DISINTEGRATION OF INVARIANT MEASURES FOR HYPERBOLIC SKEW PRODUCTS

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Abstract. We study hyperbolic skew products and the disintegration of the SRB measure into measures supported on local stable manifolds. Such a disintegration gives a method for passing from an observable v on the skew product to an observable $\bar{v}$ on the system quotiented along stable manifolds. Under mild assumptions on the system we prove that the disintegration preserves the smoothness of v, firstly in the case where v is Hölder and secondly in the case where v is $\mathcal{C}^1$.

1. Introduction

We suppose throughout that $\widehat{F} : \widehat{\Delta} \to \widehat{\Delta}$ has the form of a skew product map so $\widehat{\Delta} = \Delta \times N$ are compact metric spaces and

$$\widehat{F}(x, z) = (Fx, G(x, z))$$

where $F : \Delta \to \Delta$ and $G : \Delta \times N \to N$ are continuous. Moreover, we suppose that ν is an F-invariant Borel probability measure on Δ. Let $\pi : \widehat{\Delta} \to \Delta$ be the projection $\pi(x, z) = x$ and note that π defines a semiconjugacy between $\widehat{F}$ and F, i.e., $F \circ \pi = \pi \circ \widehat{F}$. In different sections of this note we will require the system to satisfy different degrees of regularity but the basic setting is for F to be a uniformly expanding map and for $\widehat{F}$ to be uniformly contracting in the fibre direction in the sense that

$$\operatorname{diam} \widehat{F}^n \pi^{-1}(x) \to 0 \quad \text{as } n \to \infty, \text{ uniformly in } x. \tag{1}$$

In order to study statistical properties of $\widehat{F}$ it is often convenient to study the statistical properties of the expanding map F and then use this to deduce the behaviour for the hyperbolic map $\widehat{F}$. This involves associating observables on $\widehat{\Delta}$ to observables on Δ and the consideration of the possible loss of regularity involved in this process. In the symbolic setting, this corresponds to the argument where one-sided observables can be used to approximate two-sided observables (see, for example [6, §1.A]).

Here we pursue a different approach, inspired by [5]. Suppose that ν is an F-invariant probability measure on Δ. A standard construction (see Section 2) yields an $\widehat{F}$-invariant probability measure η on $\widehat{\Delta}$ such that $\nu = \pi_* \eta$. We are interested in a disintegration $\{\eta_x\}_{x \in \Delta}$ of η in the sense that each η_x is a Borel probability measure on $\widehat{\Delta}$ supported on $\pi^{-1}(x)$ and

$$\eta(v) = \int_{\Delta} \eta_x(v) \; d\nu(x)$$

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for all continuous $v : \widehat{\Delta} \to \mathbb{R}$. Then, in a natural way, $x \mapsto \bar{v} = \eta_x(v)$ is the observable on Δ associated to the observable $v : \widehat{\Delta} \to \mathbb{R}$.

The existence (and uniqueness) of such a disintegration goes back to Rohlin [8] in a rather more general context. The purpose of this note is to study the regularity of the disintegration $\{\eta_x\}_{x\in\Delta}$ in the sense that the regularity of v is inherited by $\bar{v}$. Such questions are important when studying rates of mixing for hyperbolic systems. For example, in [5] exponential mixing is first proved for a hyperbolic semiflow and then lifted to the hyperbolic flow using a regularity result for the disintegration as described above. In their setting the measure η has a smooth density and so the regularity of the disintegration is immediate [5, Lemma 4.3]. We consider the case where ν is absolutely continuous, but make no such assumption on η. At first glance the situation appears rather bad since in general the invariant measure η could be singular along the stable manifolds. This turns out not to be a problem and good regularity of the disintegration is still possible in these situations. Such a result, in the case when the invariant density is singular along stable manifolds, is required in [1, 2, 4]. In the situations studied in those references, there is a $\mathcal{C}^k$ global stable foliation where $k = 1 + \alpha$ or $k = 2$. After a $\mathcal{C}^k$ change of coordinates, we obtain a skew product map $\hat{F}$ such that F and G are $\mathcal{C}^k$, and our main results exploit this information. In general such good regularity of the stable foliation cannot be expected but in many cases, for instance under domination conditions or in low dimensions, the regularity is good (see, for example [1, 4]).

