonuniformly hyperbolic, transformations in Sections 3, 4 and 5 respectively.

2 Statement of the main results

In this section, we state our main results for uniformly expanding maps (Subsection 2.1), nonuniformly expanding maps (Subsection 2.2), and nonuniformly hyperbolic transformations (Subsection 2.3).

2.1 Uniformly expanding maps

Let (Y, m) be a probability space, and $F : Y \to Y$ be a nonsingular transformation. Let d be a metric on Y such that $\operatorname{diam} Y \le 1$.

Suppose that α is an at most countable measurable partition of Y, and that F restricts to a measure-theoretic bijection from a onto Y for each $a \in \alpha$.

Let $\zeta = \frac{dm}{dm\circ F}$ be the inverse Jacobian of F with respect to m. Assume that there are constants $\lambda > 1$, $K > 0$ and $\eta \in (0, 1]$ such that for x, y in the same partition element

$$d(Fx, Fy) \ge \lambda d(x, y) \quad \text{and} \qquad |\log \zeta(x) - \log \zeta(y)| \le K d(Fx, Fy)^\eta. \tag{2.1}$$

Let $P_m : L^1(Y) \to L^1(Y)$ be the transfer operator corresponding to F and m, so $\int_Y P_m\phi\, \psi \, dm = \int_Y \phi\, \psi \circ F \, dm$ for all $\phi \in L^1$ and $\psi \in L^\infty$. Then $P_m\phi$ is given explicitly by

$$(P_m\phi)(y) = \sum_{a\in\alpha} \zeta(y_a)\phi(y_a),$$

where y_a is the unique preimage of y under F lying in a.

Given $\phi : Y \to \mathbb{R}$, define

$$|\phi|_\eta = \sup_{x\neq y} \frac{|\phi(x)-\phi(y)|}{d(x,y)^\eta} \qquad \text{and} \qquad \|\phi\|_\eta = |\phi|_\infty + |\phi|_\eta.$$

Let C^η denote the Banach space of observables $\phi : Y \to \mathbb{R}$ such that $\|\phi\|_\eta < \infty$.

It is well-known that there exist constants $C > 0$, $\gamma \in (0,1)$, such that $\|P_m^n\phi\|_\eta \leq C\gamma^n\|\phi\|_\eta$ for all $\phi \in C^\eta$ with $\int_Y \phi\, dm = 0$ and all $n \geq 1$. Our main result is:

Theorem 2.1 *There exist constants $C > 0$, $\gamma \in (0,1)$ depending continuously on λ, K and η, such that*

$$\|P_m^n\phi\|_\eta \leq C\gamma^n|\phi|_\eta,$$

for all $\phi \in C^\eta$ with $\int_Y \phi\, dm = 0$, and all $n \geq 1$.

Remark 2.2 For example, take $R = 2K/(1-\lambda^{-\eta})$ and $\xi = \frac{1}{2}e^{-R}(1-\lambda^{-\eta})$. Then Theorem 2.1 holds with $C = 4e^R(1+R)$ and $\gamma = 1-\xi$.

Next, let $\mathcal{M}$ be the collection of probability measures on Y that are equivalent to m and satisfy $L_\mu < \infty$ where $L_\mu = |\log \frac{d\mu}{dm}|_\eta$. Given $\mu \in \mathcal{M}$, define $\zeta_\mu = \frac{d\mu}{d\mu\circ F}$ and let P_μ be the corresponding transfer operator.

Proposition 2.3 *For all x, y in the same partition element,*

$$|\log\zeta_\mu(x) - \log\zeta_\mu(y)| \leq K_\mu d(Fx,Fy)^\eta,$$

where $K_\mu = K + (\lambda^{-\eta}+1)L_\mu$.

Proof Note that $\log\zeta_\mu = \log\zeta + h - h\circ F$ where $h = \log\frac{d\mu}{dm}$. Hence $|\log\zeta_\mu(x) - \log\zeta_\mu(y)| \leq |\log\zeta(x) - \log\zeta(y)| + |h|_\eta d(x,y)^\eta + |h|_\eta d(Fx,Fy)^\eta \leq (K + L_\mu\lambda^{-\eta} + L_\mu)d(Fx,Fy)^\eta$. ∎

In other words, the hypotheses of Theorem 2.1 are satisfied with m and K replaced by μ and K_μ. Hence, we obtain:

Corollary 2.4 *Let $\mu \in \mathcal{M}$. There exist constants $C > 0$, $\gamma \in (0,1)$ depending continuously on λ, K_μ and η, such that*

$$\|P_\mu^n\phi\|_\eta \leq C\gamma^n|\phi|_\eta,$$

for all $\phi \in C^\eta$ with $\int_Y \phi\, d\mu = 0$, and all $n \geq 1$. ∎

Of special interest is the case where μ is the unique absolutely continuous F-invariant probability measure. For this special case, we prove:

Proposition 2.5 *The invariant probability measure μ lies in $\mathcal{M}$, and there is a constant R depending continuously on λ, K and η (chosen as in Remark 2.2 say) such that*

$$e^{-R} \le \frac{d\mu}{dm} \le e^{R}, \qquad \Big|\log \frac{d\mu}{dm}\Big|_\eta \le R.$$

In particular, the constants C and γ in Corollary 2.4 depend continuously on λ, K and η.

Remark 2.6 A standard extension of these results is to treat observables $\phi : Y \to \mathbb{R}$ that are piecewise Hölder (relative to the partition α) and possibly unbounded. Provided $P_m\phi \in C^\alpha$, our results go through unchanged (with obvious modifications to the constant C). For instances of this extension, we refer to [23, Lemma 2.2] or [19, Proposition 4.7].

2.2 Nonuniformly expanding maps

Let $F : Y \to Y$ be a uniformly expanding map with probability measure m (not necessarily invariant), constants λ, K and η, and partition α, as in Subsection 2.1. Let $\tau : Y \to \mathbb{Z}^+$ be an integrable function that is constant on partition elements. Define the Young tower [32]

$$\Delta = \{(y,\ell) \in Y \times \mathbb{Z} : 0 \le \ell \le \tau(y) - 1\}$$

and $f : \Delta \to \Delta$,

$$f(y,\ell) = \begin{cases} (y,\ell+1), & \ell \le \tau(y) - 2, \\ (Fy, 0), & \ell = \tau(y) - 1. \end{cases}$$

Let $\bar\tau = \int_Y \tau\, dm$. Let m_Δ be the probability measure on Δ given by $m_\Delta(A \times \{\ell\}) = \bar\tau^{-1} m(A)$ for all $\ell \ge 0$ and measurable $A \subset \{y \in Y : \tau(y) \ge \ell + 1\}$.

Let d_Δ be the metric on Δ given by

$$d_\Delta((y,\ell),(y',\ell')) = \begin{cases} 1, & \ell \ne \ell' \\ d(y,y'), & \ell = \ell' \end{cases}.$$

Given $\phi : \Delta \to \mathbb{R}$, define $|\phi|_\eta = \sup_{x,y\in\Delta} \frac{|\phi(x)-\phi(y)|}{d_\Delta(x,y)^\eta}$ and $\|\phi\|_\eta = |\phi|_\eta + |\phi|_\infty$.

Let $L : L^1(\Delta) \to L^1(\Delta)$ denote the transfer operator corresponding to f and m_Δ, so $\int_\Delta L\phi\,\psi\, dm_\Delta = \int_\Delta \phi\,\psi\circ f\, d\mu$ for all $\phi \in L^1$, $\psi \in L^\infty$.

When the measure m on Y is F-invariant, m_Δ is an ergodic f-invariant probability measure on Δ and m_Δ is mixing under f if and only if $\gcd\{\tau(a) : a \in \alpha\} = 1$. Accordingly, we say that the tower $f : \Delta \to \Delta$ is *mixing* if $\gcd\{\tau(a) : a \in \alpha\} = 1$, and *nonmixing* otherwise, even though we do not assume that m_Δ is f-invariant.

Mixing Young towers In the mixing case, there exist $\delta > 0$ and a finite set of positive integers $\{I_k\}$ with $\gcd\{I_k\} = 1$ such that $m(\{y \in Y : \tau(y) = I_k\}) \geq \delta$.

Theorem 2.7 *Let $\phi : \Delta \to \mathbb{R}$ be an observable with $\|\phi\|_\eta < \infty$ and $\int_\Delta \phi \, dm_\Delta = 0$.*

- *Suppose that $m(\tau \geq n) \leq C_\tau n^{-\beta}$ for some $\beta > 1$ and all $n > 0$. Then there exists a constant $C > 0$ depending continuously on λ, K, η, $\max\{I_k\}$, δ, β and C_τ, such that for all $n \geq 0$*

$$\int_\Delta |L^n \phi| \, dm_\Delta \leq C \|\phi\|_\eta n^{-(\beta-1)}.$$

- *Suppose that $m(\tau \geq n) \leq C_\tau e^{-An^\gamma}$ for some $A > 0$, $0 < \gamma \leq 1$ and all $n > 0$. Then there exist constants $B > 0$ and $C > 0$ depending continuously on λ, K, η, $\max\{I_k\}$, δ, A, γ and C_τ, such that for all $n \geq 0$*

$$\int_\Delta |L^n \phi| \, dm_\Delta \leq C \|\phi\|_\eta e^{-Bn^\gamma}.$$

Nonmixing Young towers In the nonmixing case, define

$$d = \gcd\{j \geq 1 : m(\{y \in Y : \tau(y) = j\}) > 0\} \geq 2.$$

There exist $\delta > 0$ and a finite set of positive integers $\{I_k\}$ with $\gcd\{I_k\} = d$ such that $m(\{y \in Y : \tau(y) = I_k\}) \geq \delta$.

Theorem 2.8 *Let $\phi : \Delta \to \mathbb{R}$ be an observable with $\|\phi\|_\eta < \infty$ and $\int_\Delta \phi \, dm_\Delta = 0$. Then Theorem 2.7 holds with $\int_\Delta |L^n \phi| \, dm_\Delta$ replaced by $\int_\Delta \big|\sum_{k=0}^{d-1} L^{nd+k} \phi\big| \, dm_\Delta$.*

Theorem 2.8 has the following equivalent reformulation which gives uniform mixing rates up to a cycle of length d. We state the reformulation for the case of (stretched) exponential mixing. The polynomial mixing case goes the same way.

Write $\Delta = E_1 \cup \cdots \cup E_d$ where $f(E_j) = E_{j+1 \bmod d}$ and $f^d : E_j \to E_j$ is a mixing tower for $j = 1, \dots, d$.

Corollary 2.9 *Suppose that we are in the situation of Theorem 2.8 and that $m(\tau \geq n) \leq C_\tau e^{-An^\gamma}$ as in the second part of Theorem 2.7. Fix $j = 1, \dots, d$. Then there exist uniform constants B, $C > 0$ as in Theorem 2.7 such that*

$$\Big| \int_\Delta \phi \, \psi \circ f^{nd} \, dm_\Delta - \int_\Delta \phi \, dm_\Delta \int_\Delta \psi \, dm_\Delta \Big| \leq C \|\phi\|_\eta |\psi|_\infty e^{-Bn^\gamma},$$

for all $n \geq 1$ and all $\phi, \psi \in L^\infty$ supported in E_j with $\|\phi\|_\eta < \infty$.

Examples In [19, 20], we verified for specific families of nonuniformly expanding maps that the corresponding induced maps F are uniformly expanding, as in Subsection 2.1, with uniform constants λ, K, η. A key ingredient in this verification is the work of [3, 5, 7, 15] on strong statistical stability (where the density of the invariant measure varies continuously in L^1). It follows from this abstract framework (specifically condition (U1) in [7]) that the data $d = \gcd\{I_k\} \geq 1$ and $\delta > 0$ associated with the inducing time τ varies continuously in the mixing case and upper semicontinuously in general (so d can decrease under small perturbations but cannot increase). Hence for the examples in [19, 20], uniform estimates on decay of correlations follow immediately from Theorems 2.7 and 2.8.

Specifically, we obtain uniform polynomial decay of correlations for intermittent maps [20, Example 4.9], uniform exponential decay of correlations (up to a finite cycle) for smooth unimodal and multimodal maps satisfying the Collet-Eckmann condition [20, Example 4.10], and uniform stretched exponential decay of correlations for Viana maps [20, Example 4.11].

2.3 Nonuniformly hyperbolic transformations

Let $T : M \to M$ be a diffeomorphism (possibly with singularities) defined on a Riemannian manifold (M, d). Fix a subset $Y \subset M$. It is assumed that there is a "product structure": namely a family of "stable disks" $\{W^s\}$ that are disjoint and cover Y, and a family of "unstable disks" $\{W^u\}$ that are disjoint and cover Y. Each stable disk intersects each unstable disk in precisely one point. The stable and unstable disks containing y are labelled $W^s(y)$ and $W^u(y)$.

Suppose that there is a partition $\{Y_j\}$ of Y and integers $\tau(j) \geq 1$ with $\gcd\{\tau(j)\} = 1$ such that $T^{\tau(j)}(W^s(y)) \subset W^s(T^{\tau(j)}y)$ for all $y \in Y_j$. Define the return time function $\tau : Y \to \mathbb{Z}^+$ by $\tau|_{Y_j} = \tau(j)$ and the induced map $F : Y \to Y$ by $F(y) = T^{\tau(y)}(y)$.

Let s denote the *separation time* with respect to the map $F : Y \to Y$. That is, if $y, z \in Y$, then $s(y, z)$ is the least integer $n \geq 0$ such that $F^n x$, $F^n y$ lie in distinct partition elements of Y.

(P1) There exist constants $K_0 \geq 1$, $\rho_0 \in (0, 1)$ such that

   (i) If $z \in W^s(y)$, then $d(F^n y, F^n z) \leq K_0 \rho_0^n$,

   (ii) If $z \in W^u(y)$, then $d(F^n y, F^n z) \leq K_0 \rho_0^{s(y,z)-n}$,

   (iii) If $y, z \in Y$, then $d(T^j y, T^j z) \leq K_0(d(y, z) + d(Fy, Fz))$ for all $0 \leq j < \min\{\tau(y), \tau(z)\}$.

Let $\bar{Y} = Y/\sim$ where $y \sim z$ if $y \in W^s(z)$ and define the partition $\{\bar{Y}_j\}$ of $\bar{Y}$. We obtain a well-defined return time function $\tau : \bar{Y} \to \mathbb{Z}^+$ and induced map $\bar{F} : \bar{Y} \to \bar{Y}$. Suppose that the map $\bar{F} : \bar{Y} \to \bar{Y}$ and partition $\alpha = \{\bar{Y}_j\}$ separate points in $\bar{Y}$, and let s denote also the separation time on $\bar{Y}$. Fix $\theta \in (0, 1)$. Then $d_\theta(y, z) = \theta^{s(y,z)}$ defines a metric on $\bar{Y}$. Suppose further that $\bar{F} : \bar{Y} \to \bar{Y}$ is a uniformly expanding map in the sense of Subsection 2.1 on the metric space $(\bar{Y}, d_\theta)$, with partition α and constants $\lambda = 1/\theta > 1$, $K > 0$, $\eta = 1$. Let $\bar{\mu}_Y$ denote the $\bar{F}$-invariant probability measure on $\bar{Y}$ from Proposition 2.5. We assume that $\tau : \bar{Y} \to \mathbb{Z}^+$ is integrable. We suppose also that there is an F-invariant probability measure μ_Y on Y such that $\bar{\pi}_*\mu_Y = \bar{\mu}_Y$ where $\bar{\pi} : Y \to \bar{Y}$ is the quotient map.

As in Subsection 2.2, starting from $\bar{F} : \bar{Y} \to \bar{Y}$ and $\tau : \bar{Y} \to \mathbb{Z}^+$, we can form the *quotient tower* $\bar{f} : \bar{\Delta} \to \bar{\Delta}$ with $\bar{f}$-invariant mixing probability measure $\bar{\mu}_\Delta$. Similarly, starting from $F : Y \to Y$ and $\tau : Y \to \mathbb{Z}^+$, we form the tower $f : \Delta \to \Delta$ such that $F = f^\tau : Y \to Y$ with f-invariant mixing probability measure μ_Δ.

Define the semiconjugacy $\pi : \Delta \to M$, $\pi(y, \ell) = T^\ell y$. Then $\mu = \pi_*\mu_\Delta$ is a T-invariant mixing probability measure on M.

As in Subsection 2.2, we restrict to the cases $\mu(\tau > n) = O(n^{-\beta})$, $\beta > 1$, and $\mu(\tau > n) = O(e^{-An^\gamma})$, $A > 0$, $\gamma \in (0, 1]$.

Theorem 2.10 *Let $\eta \in (0, 1]$. Then there exist $C > 0$, $B > 0$ depending continuously on the constants in Theorem 2.7 (associated to the nonuniformly expanding map $\bar{f} : \bar{\Delta} \to \bar{\Delta}$) as well as η, ρ_0 and K_0, such that $|\int_M v\, w \circ T^n\, d\mu - \int_M v\, d\mu \int_M w\, d\mu| \le Ca_n \|v\|_\eta \|w\|_\eta$, for all $v, w \in C^\eta(M)$, $n \ge 1$, where $a_n = n^{-(\beta-1)}$ or e^{-Bn^γ} respectively.*

Remark 2.11 Note that there is no assumption about contraction rates along stable manifolds for T; all that is required is exponential contraction/expansion for the induced map $F : Y \to Y$. This is in contrast to [31] where exponential contraction is assumed for T (this restriction is also present in [6]) and [4] where polynomial contraction is assumed for T.

The method for removing such assumptions on contractivity of T is due to Gouëzel (based on ideas in [11]) and was used previously in [24, Appendix B].

3 Proof for uniformly expanding maps

In this section, we prove Theorem 2.1 and Proposition 2.5.

For $\psi : Y \to (0, \infty)$, we define $|\psi|_{\eta,\ell} = |\log \psi|_\eta$. Note that

$$e^{-|\psi|_{\eta,\ell}} \int_Y \psi\, dm \le \psi \le e^{|\psi|_{\eta,\ell}} \int_Y \psi\, dm. \tag{3.1}$$

Also, for at most countably many observables $\psi_k : Y \to (0, \infty)$,

$$\Big|\sum_k \psi_k\Big|_{\eta,\ell} \le \sup_k |\psi_k|_{\eta,\ell}. \tag{3.2}$$

Proposition 3.1 *Let $\psi : Y \to (0, \infty)$. Then $|P_m\psi|_{\eta,\ell} \le K + \lambda^{-\eta}|\psi|_{\eta,\ell}$.*

Proof For $a \in \alpha$ write $\psi_a = 1_a\psi$. Then $P_m\psi = \sum_a P_m\psi_a$. For $y \in Y$, we have $(P_m\psi_a)(y) = \zeta(y_a)\psi(y_a)$ where y_a is the unique preimage of y under F lying in a.

Let $x, y \in Y$ with preimages $x_a, y_a \in a$. Then

$$\begin{aligned}|\log(P_m\psi_a)(x) - \log(P_m\psi_a)(y)| &\le |\log\zeta(x_a) - \log\zeta(y_a)| + |\log\psi(x_a) - \log\psi(y_a)| \\ &\le Kd(Fx_a, Fy_a)^\eta + |\psi|_{\eta,\ell}\, d(x_a, y_a)^\eta \le (K + \lambda^{-\eta}|\psi|_{\eta,\ell})d(x,y)^\eta,\end{aligned}$$

and so $|P_m\psi_a|_{\eta,\ell} \le K + \lambda^{-\eta}|\psi|_{\eta,\ell}$. The result follows from (3.2). ∎

Proposition 3.2 *Let $\psi : Y \to (0,\infty)$. For each $t \in [0, e^{-|\psi|_{\eta,\ell}})$*

$$\Big|\psi - t\int_Y \psi\, dm\Big|_{\eta,\ell} \le \frac{|\psi|_{\eta,\ell}}{1 - te^{|\psi|_{\eta,\ell}}}.$$

Proof Let $\kappa(y) = \log\psi(y)$. Note that

$$\frac{d}{d\kappa}\log\Big(e^\kappa - t\int_Y \psi\, dm\Big) = \frac{e^\kappa}{e^\kappa - t\int_Y \psi\, dm} = \frac{1}{1 - te^{-\kappa}\int_Y \psi\, dm}.$$

By (3.1),

$$\frac{1}{1 - te^{-\kappa(y)}\int_Y \psi\, dm} = \frac{1}{1 - t\psi(y)^{-1}\int_Y \psi\, dm} \le \frac{1}{1 - te^{|\psi|_{\eta,\ell}}},$$

for all $y \in Y$. Hence, by the mean value theorem, for $x, y \in Y$,

$$\Big|\log\Big(e^{\kappa(x)} - t\int_Y \psi\, dm\Big) - \log\Big(e^{\kappa(y)} - t\int_Y \psi\, dm\Big)\Big| \le \frac{|\kappa(x) - \kappa(y)|}{1 - te^{|\psi|_{\eta,\ell}}} \le \frac{|\psi|_{\eta,\ell}\, d(x,y)^\eta}{1 - te^{|\psi|_{\eta,\ell}}}.$$

This completes the proof. ∎

Fix constants $R > 0$ and $\xi \in (0, e^{-R})$, such that $R(1 - \xi e^R) \ge K + \lambda^{-\eta}R$. (For example, choose R and ξ as in Remark 2.2.)

Proposition 3.3 *Let $\psi : Y \to (0,\infty)$ with $|\psi|_{\eta,\ell} \le R$. Then $|P_m\psi|_{\eta,\ell} \le R$.*

Proof By Proposition 3.1, $|P_m\psi|_{\eta,\ell} \le K + \lambda^{-\eta}R \le R$. ∎

Lemma 3.4 *Let ψ_1, $\psi_2 : Y \to (0,\infty)$ with $|\psi_1|_{\eta,\ell} \le R$, $|\psi_2|_{\eta,\ell} \le R$, and $\int_Y \psi_1\, dm = \int_Y \psi_2\, dm$. Let $\psi_j' = P_m\psi_j - \xi\int_Y \psi_j\, dm$ for $j = 1, 2$. Then*

(a) $|\psi_j'|_{\eta,\ell} \le R$ *for* $j = 1, 2$,

(b) $P_m\psi_1 - P_m\psi_2 = \psi_1' - \psi_2'$,

(c) $\int_Y \psi_1'\, dm = \int_Y \psi_2'\, dm = (1 - \xi)\int_Y \psi_1\, dm$.

Proof By Propositions 3.1 and 3.2,

$$|\psi_j'|_{\eta,\ell} = \Big|P_m\psi_j - \xi\int_Y \psi_j\,dm\Big|_{\eta,\ell} \le \frac{|P_m\psi_j|_{\eta,\ell}}{1-\xi e^{|P_m\psi_j|_{\eta,\ell}}} \le \frac{K+\lambda^{-\eta}R}{1-\xi e^R} \le R,$$

proving part (a). Parts (b) and (c) are immediate. ∎

Now we are ready to prove Theorem 2.1 taking $C = 4e^R(1+R)$ and $\gamma = 1-\xi$.

Proof of Theorem 2.1 Assume first that $|\phi|_\eta \le R$. Later we remove this restriction.

Since $\int_Y \phi\,dm = 0$, there exists $x, y \in Y$ such that $\phi(x) \le 0 \le \phi(y)$. Hence it follows from the assumption $|\phi|_\eta \le R$ that $|\phi|_\infty \le R$.

Write $\phi = \psi_0^+ - \psi_0^-$, where $\psi_0^+ = 1 + \max\{0,\phi\}$ and $\psi_0^- = 1 - \min\{0,\phi\}$. Then $\psi_0^\pm : Y \to [1,\infty)$ and $\int_Y \psi_0^+\,d\mu = \int_Y \psi_0^-\,d\mu \le 1 + |\phi|_\infty \le 1 + R$. For $x, y \in Y$,

$$\big|\log\psi_0^\pm(x) - \log\psi_0^\pm(y)\big| \le \big|\psi_0^\pm(x) - \psi_0^\pm(y)\big| \le |\phi(x) - \phi(y)|,$$

so $|\psi_0^\pm|_{\eta,\ell} \le |\phi|_\eta \le R$.