In Section 2, we recall the argument for the existence of the invariant measure for the hyperbolic system. Then, in Section 3, we present the construction of the disintegration along stable manifolds. These sections do not require specific assumptions on F, ν, or the rate of contraction in (1).

Sections 4 and 5 contain our main results on the regularity of the disintegration, firstly for the Hölder case and secondly for the $\mathcal{C}^1$ case. To prove these results, we require additional regularity assumptions on F and G, absolute continuity of ν and exponential contraction in (1).

2. Invariant Measure on $\widehat{\Delta}$

In this section, we recall the standard argument for constructing an invariant measure η for $\widehat{F} : \widehat{\Delta} \to \widehat{\Delta}$ (see for example [3, Section 6]). This construction makes use of the invariant measure ν for F together with the contracting stable foliation, but the details of the map $F : \Delta \to \Delta$ and the rate of contraction are not required.

Proposition 1. *Given $v : \widehat{\Delta} \to \mathbb{R}$, define $v_+, v_- : \Delta \to \mathbb{R}$ by setting $v_+(x) = \sup_z v(x, z)$, $v_-(x) = \inf_z v(x, z)$. Then the limits*

$$\lim_{n\to\infty} \int_\Delta (v \circ \widehat{F}^n)_+ \, d\nu \quad \text{and} \quad \lim_{n\to\infty} \int_\Delta (v \circ \widehat{F}^n)_- \, d\nu$$

exist and coincide for all v continuous. Denote the common limit by $\eta(v)$. This defines an $\widehat{F}$-invariant probability measure η on $\widehat{\Delta}$ and $\pi_*\eta = \nu$.

Proof. Let $v_n^\pm = \int_\Delta (v \circ \widehat{F}^n)_\pm \, d\nu$. We have

$$\begin{aligned}(v \circ \widehat{F}^{n+1})_+(x) &= \sup_z v \circ \widehat{F}^{n+1}(x, z) = \sup_z v \circ \widehat{F}^n(Fx, G(x, z)) \\ &\le \sup_z v \circ \widehat{F}^n(Fx, z) = (v \circ \widehat{F}^n)_+(Fx).\end{aligned}$$

By F-invariance of ν,

$$v_{n+1}^+ = \int_\Delta (v \circ \widehat{F}^{n+1})_+ \, d\nu \le \int_\Delta (v \circ \widehat{F}^n)_+ \circ F \, d\nu = \int_\Delta (v \circ \widehat{F}^n)_+ \, d\nu = v_n^+.$$

Hence v_n^+ is a monotone decreasing sequence bounded below by $-|v|_\infty$, and consequently $\lim_{n\to\infty} v_n^+$ exists. Similarly $\lim_{n\to\infty} v_n^-$ exists.

Next, using uniform continuity of v and the fact that $\operatorname{diam} \widehat{F}^n\pi^{-1}(x) \to 0$ as $n\to\infty$ for each $x\in\Delta$,

$$\begin{aligned}(v\circ\widehat{F}^n)_+(x) - (v\circ\widehat{F}^n)_-(x) &= \sup_z v\circ\widehat{F}^n(x,z) - \inf_z v\circ\widehat{F}^n(x,z) \\ &\le \sup_{z\in\widehat{F}^n\pi^{-1}(x)} v - \inf_{z\in\widehat{F}^n\pi^{-1}(x)} v \to 0.\end{aligned}$$

Hence $\lim_{n\to\infty} v_n^+ = \lim_{n\to\infty} v_n^-$.

Since $\int_{\widehat{\Delta}} v\,d\eta = \lim_{n\to\infty}\int_\Delta (v\circ\widehat{F}^n)_+\,d\nu$ we know $\int_{\widehat{\Delta}}(v_1+v_2)\,d\eta \le \int_{\widehat{\Delta}} v_1\,d\eta + \int_{\widehat{\Delta}} v_2\,d\eta$. Similarly, using $\int_{\widehat{\Delta}} v\,d\eta = \lim_{n\to\infty}\int_\Delta (v\circ\widehat{F}^n)_-\,d\nu$ it follows that $\int_{\widehat{\Delta}}(v_1+v_2)\,d\eta \ge \int_{\widehat{\Delta}} v_1\,d\eta + \int_{\widehat{\Delta}} v_2\,d\eta$. Hence $v\mapsto \int_{\widehat{\Delta}} v\,d\eta$ defines a linear functional on the space of continuous functions. Clearly $\int_{\widehat{\Delta}} v\,d\eta\ge 0$ whenever $v\ge 0$, and $\int_{\widehat{\Delta}} 1\,d\eta = 1$, so η is a probability measure. Moreover, $\widehat{F}$-invariance of η is immediate from the definition $\int_{\widehat{\Delta}} v\,d\eta = \lim_{n\to\infty}\int_\Delta (v\circ\widehat{F}^n)_+\,d\nu$. Finally, the fact that $\pi_*\eta = \nu$ is immediate from the definitions and the invariance of ν. □

Remark 2. (a) In [3, Corollary 6.4], it is shown that ergodicity of ν implies ergodicity of η.
(b) Given an ergodic F-invariant probability measure ν, Proposition 1 shows how to construct an ergodic $\widehat{F}$-invariant measure η with $\pi_*\eta=\nu$.

Conversely, suppose that we are given an ergodic $\widehat{F}$-invariant probability measure η_0. Then $\nu=\pi_*\eta_0$ is an an ergodic F-invariant probability measure and gives rise via Proposition 1 to an ergodic $\widehat{F}$-invariant probability measure η. We claim that $\eta=\eta_0$.

Indeed, suppose that η_1, η_2 are two ergodic $\widehat{F}$-invariant probability measures such that $\pi_*\eta_1=\pi_*\eta_2=\nu$. We show that $\eta_1=\eta_2$.[1] Let $v:\widehat{\Delta}\to\mathbb{R}$ be continuous and define $S_n = n^{-1}\sum_{j=0}^{n-1} v\circ\widehat{F}^j$. By the ergodic theorem, $\lim_{n\to\infty} S_n = \int v\,d\eta_i$ on a set $\widehat{E}_i\subset\widehat{\Delta}$ with $\eta_i(\widehat{E}_i)=1$ for $i=1,2$. The proof of Proposition 1 shows that $\lim_{n\to\infty} n^{-1}\sum_{j=0}^{n-1} v\circ\widehat{F}^j(x,z)$ is independent of z so for $i=1,2$, there exist sets $E_i\subset\Delta$ with $\nu(E_i)=1$ such that $\pi^{-1}E_i=\widehat{E}_i$. In particular, $\widehat{E}_1\cap\widehat{E}_2\neq\emptyset$, and hence $\int v\,d\eta_1=\int v\,d\eta_2$. Since v is an arbitrary continuous function, $\eta_1=\eta_2$ as required.

3. Disintegration

In this section, we assume the same set up as in Section 2. Let $\mathcal{U}:L^1(\Delta)\to L^1(\Delta)$ denote the Koopman operator $\mathcal{U}w = w\circ F$ corresponding to $F:\Delta\to\Delta$. Define the transfer operator $\mathcal{L}:L^1(\Delta)\to L^1(\Delta)$ given by $\int_\Delta \mathcal{U}w\,v\,d\nu = \int_\Delta w\,\mathcal{L}v\,d\nu$ where $v\in L^1(\Delta)$, $w\in L^\infty(\Delta)$.

Let 0 denote a distinguished point in N. Following [4, Proposition 4.10 and Remark 4.11], we define η_x almost everywhere as the limit as $n\to\infty$ of $(\mathcal{L}^n v_n)(x)$ where $v_n(x) = v\circ\widehat{F}^n(x,0)$. We note that the argument below is considerably more direct and general than the one in [4, Section 4.4].