Define

$$\psi_{n+1}^\pm = P_m\psi_n^\pm - \xi\int_Y \psi_n^\pm\,dm, \quad n \ge 0.$$

By Lemma 3.4(a), $|\psi_n^\pm|_{\eta,\ell} \le R$ for all $n \ge 0$. By Lemma 3.4(b,c),

$$P_m^n\phi = P_m^n\psi_0^+ - P_m^n\psi_0^- = \psi_n^+ - \psi_n^-, \tag{3.3}$$

and $\int_Y \psi_n^\pm\,dm = \gamma^n\int_Y \psi_0^\pm\,dm \le (1+R)\gamma^n$. By (3.1),

$$\psi_n^\pm \le e^R\int_Y \psi_n^\pm\,dm \le e^R(1+R)\gamma^n. \tag{3.4}$$

Next, we recall the inequality

$$|a-b| \le \max\{a,b\}\,|\log a - \log b|, \text{ for all } a, b > 0. \tag{3.5}$$

By (3.5) and the definition of $|\psi|_{\eta,\ell}$, for $x, y \in Y$,

$$\begin{aligned}\big|\psi_n^\pm(x) - \psi_n^\pm(y)\big| &\le \max(\psi_n^\pm(x), \psi_n^\pm(y))\,\big|\log\psi_n^\pm(x) - \log\psi_n^\pm(y)\big| \\ &\le e^R(1+R)\gamma^n|\psi_n^\pm|_{\eta,\ell}\,d(x,y)^\eta \le e^R R(1+R)\gamma^n d(x,y)^\eta.\end{aligned}$$

Hence, $|\psi_n^\pm|_\eta \le e^R R(1+R)\gamma^n$. By (3.3),

$$|P_m^n\phi|_\eta \le 2e^R R(1+R)\gamma^n. \tag{3.6}$$

Finally, we remove the restriction $|\phi|_\eta \le R$. Note that $u = R|\phi|_\eta^{-1}\phi$ satisfies $|u|_\eta \le R$, and therefore it follows from (3.6) that

$$|P_m^n\phi|_\eta = R^{-1}|\phi|_\eta\,|P_m^n u|_\eta \le 2e^R(1+R)\gamma^n\,|\phi|_\eta.$$

Also, $\int_Y P_m^n\phi\, dm = 0$, so $|P_m^n\phi|_\infty \le |P_m^n\phi|_\eta$. Hence

$$\|P_m^n\phi\|_\eta \le 2|P_m^n\phi|_\eta \le 4e^R(1+R)\gamma^n\,|\phi|_\eta,$$

as required. ∎

Proof of Proposition 2.5 We construct an invariant probability measure $\mu \in \mathcal{M}$ and show that $|\frac{d\mu}{dm}|_{\eta,\ell} \le R$.

By Proposition 3.3, $|P_m^n 1|_{\eta,\ell} \le R$ for all $n \ge 0$. In particular, it follows from (3.1) that $|P_m 1|_\infty \le e^R$. By (3.5),

$$|P_m 1|_\eta \le |P_m 1|_\infty |P_m 1|_{\eta,\ell} \le e^R R.$$

Also, $\int_Y (P_m 1 - 1)\, dm = 0$, so by Theorem 2.1, $\|P_m^n(P_m 1 - 1)\|_\eta \le Ce^R R\gamma^n$. Hence we can define

$$\rho = \lim_{n\to\infty} P_m^n 1 = 1 + \sum_{n=0}^{\infty} P_m^n(P_m 1 - 1) \in C^\eta.$$

It is immediate that $\int_Y \rho\, dm = 1$ and $P_m\rho = \rho$, so ρ is an invariant density. Moreover, for $x, y \in Y$,

$$|\log\rho(x) - \log\rho(y)| = \lim_{n\to\infty} |\log(P_m^n 1)(x) - \log(P_m^n 1)(y)| \le Rd(x,y)^\eta,$$

so that $|\rho|_{\eta,\ell} \le R$. ∎

Remark 3.5 In this paper, we have restricted attention to expanding maps $F : Y \to Y$ satisfying the full branch condition $Fa = Y$ for all $a \in \alpha$. This is a reasonable restriction for situations where the expanding maps are obtained by inducing nonuniformly expanding maps as in [19]. More generally, the restriction is justified by the family of examples $F_\delta : [0,1] \to [0,1]$ depicted in Figure 1 below. Note that each map preserves Lebesgue measure and is mixing. Moreover, we can take $\lambda = 2$ and $K = 0$ for all δ. Nevertheless, correlations decay arbitrarily slowly as $\delta \to 0$. (Explicit constants depending on δ can be computed from [33].)

4 Proof for nonuniformly expanding maps

In this section, we prove Theorems 2.7 and 2.8. The coupling technique from probability theory, on which our proofs are based, was introduced to dynamical systems by Young [32], and has since been used in various ways by numerous authors, including [10, 12, 33]. Our proof is in many ways similar to those in the above works, but is also different: to obtain explicit control on various constants, we developed a new (to the best of our knowledge) construction of coupling and the method to apply it.

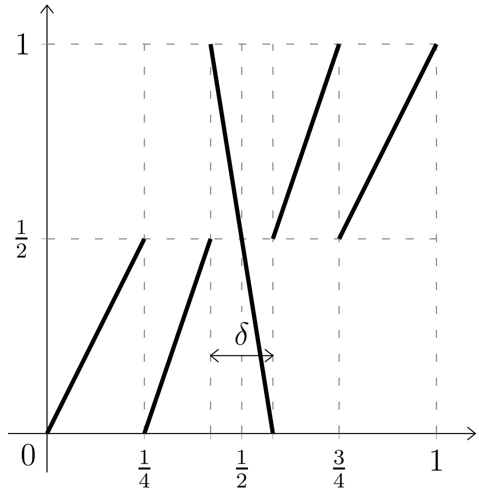

Figure 1: A family of uniformly expanding maps $F_\delta : [0,1] \to [0,1]$ with $\lambda = 2$ and $K = 0$ but with arbitrarily slow decay of correlations.

4.1 Outline of the proof

Let $\Delta_\ell = \{(y,k) \in \Delta : k = \ell\}$ denote the ℓ-th level of the tower. Our strategy is to construct a countable probability space $(W, \mathbb{P})$ and a random variable $h : W \to \mathbb{N}$ such that every sufficiently regular observable $\psi : \Delta \to [0,\infty)$ with $\int_\Delta \psi \, dm_\Delta = 1$ can be decomposed into a series $\psi = \sum_{w\in W} \psi_w$ where $\psi_w : \Delta \to [0,\infty)$ are such that $\int_\Delta \psi_w \, dm_\Delta = \mathbb{P}(w)$ and $L^{h(w)}\psi_w = \mathbb{P}(w)\bar{\tau}1_{\Delta_0}$.

Now let $\phi : \Delta \to \mathbb{R}$ and suppose that $L^N\phi = C(\psi - \psi')$ where ψ and ψ' can be decomposed as above and $C > 0$, $N \in \mathbb{N}$ are constants. We have $L^{h(w)}\psi_w = L^{h(w)}\psi'_w$, and so $L^n(\psi_w - \psi'_w) = 0$ whenever $n \ge h(w)$. Therefore

$$\begin{aligned}\int_\Delta |L^{N+n}\phi| \, dm_\Delta &\le C \sum_{w\in W: h(w)>n} \int_\Delta (L^n\psi_w + L^n\psi'_w) \, dm_\Delta \\ &= C \sum_{w\in W: h(w)>n} \int_\Delta (\psi_w + \psi'_w) \, dm_\Delta = 2C\mathbb{P}(h > n).\end{aligned}$$

In this way, decay rates for $L^n\phi$ reduce to tail estimates for h.

4.2 Recurrence to Δ_0

Given $\psi : \Delta \to [0,\infty)$, define

$$|\psi|_{\eta,\ell} = \sup_{n\ge 0} \sup_{(y,n)\ne(y',n)\in\Delta_n} \frac{|\log\psi(y,n) - \log\psi(y',n)|}{d(y,y')^\eta},$$

where $\log 0 = -\infty$ and $\log 0 - \log 0 = 0$.

As in Section 3, we fix constants $R > 0$ and $\xi \in (0, e^{-R})$, such that $R(1 - \xi e^R) \geq K + \lambda^{-\eta} R$. (For example, choose R and ξ as in Remark 2.2.) Using notation from Section 3, $(L\phi)(y, \ell) = \begin{cases} \phi(y, \ell - 1) & \ell \geq 1 \\ \sum_{a \in \alpha} \zeta(y_a)\phi(y_a, \tau(a) - 1) & \ell = 0 \end{cases}$.

Proposition 4.1 *Let $\psi : \Delta \to [0, \infty)$ with $|\psi|_{\eta,\ell} \leq R$. Then*

(a) $e^{-R}\bar{\tau} \int_{\Delta_0} \psi \, dm_\Delta \leq \psi \, 1_{\Delta_0} \leq e^R \bar{\tau} \int_{\Delta_0} \psi \, dm_\Delta$.

(b) $|L\psi|_{\eta,\ell} \leq R$.

(c) If $t \in [0, \xi]$, then $\psi' = L\psi - t\,\bar{\tau} \int_{\Delta_0} L\psi \, dm_\Delta \, 1_{\Delta_0}$ is nonnegative and $|\psi'|_{\eta,\ell} \leq R$.

Proof (a) This is the counterpart of (3.1).
(b) Let $(y, \ell), (y', \ell) \in \Delta_\ell$. If $\ell \geq 1$, then it is immediate that $|\log(L\psi)(y, \ell) - \log(L\psi)(y', \ell)| \leq Rd(y, y')^\eta$. The same calculation as in Proposition 3.1 shows that

$$|\log(L\psi)(y, 0) - \log(L\psi)(y, 0)| \leq (K + \lambda^{-\eta} R) d(y, y')^\eta \leq Rd(y, y')^\eta.$$

(c) It follows from (b) that $|L\psi|_{\eta,\ell} \leq R$. Hence, by (a), $\psi' \geq 0$. As in part (b), it is immediate that $|\log \psi'(y, \ell) - \log \psi'(y', \ell)| \leq Rd(y, y')^\eta$ for $\ell \geq 1$. Also, $\psi'(y, 0) = \chi(y) - t \int_Y \chi dm$ where $\chi : Y \to [0, \infty)$ is given by $\chi(y) = (L\psi)(y, 0)$, so it follows from Proposition 3.2 that $|\log \psi'(y, 0) - \log \psi'(y', 0)| \leq (K + \lambda^{-\eta} R)(1 - te^R)^{-1} d(y, y')^\eta \leq Rd(y, y')^\eta$. ∎

Define $N = N_1 + N_2$ where

$$N_1 = \max\{I_k^2\}, \quad N_2 = \min\bigl\{n \geq 1 : m_\Delta(\cup_{\ell \geq n} \Delta_\ell) \leq \tfrac{1}{2} e^{-R} \bar{\tau}^{-1}\bigr\}.$$

Let $\mathcal{A}$ be the set of observables $\psi : \Delta \to [0, \infty)$ such that $|\psi|_\infty \leq e^R \bar{\tau} \int_\Delta \psi \, dm_\Delta$ and $|\psi|_{\eta,\ell} \leq R$. Define $\mathcal{B} = L^N \mathcal{A}$.

Corollary 4.2 *(a) If $\psi : \Delta \to [0, \infty)$ is supported on Δ_0, and $|\psi|_{\eta,\ell} \leq R$, then $\psi \in \mathcal{A}$.*
(b) If $\psi, \psi' \in \mathcal{A}$ (or $\mathcal{B}$) and $t \geq 0$, then $L\psi$, $\psi + \psi'$ and $t\psi$ belong in $\mathcal{A}$ (or $\mathcal{B}$). In particular, $\mathcal{B} \subset \mathcal{A}$.

Proof Part (a) follows from Proposition 4.1(a). Next, let $\psi \in \mathcal{A}$. We show that $L\psi \in \mathcal{A}$; the remaining statements in part (b) are immediate. By Proposition 4.1(b), $|L\psi|_{\eta,\ell} \leq R$. Also, using the definition of $\mathcal{A}$ and Proposition 4.1(a),

$$|1_{\Delta \setminus \Delta_0} L\psi|_\infty \leq |\psi|_\infty \leq e^R \bar{\tau} \int_\Delta \psi \, dm_\Delta = e^R \bar{\tau} \int_\Delta L\psi \, dm_\Delta, \quad \text{and}$$

$$|1_{\Delta_0} L\psi|_\infty \leq e^R \bar{\tau} \int_{\Delta_0} L\psi \, dm_\Delta \leq e^R \bar{\tau} \int_\Delta L\psi \, dm_\Delta.$$

Hence $|L\psi|_\infty \leq e^R \bar{\tau} \int_\Delta L\psi \, dm_\Delta$, so $L\psi \in \mathcal{A}$. ∎

Proposition 4.3 *If $\psi \in \mathcal{A}$, then* $\max_{0\le j\le N_2}\int_{\Delta_0} L^j\psi\, dm_\Delta \ge \frac{1}{2}e^{-R}\bar{\tau}^{-1}\int_\Delta \psi\, dm_\Delta$.