Proposition 3. *For almost every $x\in\Delta$, the limit*

$$\eta_x(v) = \lim_{n\to\infty}(\mathcal{L}^n v_n)(x), \quad v_n(x) = v\circ\widehat{F}^n(x,0),$$

exists for every $v\in C^0(\widehat{\Delta})$ and defines a probability measure supported on $\pi^{-1}(x)$.

[1] We are grateful to Vitor Araújo for pointing out this argument.

Moreover, for each $v \in \mathcal{C}^0(\widehat{\Delta})$, the map $x \mapsto \eta_x(v)$ lies in $L^\infty(\Delta)$ and

$$\eta(v) = \int_\Delta \eta_x(v)\ d\nu(x). \tag{2}$$

Proof. First we consider a fixed $v \in \mathcal{C}^0(\widehat{\Delta})$. Since $\mathcal{L}\mathcal{U} = I$, we have

$$\mathcal{L}^n v_n - \mathcal{L}^{n+m} v_{n+m} = \mathcal{L}^{n+m}(\mathcal{U}^m v_n - v_{n+m}).$$

Now

$$(\mathcal{U}^m v_n)(x) - v_{n+m}(x) = v \circ \widehat{F}^n(F^m x, 0) - v \circ \widehat{F}^{n+m}(x,0).$$

By contractivity of the stable foliation, diam $\widehat{F}^n \pi^{-1}(F^m x) \to 0$ as $n \to \infty$ uniformly in m and x. Hence by uniform continuity of v,

$$|\mathcal{U}^m v_n - v_{n+m}|_\infty \to 0 \tag{3}$$

as $n \to \infty$ uniformly in m.

Since ν is F-invariant, it follows from the duality definition of $\mathcal{L}$ that $|\mathcal{L}v|_\infty \le |v|_\infty$ for all $v \in L^\infty(\Delta)$. Hence

$$\left|\mathcal{L}^n v_n - \mathcal{L}^{n+m} v_{n+m}\right|_\infty \le |\mathcal{U}^m v_n - v_{n+m}|_\infty \to 0,$$

as $n, m \to \infty$. That is, $\mathcal{L}^n v_n$ defines a Cauchy sequence in $L^\infty(\Delta)$. In particular, the limit $\eta_x(v)$ exists for almost every x. Note also that $|\eta_x(v)| \le |v|_\infty$.

It follows from separability of $\mathcal{C}^0(\widehat{\Delta})$ that the functional $v \mapsto \eta_x(v)$ defines a bounded linear functional on $\mathcal{C}^0(\widehat{\Delta})$ for almost every $x \in \Delta$. Moreover η_x is positive and normalised and hence is identified with a probability measure on $\widehat{\Delta}$. If $v|_{\pi^{-1}(x)} \equiv 0$, then $(\mathcal{L}^n v_n)(x) = 0$ for all n and so $\eta_x(v) = 0$. Hence η_x is supported on $\pi^{-1}(x)$.

Finally,

$$\int_\Delta \eta_x(v)\, d\nu(x) = \lim_{n\to\infty} \int_\Delta \mathcal{L}^n v_n\, d\nu = \lim_{n\to\infty} \int_\Delta v \circ \widehat{F}^n(x,0)\, d\nu(x).$$

Hence

$$\eta(v) - \int_\Delta \eta_x(v)\, d\nu(x) = \lim_{n\to\infty} \int_\Delta \Big((v \circ \widehat{F}^n)_+(x) - v \circ \widehat{F}^n(x,0)\Big)\, d\nu(x),$$

which again converges to zero, so $\eta(v) = \int_\Delta \eta_x(v)\, d\nu(x)$. □

Remark 4. It follows that property 2 and the first part of property 3 in [5, Definition 2.5] are automatically satisfied.

4. Hölder regularity

In this section, we continue to assume the set up in Sections 2 and 3. In addition we suppose that Δ is a Riemannian manifold (with boundary), that $F : \Delta \to \Delta$ is a $\mathcal{C}^{1+\alpha}$ uniformly expanding map, as defined below, for some $\alpha \in (0,1]$ with absolutely continuous invariant probability measure ν, and that G is Lipschitz. (Normally this situation would arise when there is a $\mathcal{C}^{1+\alpha}$ stable foliation, in which case we would have also that G is $\mathcal{C}^{1+\alpha}$, but we do not make explicit use of this extra structure.) In the case $\alpha = 1$, $\mathcal{C}^{1+\alpha}$ means $\mathcal{C}^{1+\mathrm{Lip}}$.