Proof It follows from the definition of N_2 and $\mathcal{A}$ that $m_\Delta(\cup_{\ell=N_2+1}^\infty \Delta_\ell)|\psi|_\infty \le \frac{1}{2}\int_\Delta \psi\, dm_\Delta$. Hence

$$\begin{aligned}\int_\Delta \psi\, dm_\Delta &= \int_{\bigcup_{\ell=0}^{N_2}\Delta_\ell} L^{N_2}\psi\, dm_\Delta + \int_{\bigcup_{\ell=N_2+1}^{\infty}\Delta_\ell} L^{N_2}\psi\, dm_\Delta \\ &\le \int_{\bigcup_{\ell=0}^{N_2}\Delta_\ell} L^{N_2}\psi\, dm_\Delta + \frac{1}{2}\int_\Delta \psi\, dm_\Delta,\end{aligned}$$

so $\int_{\bigcup_{\ell=0}^{N_2}\Delta_\ell} L^{N_2}\psi\, dm_\Delta \ge \frac{1}{2}\int_\Delta \psi\, dm_\Delta$.

Next, if $\ell \le N_2$ then $(L^{N_2}\psi)(y,\ell) = (L^{N_2-\ell}\psi)(y,0)$, and so $\int_{\Delta_\ell} L^{N_2}\psi\, dm_\Delta \le m_\Delta(\Delta_\ell)|L^{N_2-\ell}\psi\, 1_{\Delta_0}|_\infty \le m_\Delta(\Delta_\ell)\max_{0\le j\le N_2}|L^j\psi\, 1_{\Delta_0}|_\infty$. Hence, by Proposition 4.1(a,b),

$$\int_{\bigcup_{\ell=0}^{N_2}\Delta_\ell} L^{N_2}\psi\, dm_\Delta \le \max_{0\le j\le N_2}|L^j\psi\, 1_{\Delta_0}|_\infty \le e^R\bar{\tau}\max_{0\le j\le N_2}\int_{\Delta_0} L^j\psi\, dm_\Delta.$$

The result follows. ∎

Proposition 4.4 *If $|\psi|_{\eta,\ell} \le R$, then $\int_{\Delta_0} L^n\psi\, dm_\Delta \ge (e^{-R}\delta)^n\int_{\Delta_0}\psi\, dm_\Delta$ for all $n \ge N_1$.*

Proof By Proposition 4.1(a), $\inf_{\Delta_0}\psi \ge e^{-R}\bar{\tau}\int_{\Delta_0}\psi\, dm_\Delta$. By our assumptions, $m_\Delta(\{x\in\Delta_0 : f^{I_k}x\in\Delta_0\}) \ge \delta/\bar{\tau}$ for every I_k. Hence

$$\begin{aligned}\int_{\Delta_0} L^{I_k}\psi\, dm_\Delta &= \int_\Delta \psi\ 1_{\Delta_0}\circ f^{I_k}\, dm_\Delta \ge \int_{\Delta_0}\psi\ 1_{\Delta_0}\circ f^{I_k}\, dm_\Delta \\ &\ge \inf_{\Delta_0}\psi\ m_\Delta(\{x\in\Delta_0 : f^{I_k}x\in\Delta_0\}) \ge e^{-R}\delta\int_{\Delta_0}\psi\, dm_\Delta.\end{aligned}$$

By [27], every $n \ge N_1$ can be written as $n = \sum_k n_k I_k$, where n_k are nonnegative integers. By Proposition 4.1(b), it follows inductively that

$$\int_{\Delta_0} L^n\psi\, dm_\Delta \ge (e^{-R}\delta)^{\sum_k n_k}\int_{\Delta_0}\psi\, dm_\Delta \ge (e^{-R}\delta)^n\int_{\Delta_0}\psi\, dm_\Delta,$$

as required. ∎

Lemma 4.5 *If $\psi \in \mathcal{B}$, then $\int_{\Delta_0}\psi\, dm_\Delta \ge \epsilon\int_\Delta \psi\, dm_\Delta$, where $\epsilon = \frac{1}{2}e^{-R}\bar{\tau}^{-1}(e^{-R}\delta)^N$.*

Proof By definition of $\mathcal{B}$, there exists $\psi' \in \mathcal{A}$ such that $L^{N_1+N_2}\psi' = \psi$. By Proposition 4.3, there exists $j \le N_2$ such that $\int_{\Delta_0} L^j\psi'\,dm_\Delta \ge \frac{1}{2}e^{-R}\bar{\tau}^{-1}\int_\Delta \psi'\,dm_\Delta$. By Proposition 4.4 (taking $n = N_1 + N_2 - j \ge N_1$),

$$\begin{aligned}\int_{\Delta_0}\psi\,dm_\Delta &= \int_{\Delta_0} L^{N_1+N_2}\psi'\,dm_\Delta \ge (e^{-R}\delta)^{N_1+N_2-j}\int_{\Delta_0} L^j\psi'\,dm_\Delta \\ &\ge \frac{1}{2}e^{-R}\bar{\tau}^{-1}(e^{-R}\delta)^{N_1+N_2}\int_\Delta \psi'\,dm_\Delta = \epsilon\int_\Delta \psi\,dm_\Delta,\end{aligned}$$

as required. ∎

4.3 Decomposition in $\mathcal{B}$

Next, we introduce constants $p_n,\ t_n \in [0,1]$,

$$\begin{aligned}&t_1 = 1-\epsilon, \quad t_n = \min\{t_1,\ e^R\bar{\tau}m_\Delta(\cup_{\ell=n}^\infty \Delta_\ell)\},\ n \ge 2, \\ &p_{-1} = \xi\epsilon, \quad p_0 = (1-\xi)\epsilon, \quad p_n = t_n - t_{n+1},\ n \ge 1.\end{aligned}$$

The monotonicity of the sequence t_n ensures that $p_n \ge 0$ for all n. Note that $\sum_{n=-1}^\infty p_n = 1$.

Let $E_0 = \Delta_0$ and $E_k = \{(y,\ell) \in \Delta : \ell = \tau(y) - k,\ \ell \ge 1\}$ for $k \ge 1$. Then $\{E_0, E_1, \ldots\}$ defines a partition of Δ and $m_\Delta(E_k) = m_\Delta(\Delta_k)$ for all k.

Proposition 4.6 *If $\psi \in \mathcal{B}$ with $\int_\Delta \psi\,dm_\Delta = 1$, then $\int_{\bigcup_{\ell=n}^\infty E_\ell}\psi\,dm_\Delta \le t_n$, for $n \ge 1$.*

Proof By Lemma 4.5, $\int_{\bigcup_{\ell=n}^\infty E_\ell}\psi\,dm_\Delta \le \int_{\bigcup_{\ell=1}^\infty E_\ell}\psi\,dm_\Delta \le 1-\epsilon = t_1$ for all $n \ge 1$.

By definition of $\mathcal{B}$, for $n \ge 2$ we have in addition that $\int_{\bigcup_{\ell=n}^\infty E_\ell}\psi\,dm_\Delta \le m_\Delta(\cup_{\ell=n}^\infty \Delta_\ell)|\psi|_\infty \le e^R\bar{\tau}m_\Delta(\cup_{\ell=n}^\infty\Delta_\ell)$. The result follows by definition of t_n. ∎

Proposition 4.7 *Let $p_j,\ q_j \in [0,\infty)$ be sequences such that $\sum_{j=0}^\infty p_j = \sum_{j=0}^\infty q_j < \infty$ and $\sum_{j=0}^k q_j \ge \sum_{j=0}^k p_j$ for all $k \ge 0$. Then there exist $s_{k,j} \in [0,1]$, $0 \le j \le k$, such that $\sum_{j=0}^k s_{k,j}q_j = p_k$ for all $k \ge 0$ and $\sum_{k=j}^\infty s_{k,j} = 1$ for all $j \ge 0$.*

Proof We assume that $q_j > 0$ for all j; otherwise set $s_{k,j} = \delta_{k,j}$ for $k \le j$ whenever $q_j = 0$.

For $k = 0$, choose $s_{0,0} = p_0/q_0$. Next let $k \ge 1$, and suppose inductively that $s_{k',j}$ have been constructed for $0 \le j \le k' \le k-1$, such that $\sum_{j=0}^{k'} s_{k',j}q_j = p_{k'}$ for $k' \le k-1$ and $\sum_{k'=j}^{k-1} s_{k',j} \le 1$ for $j \le k-1$.

Define $s_{k,0}, s_{k,1}, \ldots, s_{k,k} \in [0,1]$ (in this order) by

$$s_{k,j} = \min\Big\{1 - \sum_{k'=j}^{k-1} s_{k',j},\ \frac{p_k - \sum_{j'=0}^{j-1} s_{k,j'}q_{j'}}{q_j}\Big\}, \quad j = 0,1,\ldots,k.$$

By construction, $\sum_{j=0}^{k} s_{k,j}q_j \le p_k$. If $\sum_{j=0}^{k} s_{k,j}q_j < p_k$, then necessarily $\sum_{k'=j}^{k} s_{k',j} = 1$ for all $j \le k$, and so

$$\sum_{k'=0}^{k} q_{k'} = \sum_{k'=0}^{k}\sum_{j=0}^{k'} s_{k',j}q_j = \sum_{j=0}^{k} s_{k,j}q_j + \sum_{k'=0}^{k-1} p_{k'} < \sum_{k'=0}^{k} p_{k'},$$

which is a contradiction. Hence $\sum_{j=0}^{k} s_{k,j}q_j = p_k$.

By the above construction, $\sum_{j=0}^{k} s_{k,j}q_j = p_k$ for $k \ge 0$ and $\sum_{k=j}^{\infty} s_{k,j} \le 1$ for $j \ge 0$. Since also $\sum_{j=0}^{\infty} p_j = \sum_{j=0}^{\infty} q_j < \infty$, we conclude that $\sum_{k=j}^{\infty} s_{k,j} = 1$. ∎

Lemma 4.8 *Let $\psi : \Delta \to [0,\infty)$ be such that $L^n\psi \in \mathcal{B}$ for some $n \ge 0$. Then $\psi = \sum_{k=-1}^{\infty} \psi_k$, where $\psi_k : \Delta \to [0,\infty)$ are such that*

(i) $L^n\psi_{-1} = p_{-1}\bar{\tau}\int_\Delta \psi\, dm_\Delta 1_{\Delta_0}$, (ii) $L^{k+n}\psi_k \in \mathcal{A}$ for all $k \ge 0$,

(iii) $\int_\Delta \psi_k\, dm_\Delta = p_k \int_\Delta \psi\, dm_\Delta$ for all $k \ge -1$.

Proof First we consider the case $n = 0$. Suppose without loss that $\int_\Delta \psi\, dm_\Delta = 1$. Define $\psi_{-1} = p_{-1}\bar{\tau}1_{\Delta_0}$ in accordance with properties (i) and (iii).

By Lemma 4.5, $\int_{\Delta_0} \psi\, dm_\Delta \ge \epsilon$. Hence $t = p_{-1}/\int_{\Delta_0}\psi\, dm_\Delta = \xi\epsilon/\int_{\Delta_0}\psi\, dm_\Delta \in [0,\xi]$. Since $\psi \in \mathcal{B} \subset L\mathcal{A}$, it follows from Proposition 4.1(c) that

$$\psi' = \psi - t\bar{\tau}\textstyle\int_{\Delta_0}\psi\, dm_\Delta\, 1_{\Delta_0} = \psi - p_{-1}\bar{\tau}1_{\Delta_0} = \psi - \psi_{-1}$$

is nonnegative and $|\psi'|_{\eta,\ell} \le R$. Setting $g_0 = \psi' 1_{\Delta_0}$, we obtain that $\psi 1_{\Delta_0} = \psi_{-1} + g_0$ where g_0 is nonnegative and $|g_0|_{\eta,\ell} \le R$. By Corollary 4.2(a), $g_0 \in \mathcal{A}$.

Define $g_k = \psi 1_{E_k}$ for $k \ge 1$. Note that $L^k g_k$ is supported on Δ_0 and $|L^k g_k|_{\eta,\ell} \le R$. By Corollary 4.2(a), $L^k g_k \in \mathcal{A}$.

Now $\psi = \psi 1_{\Delta_0} + \sum_{k=1}^{\infty} g_k = \psi_{-1} + \sum_{k=0}^{\infty} g_k$. By Proposition 4.6,

$$p_{-1} + \sum_{j=0}^{k}\int_\Delta g_j\, dm_\Delta = 1 - \sum_{j=k+1}^{\infty}\int_\Delta g_j\, dm_\Delta \ge 1 - t_{k+1} = \sum_{j=-1}^{k} p_j.$$

Setting $q_k = \int_\Delta g_k\, dm_\Delta$, we have that $\sum_{j=0}^{k} q_j \ge \sum_{j=0}^{k} p_j$ for all $k \ge 0$. Choose $s_{k,j} \in [0,1]$ as in Proposition 4.7, and define $\psi_k : \Delta \to [0,\infty)$, $k \ge 0$, by

$$\psi_k = \textstyle\sum_{j=0}^{k} s_{k,j}g_j.$$

By construction, condition (iii) holds for all k. Condition (ii) is satisfied by Corollary 4.2(b). Finally, by Proposition 4.7,

$$\sum_{k=0}^{\infty}\psi_k = \sum_{k=0}^{\infty}\sum_{j=0}^{k} s_{k,j}g_j = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} s_{k,j}g_j = \sum_{j=0}^{\infty} g_j = \psi - \psi_{-1},$$

completing the proof for $n = 0$.

Now suppose $L^n\psi \in \mathcal{B}$ for some $n \geq 1$. Setting $\psi' = L^n\psi$ and applying the result for $n = 0$, we can write $\psi' = \sum_{k=-1}^{\infty} \psi'_k$ where ψ'_k satisfy properties (i)–(iii). Define $\psi_k = \big(\frac{\psi'_k}{\psi'} \circ f^n\big)\psi$ with the convention that $0/0 = 0$. Then $L^n\psi_k = \frac{\psi'_k}{\psi'}L^n\psi = \psi'_k$, so properties (i)–(iii) are passed down from ψ'_k to ψ_k. Also $\sum_{k=-1}^{\infty}\psi_k = \big(\frac{\psi'}{\psi'}\circ f^n\big)\psi = \psi$. ∎

Let W be the countable set of all finite words in the alphabet $\mathbb{N} = \{0, 1, 2, \ldots\}$ including the zero length word, and let W_k be the subset consisting of words of length k. Let $\mathbb{P}$ be the probability measure on W given for $w = w_1 \cdots w_k \in W$ by $\mathbb{P}(w) = p_{-1}p_{w_1} \cdots p_{w_k}$. Define $h : W \to \mathbb{N}$ by $h(w) = \Sigma w + N|w|$, where $\Sigma w = w_1 + \cdots + w_k$ and $|w| = k$ for $w = w_1 \cdots w_k$.

Proposition 4.9 *Let $\psi \in \mathcal{B}$ with $\int_\Delta \psi \, dm_\Delta = 1$. Then $\psi = \sum_{w\in W}\psi_w$, where $\psi_w : \Delta \to [0, \infty)$ are such that $\int_\Delta \psi_w \, dm_\Delta = \mathbb{P}(w)$ and $L^{h(w)}\psi_w = \mathbb{P}(w)\bar{\tau}1_{\Delta_0}$.*

Proof Write $\psi = \sum_{k=-1}^{\infty}\psi_k$ as in Lemma 4.8 (with $n = 0$). By properties (iii) and (i), $\int_\Delta \psi_k \, dm_\Delta = p_k$ for all $k \geq -1$, and $\psi_{-1} = p_{-1}\bar{\tau}1_{\Delta_0}$.

Also $L^{k+N}\psi_k \in \mathcal{B}$ by property (ii), allowing us to apply Lemma 4.8 to each ψ_k (with $n = k + N$), yielding

$$\psi = \psi_{-1} + \sum_{k=0}^{\infty}\psi_k = \psi_{-1} + \sum_{k=0}^{\infty}\Big(\psi_{-1,k} + \sum_{j=0}^{\infty}\psi_{j,k}\Big),$$

where

$$\int_\Delta \psi_{j,k} \, dm_\Delta = p_j \int_\Delta \psi_k \, dm_\Delta = p_j p_k,$$

$$L^{k+N}\psi_{-1,k} = p_{-1}\bar{\tau}\int_\Delta \psi_k \, dm_\Delta \, 1_{\Delta_0} = p_{-1}p_k\bar{\tau}1_{\Delta_0}.$$

At the next step,

$$\psi = \psi_{-1} + \sum_{k=0}^{\infty}\psi_{-1,k} + \sum_{j,k=0}^{\infty}\Big(\psi_{-1,j,k} + \sum_{i=0}^{\infty}\psi_{i,j,k}\Big),$$

$$= \psi_{-1} + \sum_{w\in W_1}\psi_{-1,w} + \sum_{w\in W_2}\psi_{-1,w} + \sum_{i,j,k=0}^{\infty}\psi_{i,j,k},$$

where

$$\int_\Delta \psi_{i,j,k} = p_i p_j p_k, \quad L^{j+k+2N}\psi_{-1,j,k} = p_{-1}p_j p_k\bar{\tau}1_{\Delta_0}.$$

In particular, for the terms $\psi_{-1,w}$ with $w \in W_0 \cup W_1 \cup W_2$, we have the required properties $\int_\Delta \psi_{-1w} \, dm_\Delta = \mathbb{P}(w)$ and $L^{h(w)}\psi_{-1,w} = \mathbb{P}(w)\bar{\tau}1_{\Delta_0}$.

In this way we obtain $\psi = \sum_{w\in W}\psi_{-1w}$ where $\int_\Delta \psi_{-1w} \, dm_\Delta = \mathbb{P}(w)$ and $L^{h(w)}\psi_{-1,w} = \mathbb{P}(w)\bar{\tau}1_{\Delta_0}$. ∎

4.4 Proof of Theorems 2.7 and 2.8

Let $\phi : \Delta \to \mathbb{R}$ be an observable as in Theorem 2.7, i.e. $\|\phi\|_\eta < \infty$ and $\int_\Delta \phi \, dm_\Delta = 0$. Define $\tilde{\psi}, \tilde{\psi}' : \Delta \to [0, \infty)$ by

$$\tilde{\psi} = 1 + \frac{\phi}{\|\phi\|_\eta (1 + R^{-1})}, \qquad \tilde{\psi}' \equiv 1.$$

Then $L^n \phi = \|\phi\|_\eta (1 + R^{-1})(\psi - \psi')$, where $\psi = L^N \tilde{\psi}$, $\psi' = L^N \tilde{\psi}'$.

Now $\int_\Delta \psi \, dm_\Delta = \int_\Delta \psi' \, dm_\Delta = 1$. Next,

$$|\tilde{\psi}|_\infty \le 1 + \frac{1}{1 + R^{-1}} \le 1 + R \le e^R \le \bar{\tau} e^R.$$

Also, $|\tilde{\psi}| \ge 1 - (1 + R^{-1})^{-1} = (1 + R)^{-1}$ and $|\tilde{\psi}|_\eta \le (1 + R^{-1})^{-1}$, so for $x, y \in \Delta$,

$$|\log \tilde{\psi}(x) - \log \psi(y)| \le |\tilde{\psi}^{-1}|_\infty \, |\tilde{\psi}(x) - \tilde{\psi}(y)| \le \frac{R+1}{1 + R^{-1}} d_\Delta(x, y) = R d_\Delta(x, y).$$

Thus $|\tilde{\psi}|_{\eta,\ell} \le R$. We have shown that $\tilde{\psi} \in \mathcal{A}$, and hence $\psi \in \mathcal{B}$. Clearly, $\psi' \in \mathcal{B}$.

We have shown that ψ and ψ' satisfy the hypotheses of Proposition 4.9 and hence admit the decompositions given in the conclusion of Proposition 4.9. We are therefore in the situation described in Subsection 4.1 (with $C = \|\phi\|_\eta (1 + R^{-1})$, and the argument there shows that

$$\int_\Delta |L^{N+n} \phi| \, dm_\Delta \le 2 \|\phi\|_\eta (1 + R^{-1}) \mathbb{P}(h > n).$$

To prove Theorem 2.7, it remains to estimate the decay of $\mathbb{P}(h > n)$.

Recall that W_k is the subset of W consisting of words of length k. Then $\mathbb{P}(W_k) = (1 - p_{-1})^k p_{-1}$. Elements of W_k have the form $w_1 \cdots w_k$ where $w_1, \ldots, w_k$ can be regarded as independent identically distributed random variables, drawn from $\mathbb{N}$ with distribution $\mathbb{P}(w_1 = n) = p_n / (1 - p_{-1})$. Also, $\mathbb{P}(|w| \ge n) = (1 - p_{-1})^n$.

Polynomial tails

Proposition 4.10 *Suppose that there exists $C_\tau > 0$ and $\beta > 1$ such that $m(\tau \ge n) \le C_\tau n^{-\beta}$ for $n \ge 1$.*

Then $\mathbb{P}(h \ge n) \le C n^{-(\beta - 1)}$ for $n \ge 1$, where C depends continuously on C_τ, β, R, N and p_{-1}.

Proof Let $\tilde{t}_n = \bar{\tau} e^R m_\Delta(\bigcup_{\ell=n}^\infty \Delta_\ell)$. Then

$$p_n = t_n - t_{n+1} \le \tilde{t}_n - \tilde{t}_{n+1} = \bar{\tau} e^R m_\Delta(\Delta_n) = e^R m(\tau \ge n) \le e^R C_\tau n^{-\beta}.$$

Using the inequality $\sum_{j\ge n} j^{-\beta} \le n^{-\beta} + \int_n^\infty x^{-\beta}\,dx \le \beta n^{-(\beta-1)}/(\beta-1)$, we obtain

$$\mathbb{P}(w_1 \ge n) = (1-p_{-1})^{-1}\sum_{j\ge n} p_j \le C_\tau e^R (1-p_{-1})^{-1}\frac{\beta n^{-(\beta-1)}}{\beta-1} = C_1 n^{-(\beta-1)},$$

where $C_1 = C_\tau e^R(1-p_{-1})^{-1}\beta(\beta-1)^{-1}$. It follows that for $w\in W$, $k\ge 1$,

$$\begin{aligned}\mathbb{P}(\Sigma w \ge n \,|\, w\in W_k) &= \mathbb{P}(w_1+\cdots+w_k \ge n)\\ &\le \sum_{j=1}^k \mathbb{P}(w_j \ge n/k) = k\mathbb{P}(w_1\ge n/k) \le C_1 k^\beta n^{-(\beta-1)}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{P}(\Sigma w\ge n) &= \sum_{k=1}^\infty \mathbb{P}(\Sigma w\ge n\,|\, w\in W_k)\,\mathbb{P}(W_k)\\ &\le C_1 n^{-(\beta-1)}\sum_{k=1}^\infty k^\beta(1-p_{-1})^k p_{-1} = C_1' n^{-(\beta-1)},\end{aligned}$$

where $C_1' = C_1 p_{-1}\sum_{k=1}^\infty k^\beta (1-p_{-1})^k$. Finally,

$$\begin{aligned}\mathbb{P}(h(w)\ge n) &= \mathbb{P}(\Sigma w + N|w| \ge n)\\ &\le \mathbb{P}(\Sigma w\ge n/2) + \mathbb{P}(|w|\ge n/(2N)) \le C_1' 2^{\beta-1} n^{-(\beta-1)} + (1-p_{-1})^{n/(2N)}.\end{aligned}$$

The result follows. ∎

(Stretched) exponential tails

Proposition 4.11 *Let $X_1,\ldots,X_k$ be nonnegative random variables. Suppose that there exist $\alpha>0$, $\gamma\in(0,1]$, such that*

$$\mathbb{P}(X_j\ge t\,|\, X_1=x_1,\ldots,X_{j-1}=x_{j-1}) \le Ce^{-\alpha t^\gamma}$$

for all $t\ge 0$, $1\le j\le k$ and $x_1,\ldots,x_{j-1}\ge 0$. Then for all $\beta\in(0,\alpha/2]$, $t\ge 0$,

$$\mathbb{P}(X_1+\cdots+X_k\ge t) \le (1+\beta C_1)^k e^{-\beta t^\gamma},$$

where C_1 depends continuously on C, γ and α.