We write $\|x - x'\|$ and $\|z - z'\|$ for distance on Δ and N.

Definition 5. Let $\alpha \in (0,1]$. The map $F : \Delta \to \Delta$ is *uniformly expanding* if there is an open and dense subset $\Delta_0 \subset \Delta$ with an at most countable partition into open sets U_i such that $F|_{U_i} : U_i \to \Delta_0$ is a $C^{1+\alpha}$ diffeomorphism onto Δ_0 and extends to a homeomorphism from $\bar{U}_i$ onto Δ for each i. Moreover, let $\mathcal{H}_n$ denote the set of inverse branches of F^n and write $J_h := |\det(Dh)|$. We require that there exist constants C_J, C_λ, $\lambda > 0$ such that

$$|Dh(x)| \le C_\lambda e^{-\lambda n}, \quad |\log J_h(x) - \log J_h(x')|/\|x - x'\|^\alpha \le C_J, \tag{4}$$

for all $h \in \mathcal{H}_n, n \in \mathbb{N}$, $x, x' \in \Delta_0$, $x \ne x'$.

Let d be a further metric on Δ with the property that there is a constant $C_1 > 0$ such that $\|x - x'\| \le C_1 d(x, x')$ for all $x, x' \in \Delta$. Write $\widehat{F}^n(x, z) = (F^n x, G_n(x, z))$. Set $\|v\|_{\mathcal{B}_\alpha(\Delta)} = |v|_\infty + |v|_{\mathcal{B}_\alpha(\Delta)}$ where $|v|_\infty = \sup_{x\in\Delta} |v(x)|$ and $|v|_{\mathcal{B}_\alpha(\Delta)} = \sup_{x,x'\in\Delta_0:x\ne x'} |v(x) - v(x')|/d(x, x')^\alpha$. Define $\mathcal{B}_\alpha(\Delta)$ to be the Banach space of functions $v : \Delta \to \mathbb{R}$ with $\|v\|_{\mathcal{B}_\alpha(\Delta)} < \infty$. Similarly, define $\mathcal{B}_\alpha(\widehat{\Delta})$ to be the space of functions $v : \widehat{\Delta} \to \mathbb{R}$ with $\|v\|_{\mathcal{B}_\alpha(\widehat{\Delta})} = |v|_\infty + |v|_{\mathcal{B}_\alpha(\widehat{\Delta})} < \infty$, where $|v|_\infty = \sup_{x\in\widehat{\Delta}} |v(x)|$ and

$$|v|_{\mathcal{B}_\alpha(\widehat{\Delta})} = \sup_{\substack{(x,z),(x',z')\in\widehat{\Delta}_0 \\ (x,z)\ne(x',z')}} \frac{|v(x, z) - v(x', z')|}{(d(x, x') + \|z - z'\|)^\alpha}.$$

Note that $\|vw\|_{\mathcal{B}_\alpha(\Delta)} \le \|v\|_{\mathcal{B}_\alpha(\Delta)} \|w\|_{\mathcal{B}_\alpha(\Delta)}$ for all $v, w \in \mathcal{B}_\alpha(\Delta)$ and similarly on $\widehat{\Delta}$.

If $d(x, x') = \|x-x'\|$, then $\mathcal{B}_\alpha(\Delta) = \mathcal{C}^\alpha(\Delta)$. In this case we write $|v|_\alpha = |v|_{\mathcal{B}_\alpha(\Delta)}$ and $\|v\|_\alpha = \|v\|_{\mathcal{B}_\alpha(\Delta)}$. In general $\mathcal{B}_\alpha(\Delta) \supset \mathcal{C}^\alpha(\Delta)$ and similarly $\mathcal{B}_\alpha(\widehat{\Delta}) \supset \mathcal{C}^\alpha(\widehat{\Delta})$. Hence, the formulation allows for larger spaces of functions, including those which are Lipschitz with respect to a symbolic metric.