Proof Note that $\mathbb{E}(e^{\beta X_1^\gamma}) = \int_0^\infty \mathbb{P}(e^{\beta X_1^\gamma}\ge t)\,dt = 1+\int_1^\infty \mathbb{P}(e^{\beta X_1^\gamma}\ge t)\,dt$. Making the substitution $t=e^{\beta s^\gamma}$, we obtain

$$\mathbb{E}(e^{\beta X_1^\gamma}) = 1+\beta\gamma\int_0^\infty s^{\gamma-1}e^{\beta s^\gamma}\mathbb{P}(X_1\ge s)\,ds \le 1+C\beta\gamma\int_0^\infty s^{\gamma-1}e^{-(\alpha-\beta)s^\gamma}\,ds \le 1+\beta C_1,$$

where $C_1 = C\gamma\int_0^\infty s^{\gamma-1}e^{-\frac{1}{2}\alpha s^\gamma}\,ds$. Similarly, $\mathbb{E}(e^{\beta X_j^\gamma}\,|\,X_1,\dots,X_{j-1}) \le 1+\beta C_1$. Hence

$$\begin{aligned}\mathbb{E}\big(e^{\beta(X_1+\cdots+X_k)^\gamma}\big) &\le \mathbb{E}\big(e^{\beta(X_1^\gamma+\cdots+X_k^\gamma)}\big) = \mathbb{E}[\mathbb{E}(e^{\beta(X_1^\gamma+\cdots+X_k^\gamma)}\,|\,X_1,\dots,X_{k-1})] \\ &= \mathbb{E}[e^{\beta(X_1^\gamma+\cdots+X_{k-1}^\gamma)}\mathbb{E}(e^{\beta X_k^\gamma}\,|\,X_1,\dots,X_{k-1})] \\ &\le (1+\beta C_1)\mathbb{E}(e^{\beta(X_1^\gamma+\cdots+X_{k-1}^\gamma)}) \le \cdots \le (1+\beta C_1)^k.\end{aligned}$$

The result follows from Markov's inequality. ∎

Proposition 4.12 *Suppose that there exist C_τ, $A > 0$, $\gamma \in (0,1]$ such that $m(\tau \ge n) \le C_\tau e^{-An^\gamma}$ for $n \ge 1$.*

Then $\mathbb{P}(h \ge n) \le Ce^{-Bn^\gamma}$ for all $n \ge 1$, where $C > 0$ and $B \in (0,A)$ depend continuously on C_τ, A, γ, R, N and p_{-1}.

Proof Following the proof of Proposition 4.10, $p_n \le e^R m(\tau \ge n) \le e^R C_\tau e^{-An^\gamma}$. Using that $x^q \le (2q)^q e^{x/2}$ for all $x, q > 0$,

$$\begin{aligned}\sum_{j\ge n} e^{-Aj^\gamma} &\le e^{-An^\gamma} + \int_n^\infty e^{-At^\gamma}\,dt = e^{-An^\gamma} + \gamma^{-1}A^{-1/\gamma}\int_{An^\gamma}^\infty e^{-s}s^{\frac{1}{\gamma}-1}\,ds \\ &\le e^{-An^\gamma} + C_{A,\gamma}\int_{An^\gamma}^\infty e^{-s/2}\,ds \le 3C_{A,\gamma}\,e^{-\frac{1}{2}An^\gamma},\end{aligned}$$

where $C_{A,\gamma} \ge 1$ is a constant depending continuously on A,γ. Hence

$$\mathbb{P}(w_1 \ge n) = (1-p_{-1})^{-1}\sum_{j\ge n} p_j \le 3(1-p_{-1})^{-1}e^R C_\tau C_{A,\gamma}e^{-\frac{1}{2}An^\gamma}.$$

By Proposition 4.11, for $B \in (0, \frac{1}{4}A]$,

$$\mathbb{P}(\Sigma w \ge n\,|\,w \in W_k) = \mathbb{P}(w_1+\cdots+w_k \ge n) \le (1+BC_1)^k e^{-Bn^\gamma},$$

where C_1 depends continuously on C_τ, A, γ, R, p_{-1}.

Let $r = (1+BC_1)(1-p_{-1})$ and choose B small enough that $r < 1$. Then

$$\mathbb{P}(\Sigma w \ge n) = \sum_{k=0}^\infty \mathbb{P}(\Sigma w \ge n\,|\,w \in W_k)\,\mathbb{P}(W_k) \le e^{-Bn^\gamma}p_{-1}\sum_{k=0}^\infty r^k = C' e^{-Bn^\gamma},$$

where $C' = p_{-1}(1-r)^{-1}$.

Finally,

$$\begin{aligned}\mathbb{P}(h(w) \ge n) &= \mathbb{P}(\Sigma w + N|w| \ge n) \\ &\le \mathbb{P}(\Sigma w \ge n/2) + \mathbb{P}(|w| \ge n/(2N)) \le C' e^{-Bn^\gamma/2^\gamma} + (1-p_{-1})^{n/(2N)}.\end{aligned}$$

The result follows. ∎

Proof of Theorem 2.8 As in the proof of Theorem 2.7, we can write $\phi = C_0(\psi - \psi')$, where $C_0 = \|\phi\|_\eta(1 + R^{-1})$, and $\psi, \psi' \in \mathcal{A}$ with $\int_\Delta \psi\, dm_\Delta = \int_\Delta \psi'\, dm_\Delta = 1$. By Corollary 4.2(b), $|L^n\psi|_{\eta,\ell}, |L^n\psi'|_{\eta,\ell} \le R$ and $|L^n\psi|_\infty, |L^n\psi'|_\infty \le \bar\tau e^R$ for all $n \ge 0$.

Next

$$\big|(L^n\psi)(x) - (L^n\psi)(y)\big| \le |L^n\psi|_\infty \big|\log(L^n\psi)(x) - \log(L^n\psi)(y)\big|,$$

so $|L^n\psi|_\eta \le \bar\tau e^R R$. Similarly, $|L^n\psi'|_\eta \le \bar\tau e^R R$. Hence

$$\|L^n\phi\|_\eta \le C_0(\|\psi\|_\eta + \|\psi'\|_\eta) \le C_0(2\bar\tau e^R + 2\bar\tau e^R R) \le C_1\|\phi\|_\eta,$$

where $C_1 = 2\bar\tau e^R(1 + R)(1 + R^{-1})$. Let $\tilde\phi = \sum_{k=0}^{d-1} L^k\phi$. Then $\|\tilde\phi\|_\eta \le C_1 d\|\phi\|_\eta$.

For $r = 0, \dots, d-1$, define $\Delta(r) = \{(y,\ell) \in \Delta : \ell \equiv r \bmod d\}$. Then $f^d : \Delta(r) \to \Delta(r)$ is a mixing Young tower with data $\{I_k/d\}$, δ, replacing the data $\{I_k\}$, δ, for Δ.

Note that $\sum_{k=0}^{d-1} 1_{\Delta(r)} \circ f^k \equiv 1$. Hence for $r = 0, \dots, d-1$,

$$\int_{\Delta(r)} \tilde\phi\, dm_\Delta = \sum_{k=0}^{d-1} \int_\Delta 1_{\Delta(r)} L^k\phi\, dm_\Delta = \sum_{k=0}^{d-1} \int_\Delta 1_{\Delta(r)} \circ f^k\, \phi\, dm_\Delta = \int_\Delta \phi\, dm_\Delta = 0.$$

Thus for each $r = 0, \dots, d-1$, we are in the situation of Theorem 2.7 with Δ, f, ϕ replaced by $\Delta(r)$, f^d, $\tilde\phi$. In the case of polynomial tails,

$$\begin{aligned}\int_\Delta \Big|\sum_{k=0}^{d-1} L^{nd+k}\phi\Big|\, dm_\Delta &= \sum_{r=0}^{d-1} \int_{\Delta(r)} \Big|\sum_{k=0}^{d-1} L^{nd+k}\phi\Big|\, dm_\Delta = \sum_{r=0}^{d-1} \int_{\Delta(r)} |L^{nd}\tilde\phi|\, dm_\Delta \\ &\le C\|\tilde\phi\|_\eta (nd)^{-(\beta-1)} \le CC_1 d^{-(\beta-1)}\|\phi\|_\eta\, n^{-(\beta-1)},\end{aligned}$$

and similarly for the (stretched) exponential case. ∎

5 Proof for nonuniformly hyperbolic transformations

In this section we prove Theorem 2.10.

The separation time for $\bar F : \bar Y \to \bar Y$ extends to a separation time on $\bar\Delta$: define $s((y,\ell),(y',\ell')) = s(y,y')$ if $\ell = \ell'$ and 0 otherwise. Recall that we fixed $\theta \in (0,1)$. Define the metric d_θ on $\bar\Delta$ by setting $d_\theta(p,q) = \theta^{s(p,q)}$.

Recall that the transfer operator P corresponding to $\bar F : \bar Y \to \bar Y$ and $\bar\mu_Y$ has the form $(P\phi)(y) = \sum_{a\in\alpha} \zeta(y_a)\phi(y_a)$. Also, $(P^n\phi)(y) = \sum_{a\in\alpha_n} \zeta_n(y_a)\phi(y_a)$ where $\alpha_n = \bigvee_{k=0}^{n-1} \bar F^{-k}\alpha$ is the partition of $\bar Y$ into n-cylinders and $\zeta_n = \zeta\,\zeta\circ\bar F \cdots \zeta\circ\bar F^{n-1}$.

Proposition 5.1 *Let $a \in \alpha_n$ and $y, y' \in a$. Then (a) $K_1^{-1}\bar\mu_Y(a) \le \zeta_n(y) \le K_1\bar\mu_Y(a)$, (b) $|\zeta_n(y) - \zeta_n(y')| \le K_1\bar\mu_Y(a)d_\theta(\bar F^n y, \bar F^n y')$, where $K_1 = e^{(1-\theta)^{-1}K}(1-\theta)^{-1}K$.*

Proof It follows from (2.1) that

$$|\log\zeta_n(y)-\log\zeta_n(y')|\le(1-\theta)^{-1}Kd_\theta(\bar F^ny,\bar F^ny').$$

Hence $\sup_a\zeta_n\le e^{(1-\theta)^{-1}K}\inf_a\zeta_n$ and

$$\inf_a\zeta_n=\inf P^n1_a\le\textstyle\int_{\bar Y}P^n1_a\,d\bar\mu_Y=\int_{\bar Y}1_a\,d\bar\mu_Y=\bar\mu_Y(a).$$

Thus $\sup_a\zeta_n\le K_1\bar\mu_Y(a)$. Similarly, $\inf_a\zeta_n\ge K_1^{-1}\bar\mu_Y(a)$. Finally,

$$|\zeta_n(y)-\zeta_n(y')|\le\sup_a\zeta_n\,|\log\zeta_n(y)-\log\zeta_n(y')|\le K_1\bar\mu_Y(a)d_\theta(\bar F^ny,\bar F^ny').$$

∎

The transfer operator L corresponding to $\bar f:\bar\Delta\to\bar\Delta$ and $\bar\mu_\Delta$ can be written as

$$(L\phi)(p)=\sum_{\bar fq=p}g(q)\phi(q),\qquad\text{where}\qquad g(y,\ell)=\begin{cases}\zeta(y), & \ell=\tau(y)-1,\\ 1, & \ell<\tau(y)-1\end{cases}.$$

Then $(L^n\phi)(p)=\sum_{\bar f^nq=p}g_n(q)\phi(q)$ where $g_n=g\,g\circ\bar f\cdots g\circ\bar f^{n-1}$.

Define $\bar\Delta_0=\{(y,\ell)\in\bar\Delta\colon\ell=0\}$ and $\Delta_0=\{(y,\ell)\in\Delta\colon\ell=0\}$.

Proposition 5.2 *Let $p,p'\in\bar\Delta$ with $s(p,p')\ge n\ge1$. Then (a) $\sum_{\bar f^nq=p}g_n(q)=1$, (b) $|g_n(p)-g_n(p')|\le K_1^2g_n(p)d_\theta(\bar f^np,\bar f^np')$.*

Proof Part (a) is immediate since $L^n1=1$. Let $r(p)=\#\{j\in\{1,\dots,n\}:\bar f^jp\in\bar\Delta_0\}$. Note that $r(p)=r(p')$. If $r(p)=0$, then $g_n(p)=g_n(p')=1$ and (b) holds trivially. Otherwise, we can write $p=(y,\ell)$, $p'=(y',\ell)$ with $y,y'\in\bar Y$ and $\ell\ge0$. Then $g_n(p)=\zeta_{r(p)}(y)$ and $g_n(p')=\zeta_{r(p)}(y')$.

Let $a\in\alpha_{r(p)}$ be the cylinder containing y and y'. Then by Proposition 5.1,

$$\begin{aligned}|g_n(p)-g_n(p')|&=|\zeta_{r(p)}(y)-\zeta_{r(p)}(y')|\le K_1\bar\mu_Y(a)d_\theta(\bar F^{r(p)}y,\bar F^{r(p)}y')\\&\le K_1^2\zeta_{r(p)}(y)d_\theta(\bar f^np,\bar f^np')=K_1^2g_n(p)d_\theta(\bar f^np,\bar f^np'),\end{aligned}$$

proving (b). ∎

Nonuniform expansion/contraction Recall that $\pi:\Delta\to M$ denotes the projection $\pi(y,\ell)=T^\ell y$. For $p=(x,\ell),q=(y,\ell)\in\Delta$, we write $q\in W^s(p)$ if $y\in W^s(x)$ and $q\in W^u(p)$ if $y\in W^u(x)$. Conditions (P1) translate as follows.

(P2) There exist constants $K_0>0$, $\rho_0\in(0,1)$ such that for all $p,q\in\Delta$, $n\ge1$,

(i) If $q\in W^s(p)$, then $d(\pi f^np,\pi f^nq)\le K_0\rho_0^{\kappa_n(p)}$, and

(ii) If $q\in W^u(p)$, then $d(\pi f^np,\pi f^nq)\le K_0\rho_0^{s(p,q)-\kappa_n(p)}$,

where $\kappa_n(p)=\#\{j=1,\dots,n:f^jp\in\Delta_0\}$ is the number of returns of p to Δ_0 by time n. It is immediate from conditions (P2) and the product structure on Y that

$$d(\pi f^np,\pi f^nq)\le 2K_0\rho_0^{\min\{\kappa_n(p),s(p,q)-\kappa_n(p)\}}\quad\text{for all }p,q\in\Delta,\ n\ge1.\tag{5.1}$$

Approximation of observables Given C^η observables $v, w : M \to \mathbb{R}$, let $\phi = v \circ \pi$, $\psi = w \circ \pi : \Delta \to \mathbb{R}$ be the lifted observables. For each $n \geq 1$, define $\tilde{\phi}_n : \Delta \to \mathbb{R}$,

$$\tilde{\phi}_n(p) = \inf\{\phi(f^n q) : s(p, q) \geq 2\kappa_n(p)\}.$$

Proposition 5.3 *The function $\tilde{\phi}_n$ lies in $L^\infty(\Delta)$ and projects down to a Lipschitz observable $\bar{\phi}_n : \bar{\Delta} \to \mathbb{R}$. Moreover, setting $K_2 = 1 + K_1^2 + 2^\eta K_0^\eta$, $\rho = \rho_0^\eta$ and $\theta = \rho$,*

(a) $|\bar{\phi}_n|_\infty = |\tilde{\phi}_n|_\infty \leq |v|_\infty$, *(b)* $|\phi \circ f^n(p) - \tilde{\phi}_n(p)| \leq 2^\eta K_0^\eta \|v\|_{C^\eta} \rho^{\kappa_n(p)}$ *for* $p \in \Delta$,

(c) $\|L^n \bar{\phi}_n\|_\theta \leq K_2 \|v\|_{C^\eta}$, *for all* $n \geq 1$.

Proof This is standard, see for example [24, Proposition B.5]. We give the details for completeness. If $s(p, q) \geq 2\kappa_n(p)$, then $\tilde{\phi}_n(p) = \tilde{\phi}_n(q)$. It follows that $\tilde{\phi}_n$ is piecewise constant on a measurable partition of Δ, and hence is measurable, and that $\bar{\phi}_n$ is well-defined. Part (a) is immediate.

Recall that $\phi = v \circ \pi$ where $v : M \to \mathbb{R}$ is C^η. Let $p \in \Delta$. By (5.1) and the definition of $\tilde{\phi}_n$,

$$\begin{aligned} |\phi \circ f^n(p) - \tilde{\phi}_n(p)| &= |v(\pi f^n p) - v(\pi f^n q)| \leq \|v\|_{C^\eta} d(\pi f^n p, \pi f^n q)^\eta \\ &\leq 2^\eta K_0^\eta \rho^{\min\{\kappa_n(p), s(p,q) - \kappa_n(p)\}}, \end{aligned}$$

where q is such that $s(p, q) \geq 2\kappa_n(p)$. In particular, $s(p, q) - \kappa_n(p) \geq \kappa_n(p)$, so we obtain part (b).

For part (c), first note that $|L^n \bar{\phi}_n|_\infty \leq |\bar{\phi}_n|_\infty \leq |v|_\infty$. Let $\bar{p} = (y, \ell) \in \bar{\Delta}$ and $\bar{p}' = (y', \ell') \in \bar{\Delta}$. If $d_\theta(\bar{p}, \bar{p}') = 1$, then

$$\big|(L^n \bar{\phi})(\bar{p}) - (L^n \bar{\phi})(\bar{p}')\big| \leq 2|v|_\infty = 2|v|_\infty d_\theta(\bar{p}, \bar{p}').$$

Otherwise, we can write

$$(L^n \bar{\phi}_n)(\bar{p}) - (L^n \bar{\phi}_n)(\bar{p}') = I_1 + I_2,$$

where

$$I_1 = \sum_{\bar{f}^n \bar{q} = \bar{p}} g_n(\bar{q})\big(\bar{\phi}_n(\bar{q}) - \bar{\phi}_n(\bar{q}')\big), \qquad I_2 = \sum_{\bar{f}^n \bar{q} = \bar{p}} \big(g_n(\bar{q}) - g_n(\bar{q}')\big)\bar{\phi}_n(\bar{q}').$$

As usual, preimages $\bar{q}, \bar{q}'$ are matched up so that $s(\bar{q}, \bar{q}') = \kappa_n(\bar{q}) + s(\bar{p}, \bar{p}')$.

By Proposition 5.2,

$$|I_2| \leq K_1^2 |v|_\infty \textstyle\sum_{\bar{f}^n \bar{q} = \bar{p}} g_n(\bar{q}) d_\theta(\bar{f}^n \bar{q}, \bar{f}^n \bar{q}') = K_1^2 |v|_\infty d_\theta(\bar{p}, \bar{p}').$$

We claim that $|\bar{\phi}_n(\bar{q}) - \bar{\phi}_n(\bar{q}')| \leq 2^\eta K_0^\eta \|v\|_{C^\eta} \rho^{s(\bar{p}, \bar{p}')}$. Taking $\theta = \rho$, it then follows from Proposition 5.2(a) that $|I_1| \leq 2^\eta K_0^\eta \|v\|_{C^\eta} d_\theta(\bar{p}, \bar{p}')$.

It remains to verify the claim. Choose $q, q' \in \Delta$ that project onto $\bar{q}, \bar{q}' \in \bar{\Delta}$, so

$$s(q, q') = s(\bar{q}, \bar{q}') = \kappa_n(\bar{q}) + s(\bar{p}, \bar{p}').$$

Write $\bar{\phi}_n(\bar{q}) - \bar{\phi}_n(\bar{q}') = \phi \circ f^n(\hat{q}) - \phi \circ f^n(\hat{q}')$, where $\hat{q}, \hat{q}' \in \Delta$ satisfy

$$s(\hat{q}, q) \geq 2\kappa_n(\bar{q}) \qquad \text{and} \qquad s(\hat{q}', q') \geq 2\kappa_n(\bar{q}).$$

Since $\bar{\phi}_n(\bar{q}) = \bar{\phi}_n(\bar{q}')$ if $s(\bar{q}, \bar{q}') \geq 2\kappa_n(\bar{q})$, we may suppose without loss that

$$s(\hat{q}, \hat{q}') = s(\bar{q}, \bar{q}') \leq 2\kappa_n(\bar{q}) = 2\kappa_n(\hat{q}).$$

Then

$$s(\bar{p}, \bar{p}') = s(\hat{q}, \hat{q}') - \kappa_n(\hat{q}) \leq \kappa_n(\hat{q}).$$

As in part (b),

$$|\phi \circ f^n(\hat{q}) - \phi \circ f^n(\hat{q}')| \leq 2^\eta K_0^\eta \|v\|_{C^\eta} \rho^{\min\{\kappa_n(\hat{q}),\, s(\hat{q},\hat{q}') - \kappa_n(\hat{q})\}} = 2^\eta K_0^\eta \|v\|_{C^\eta} \rho^{s(\bar{p},\bar{p}')}.$$

This completes the proof of the claim. ∎

Corollary 5.4 *Suppose $\{b_n\}$, $n \geq 0$ is a nonnegative non-increasing sequence, and $|L^n\phi|_1 \leq b_n \|\phi\|_\theta$ for all all n and all mean zero d_θ-Lipschitz functions $\phi : \bar{\Delta} \to \mathbb{R}$. Then*

$$\Big| \int_M v\, w \circ T^n \, d\mu - \int_M v \, d\mu \int_M w \, d\mu \Big| \leq \big(2^\eta K_0^\eta |\rho^{\kappa_{[n/2]}}|_1 + 2K_2\, b_{[n/2]}\big) \|v\|_{C^\eta} \|w\|_{C^\eta}.$$

Proof Suppose without loss that v is mean zero. Since $\pi : \Delta \to M$ is a semiconjugacy and $\mu = \pi_*\mu_\Delta$, it is equivalent to estimate $\int_\Delta \phi\, \psi \circ f^n \, d\mu_\Delta$, where $\phi, \psi \colon \Delta \to \mathbb{R}$, $\phi = v \circ \pi$ and $\psi = w \circ \pi$. Assume for simplicity that n is even; the proof for n odd requires little modification. Let $\ell \geq 1$, and write

$$\int_\Delta \phi\, \psi \circ f^n \, d\mu_\Delta = \int_\Delta \phi \circ f^\ell \, \psi \circ f^{\ell+n} \, d\mu_\Delta = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_\Delta (\phi \circ f^\ell - \tilde{\phi}_\ell)\, \psi \circ f^{\ell+n} \, d\mu_\Delta, \quad I_2 = \int_\Delta \tilde{\phi}_\ell\, (\psi \circ f^{n/2} - \tilde{\psi}_{n/2}) \circ f^{\ell+n/2} \, d\mu_\Delta,$$

$$I_3 = \int_\Delta \Big(\tilde{\phi}_\ell - \int_\Delta \tilde{\phi}_\ell \, d\mu_\Delta\Big) \tilde{\psi}_{n/2} \circ f^{\ell+n/2} \, d\mu_\Delta, \quad I_4 = \int_\Delta \tilde{\phi}_\ell \, d\mu_\Delta \int_\Delta \tilde{\psi}_{n/2} \, d\mu_\Delta.$$

By Proposition 5.3(b), $|I_1| \le |\phi\circ f^\ell - \tilde\phi_\ell|_1|\psi|_\infty \le 2^\eta K_0^\eta|\rho^{\kappa_\ell}|_1\|v\|_{C^\eta}|w|_\infty$. By Proposition 5.3(a,b), $|I_2| \le |\tilde\phi_\ell|_\infty|\psi\circ f^{n/2} - \tilde\psi_{n/2}|_1 \le 2^\eta K_0^\eta|v|_\infty\|w\|_{C^\eta}|\rho^{\kappa_{n/2}}|_1$. By Proposition 5.3(c),

$$\begin{aligned}|I_3| &= \Big|\int_{\bar\Delta} L^{n/2}\Big(L^\ell\bar\phi_\ell - \int_{\bar\Delta}\bar\phi_\ell\,d\bar\mu_\Delta\Big)\,\bar\psi_{n/2}\,d\bar\mu_\Delta\Big| \\ &\le |L^{n/2}(L^\ell\bar\phi_\ell - \textstyle\int_{\bar\Delta}\bar\phi_\ell\,d\bar\mu_\Delta)|_1|\bar\psi_{n/2}|_\infty \le 2b_{n/2}\|L^\ell\bar\phi_\ell\|_\theta|w|_\infty \le 2K_2b_{n/2}\|v\|_{C^\eta}|w|_\infty.\end{aligned}$$

Finally, $|I_4| \le |\int_{\bar\Delta}\tilde\phi_\ell\,d\bar\mu_\Delta||w|_\infty = |\int_{\bar\Delta}(\tilde\phi_\ell - \phi\circ f^\ell)\,d\bar\mu_\Delta||w|_\infty \le 2^\eta K_0^\eta\|v\|_{C^\eta}|w|_\infty|\rho^{\kappa_\ell}|_1$ by another application of Proposition 5.3(b).