A standard consequence of Definition 5 is the existence of a constant C'_J such that

$$\sum_{h\in\mathcal{H}_n} \|J_h\|_\alpha \le C'_J, \tag{5}$$

for all $n \in \mathbb{N}$. We require in addition that there exists $n_0 \ge 1$ such that

$$\|G_{n_0}(x, z) - G_{n_0}(x, z')\| \le \|z - z'\|, \tag{6}$$

for all $(x, z), (x, z') \in \widehat{\Delta}$. Under the above assumptions we prove:

Proposition 6. *The disintegration $\{\eta_x\}_{x\in\Delta}$ is Hölder in the following sense: there exists $C > 0$ such that for any $v \in \mathcal{B}_\alpha(\widehat{\Delta})$, the function $x \mapsto \bar{v}(x) := \eta_x(v)$ lies in $\mathcal{B}_\alpha(\Delta)$ and $\|\bar{v}\|_{\mathcal{B}_\alpha(\Delta)} \le C \|v\|_{\mathcal{B}_\alpha(\widehat{\Delta})}$.*

For a bounded variation version of this result, see [7, Lemma A.7].

To prove Proposition 6, we require the following lemma.

Lemma 7. *There exists $C > 0$ such that, for all $h \in \mathcal{H}_n$, $n \in \mathbb{N}$, (x, z), $(x', z) \in \widehat{\Delta}_0$,*

$$\|G_n(hx, z) - G_n(hx', z)\| \le Cd(x, x').$$

Proof. Fix $n \in \mathbb{N}$, $h \in \mathcal{H}_n$. Let n_0 be as in (6). Since $G_m(x, z) = G_{n_0}(F^{m-n_0}x, G_{m-n_0}(x, z))$ for any $n_0 \le m \le n$, we have $G_m(hx, z) = G_{n_0}(\ell x, G_{m-n_0}(hx, z))$ where $\ell := F^{m-n_0} \circ h \in \mathcal{H}_{n-m+n_0}$. Hence

$$\begin{aligned}
&\|G_m(hx, z) - G_m(hx', z)\| \\
&\qquad \le \|G_{n_0}(\ell x, G_{m-n_0}(hx, z)) - G_{n_0}(\ell x', G_{m-n_0}(hx, z))\| \\
&\qquad\qquad + \|G_{n_0}(\ell x', G_{m-n_0}(hx, z)) - G_{n_0}(\ell x', G_{m-n_0}(hx', z))\|.
\end{aligned}$$

Using the estimates (4), (6) and the assumption on d,

$$\begin{aligned}
A_m &\le \operatorname{Lip} G_{n_0} \|\ell x - \ell x'\| + A_{m-n_0} \\
&\le \operatorname{Lip} G_{n_0} C_1 C_\lambda e^{-\lambda(n-m+n_0)} d(x, x') + A_{m-n_0},
\end{aligned}$$

where $A_m = \|G_m(hx,z) - G_m(hx',z)\|$. Write $n = kn_0 + r$ where $k \in \mathbb{N}$ and $0 \le r \le n_0 - 1$ and set $m = jn_0 + r$ where $j \le k$. Then

$$A_{jn_0+r} \le \operatorname{Lip} G_{n_0} C_1 C_\lambda e^{-\lambda(k-j+1)n_0} d(x,x') + A_{(j-1)n_0+r}.$$

Consequently, iterating the above estimate, we obtain

$$\begin{aligned} A_{kn_0+r} &\le \operatorname{Lip} G_{n_0} C_1 C_\lambda \sum_{j=1}^{n} e^{-\lambda j n_0} d(x,x') + A_r \\ &\le \operatorname{Lip} G_{n_0} C_1 C_\lambda (e^{\lambda n_0} - 1)^{-1} d(x,x') + A_r. \end{aligned}$$

The result follows since $\max_{r<n_0} A_r \le C_1 C_\lambda \max_{r<n_0} \operatorname{Lip} G_r \, d(x,x') \le C d(x,x')$. □

Recall that $\bar{v} = \lim_{n\to\infty} \mathcal{L}^n v_n$ where $v_n(x) = v \circ \widehat{F}^n(x,0)$. Since F is uniformly expanding, $\mathcal{L}^n v_n = \varphi^{-1} \sum_{h\in\mathcal{H}_n} J_h \, (\varphi v_n) \circ h$ where the density φ corresponding to ν is $\mathcal{C}^\alpha$ and bounded below.