Altogether,

$$\Big|\int_\Delta \phi\,\psi\circ f^n\,d\mu_\Delta\Big| \le \big(2^\eta K_0^\eta|\rho^{\kappa_{n/2}}|_1 + 2K_2\,b_{n/2} + 2^{\eta+1}K_0^\eta|\rho^{\kappa_\ell}|_1\big)\|v\|_{C^\eta}\|w\|_{C^\eta}.$$

Letting $\ell\to\infty$ yields the result. ∎

By Corollary 5.4 and Theorem 2.7, it remains to estimate $|\rho^{\kappa_n}|_1$. A first step towards this is:

Lemma 5.5 *$\int_{\bar\Delta}\rho^{\kappa_n}\,d\bar\mu_\Delta \le 2\bar\tau^{-1}\sum_{j>n/3}\bar\mu_Y(\tau\ge j) + n\sum_{k=0}^\infty \rho^{k+1}\bar\mu_Y(\tau_k\ge n/3)$, where $\tau_k = \sum_{j=0}^{k-1}\tau\circ\bar F^k$.*

Proof First write $\int_{\bar\Delta}\rho^{\kappa_n}\,d\bar\mu_\Delta = \sum_{k=0}^\infty\rho^k\bar\mu_\Delta(\kappa_n = k)$. Note that $\kappa_n(p) = 0$ if and only if $f^j(p)\notin\bar\Delta_0$ for all $j = 1,\dots,n$, so $\bar\mu_\Delta(\kappa_n = 0) = \bar\tau^{-1}\sum_{j\ge n}\bar\mu_Y(\tau > j)$.

When $\kappa_n(p)\ge 1$, we can define $r(p) = \min\{j\in\{1,\dots,n\} : f^jp\in\bar\Delta_0\}$ and $s(p) = \max\{j\in\{1,\dots,n\} : f^jp\in\bar\Delta_0\}$. Hence for $k\ge 1$,

$$\{\kappa_n(p) = k\} = \bigcup_{1\le r\le s\le n}\{\kappa_n(p) = k,\, r(p) = r,\, s(p) = s\}.$$

It is easy to check that $\bar\mu_\Delta\{r(p) = j\} = \bar\tau^{-1}\bar\mu_Y(\tau\ge j)$, so

$$\bar\mu_\Delta(\kappa_n(p) = k) \le \bar\tau^{-1}\sum_{j>n/3}\bar\mu_Y(\tau\ge j) + b_{n,k},$$

where

$$
\begin{aligned}
b_{n,k} &= \sum_{0\le r\le n/3}\ \sum_{2n/3\le s\le n} \bar\mu_\Delta(\kappa_n(p)=k,\, r(p)=r,\, s(p)=s) \\
&= \sum_{0\le r\le n/3}\ \sum_{2n/3\le s\le n} \bar\mu_\Delta(\kappa_{s-r}(f^rp)=k-1,\, r(p)=r,\, s(p)=s) \\
&\le \sum_{0\le r\le n/3}\ \sum_{2n/3\le s\le n} \bar\mu_\Delta(\kappa_{s-r}(f^rp)=k-1,\, f^rp\in\bar\Delta_0,\, f^sp\in\bar\Delta_0) \\
&= \sum_{0\le r\le n/3}\ \sum_{2n/3\le s\le n} \bar\mu_\Delta(\kappa_{s-r}(p)=k-1,\, p\in\bar\Delta_0,\, f^{s-r}p\in\bar\Delta_0) \\
&\le n\sum_{j\ge n/3} \bar\mu_\Delta(\kappa_j(p)=k-1,\, p\in\bar\Delta_0,\, f^jp\in\bar\Delta_0) \\
&= n\bar\tau^{-1}\sum_{j\ge n/3} \bar\mu_Y(y\in\bar Y,\, f^jy\in\bar Y,\, \tau_{k-1}(y)=j) \le n\bar\tau^{-1}\bar\mu_Y(y\in\bar Y:\tau_{k-1}(y)\ge n/3).
\end{aligned}
$$

This completes the proof. ∎

Proof of Theorem 2.10 We restrict from now on to the cases of polynomial tails and stretched exponential tails. The sum $\sum_{j\ge n}\bar\mu_Y(\tau>j)$ is estimated in the same way as $\mathbb{P}(w_1\ge n)$ in the proofs of Propositions 4.10 and 4.11, so it remains to show that $n\sum_{k=0}^\infty \rho^k\bar\mu_Y(\tau_k\ge n)$ satisfies the required estimate.

In the case of polynomial tails, $\bar\mu_Y(\tau_k\ge n)\le k\bar\mu_Y(\tau\ge n/k)\le C_\tau k^{\beta+1}n^{-\beta}$, and so $n\sum_{k=1}^\infty \rho^k\bar\mu_Y(\tau_k\ge n)\le C_2 n^{-(\beta-1)}$ where $C_2=C_\tau\sum_{k=1}^\infty\rho^k k^{\beta+1}$.

It remains to treat the stretched exponential case. Writing $X_j=\tau\circ F_j$,

$$
\begin{aligned}
\bar\mu_Y(X_0=j_0,\dots,X_k=j_k) &= \int_{\bar Y} 1_{\{\tau\circ F^k=j_k\}}1_{\{X_0=j_0,\dots,X_{k-1}=j_{k-1}\}}\,d\bar\mu_Y \\
&= \int_{\bar Y} 1_{\{\tau=j_k\}}P^k 1_{\{X_0=j_0,\dots,X_{k-1}=j_{k-1}\}}\,d\bar\mu_Y \\
&\le \bar\mu_Y(\tau=j_k)|P^k1_{\{X_0=j_0,\dots,X_{k-1}=j_{k-1}\}}|_\infty
\end{aligned}
$$

By Proposition 5.1(a),

$$
\begin{aligned}
(P^k1_{\{X_0=j_0,\dots,X_{k-1}=j_{k-1}\}})(y) &= \sum_{a\in\alpha_k}\zeta_k(y_a)1_{\{X_0=j_0,\dots,X_{k-1}=j_{k-1}\}} \\
&\le K_1\sum_{a\in\alpha_k}\bar\mu_Y(a)1_{\{\tau(a)=j_0,\dots,\tau(F^{k-1}a)=j_{k-1}\}} \\
&= K_1\bar\mu_Y(\tau=j_0,\dots,\tau\circ F^{k-1}=j_{k-1}).
\end{aligned}
$$

Hence

$$
\bar\mu_Y(X_0=j_0,\dots,X_k=j_k)\le K_1\bar\mu_Y(\tau=j_k)\bar\mu_Y(X_0=j_0,\dots,X_{k-1}=j_{k-1}),
$$

and so

$$\bar{\mu}_Y(X_k \geq n \,|\, X_0 = j_0, \ldots, X_{k-1} = j_{k-1}) \leq K_1 \bar{\mu}_Y(\tau \geq n) \leq K_1 C_\tau e^{-An^\gamma}.$$

By Proposition 4.11, there exists $B \in (0, A)$ and $C_B \in (0, \rho)$ depending continuously on C_τ, γ and A such that

$$\mu_Y(\tau_k \geq n) = \mu_Y(X_0 + \cdots + X_{k-1} \geq n) \leq C_B^k e^{-Bn^\gamma},$$

Hence $\sum_{k=1}^\infty \rho^k \mu_Y(\tau_k \geq n) \leq \{\sum_{k=1}^\infty (\rho C_B)^k\} e^{-Bn^\gamma}$ as required. ∎

Acknowledgements This research was supported in part by a European Advanced Grant *StochExtHomog* (ERC AdG 320977).

References

[1] J. Aaronson. *An Introduction to Infinite Ergodic Theory.* Math. Surveys and Monographs **50**, Amer. Math. Soc., 1997.

[2] J. Aaronson, M. Denker and M. Urbański. Ergodic theory for Markov fibred systems and parabolic rational maps. *Trans. Amer. Math. Soc.* **337** (1993) 495–548.

[3] J. F. Alves. Strong statistical stability of non-uniformly expanding maps. *Nonlinearity* **17** (2004) 1193–1215.

[4] J. F. Alves and D. Azevedo. Statistical properties of diffeomorphisms with weak invariant manifolds. *Discrete Contin. Dyn. Syst.* **36** (2016) 1–41.

[5] J. F. Alves, S. Luzzatto and V. Pinheiro. Markov structures and decay of correlations for non-uniformly expanding dynamical systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005) 817–839.

[6] J. F Alves and V. Pinheiro Slow rates of mixing for dynamical systems with hyperbolic structures *J. Stat. Phys.* **131** (2008) 505–534.

[7] J. F. Alves and M. Viana. Statistical stability for robust classes of maps with non-uniform expansion. *Ergodic Theory Dynam. Systems* **22** (2002) 1–32.

[8] V. Baladi, M. Benedicks and D. Schnellmann. Whitney-Hölder continuity of the SRB measure for transversal families of smooth unimodal maps. *Invent. Math.* **201** (2015) 773–844.

[9] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lecture Notes in Math. **470**, Springer, Berlin, 1975.

[10] X. Bressaud and C. Liverani. Anosov diffeomorphisms and coupling. *Ergod. Th. Dynam. Syst.* **22** (2002) 129–152.

[11] J.-R. Chazottes and S. Gouëzel. Optimal concentration inequalities for dynamical systems. *Commun. Math. Phys.* **316** (2012) 843–889.

[12] N. Chernov and D. Dolgopyat. Brownian Brownian Motion – I. *Memoirs Amer. Math. Soc.* **198**, No. 927, 2009.

[13] M. F. Demers and H.-K. Zhang. A functional analytic approach to perturbations of the Lorentz gas. *Commun. Math. Phys.* **324** (2013) 767–830.

[14] P. Eslami. Stretched exponential mixing for $C^{1+\alpha}$ skew products with discontinuities. *Ergodic Theory Dynam. Systems* **37** (2017) 146–175.

[15] J. M. Freitas and M. Todd. The statistical stability of equilibrium states for interval maps. *Nonlinearity* **22** (2009) 259–281.

[16] S. Gouëzel. Decay of correlations for nonuniformly expanding systems. *Bull. Soc. Math. France* **134** (2006) 1–31.

[17] S. Gouëzel. Vitesse de décorrélation et théorèmes limites pour les applications non uniformément dilatantes. Ph. D. Thesis, Ecole Normale Supérieure, 2004.

[18] G. Keller and C. Liverani. Stability of the spectrum for transfer operators. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* **XXVIII** (1999) 141–152.

[19] A. Korepanov, Z. Kosloff and I. Melbourne. Averaging and rates of averaging for uniform families of deterministic fast-slow skew product systems. *Studia Math.* **238** (2017) 59–89.

[20] A. Korepanov, Z. Kosloff and I. Melbourne. Martingale-coboundary decomposition for families of dynamical systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **35** (2018) 859–885.

[21] C. Liverani. Rigorous numerical investigation of the statistical properties of piecewise expanding maps. A feasibility study. *Nonlinearity* **14** (2001) 463–490.

[22] V. Maume-Deschamps. Projective metrics and mixing properties on towers. *Trans. Amer. Math. Soc.* **353** (2001) 3371–3389.

[23] I. Melbourne and M. Nicol. Almost sure invariance principle for nonuniformly hyperbolic systems. *Commun. Math. Phys.* **260** (2005) 131–146.

[24] I. Melbourne and D. Terhesiu. Decay of correlations for nonuniformly expanding systems with general return times. *Ergodic Theory Dynam. Systems* **34** (2014) 893–918.

[25] W. Parry and M. Pollicott. *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics.* Astérique **187-188**, Société Mathématique de France, Montrouge, 1990.

[26] D. Ruelle. *Thermodynamic Formalism.* Encyclopedia of Math. and its Applications **5**, Addison Wesley, Massachusetts, 1978.

[27] E. Selmer. On the linear diophantine problem of Frobenius. *J. Reine Angew. Math.* **293/294** (1977) 1–17.

[28] Y. G. Sinaĭ. Gibbs measures in ergodic theory. *Russ. Math. Surv.* **27** (1972) 21–70.

[29] H. Sulku. Explicit correlation bounds for expanding circle maps using the coupling method. Preprint, 2013.

[30] M. Viana. Multidimensional nonhyperbolic attractors. *Inst. Hautes Études Sci. Publ. Math.* (1997) 63–96.

[31] L.-S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.* **147** (1998) 585–650.

[32] L.-S. Young. Recurrence times and rates of mixing. *Israel J. Math.* **110** (1999) 153–188.

[33] R. Zweimüller. Kuzmin, coupling, cones, and exponential mixing. *Forum Math.* **16** (2004) 447–457.