Corollary 8. *There exists $C > 0$ such that $|(\varphi v_n) \circ h|_{\mathcal{B}_\alpha(\Delta)} \le C \|v\|_{\mathcal{B}_\alpha(\widehat{\Delta})}$, for all $v \in \mathcal{B}_\alpha(\widehat{\Delta})$, $h \in \mathcal{H}_n$, $n \in \mathbb{N}$.*

Proof. Let $x, x' \in \Delta_0$. Since $\widehat{F}^n(hx,0) = (x, G_n(hx,0))$, we have that $v_n \circ h(x) = v(x, G_n(hx,0))$. Hence

$$|v_n \circ h(x) - v_n \circ h(x')| \le |v|_{\mathcal{B}_\alpha(\widehat{\Delta})} (d(x,x') + \|G_n(hx,0) - G_n(hx',0)\|)^\alpha.$$

Hence by Lemma 7, there is a constant $C \ge 1$ such that

$$|v_n \circ h(x) - v_n \circ h(x')| \le C \, |v|_{\mathcal{B}_\alpha(\widehat{\Delta})} \, d(x,x')^\alpha.$$

Clearly $|v_n \circ h|_\infty \le |v|_\infty$, so $\|v_n \circ h\|_{\mathcal{B}_\alpha(\Delta)} \le C \|v\|_{\mathcal{B}_\alpha(\widehat{\Delta})}$.

Also, $|\varphi \circ h|_\infty \le |\varphi|_\infty < \infty$ and

$$\begin{aligned} |\varphi \circ h(x) - \varphi \circ h(x')| &\le |\varphi|_\alpha \|hx - hx'\|^\alpha \le |\varphi|_\alpha \sup_{\xi\in\Delta_0} |Dh(\xi)|^\alpha \, \|x - x'\|^\alpha \\ &\le C_1^\alpha C_\lambda^\alpha \, |\varphi|_\alpha \, d(x,x')^\alpha, \end{aligned}$$

so that $\|\varphi \circ h\|_{\mathcal{B}_\alpha(\Delta)} \le C$. Finally,

$$\|(\varphi v_n) \circ h\|_{\mathcal{B}_\alpha(\Delta)} \le \|\varphi \circ h\|_{\mathcal{B}_\alpha(\Delta)} \, \|v_n \circ h\|_{\mathcal{B}_\alpha(\Delta)} \le C \, \|\varphi \circ h\|_{\mathcal{B}_\alpha(\Delta)} \, \|v\|_{\mathcal{B}_\alpha(\widehat{\Delta})}$$

as required. □

Lemma 9. *There exists $C > 0$ such that $\|\mathcal{L}^n v_n\|_{\mathcal{B}_\alpha(\Delta)} \le C \|v\|_{\mathcal{B}_\alpha(\widehat{\Delta})}$, for all $v \in \mathcal{B}_\alpha(\widehat{\Delta})$, $n \in \mathbb{N}$.*

Proof. It follows from the assumption on the metric d that $\|J_h\|_{\mathcal{B}_\alpha(\Delta)} \le C_1^\alpha \, \|J_h\|_\alpha$. Hence by Corollary 8,

$$\|J_h \, (\varphi v_n) \circ h\|_{\mathcal{B}_\alpha(\Delta)} \le \|J_h\|_{\mathcal{B}_\alpha(\Delta)} \, \|(\varphi v_n) \circ h\|_{\mathcal{B}_\alpha(\Delta)} \le C \, \|J_h\|_\alpha \, \|v\|_{\mathcal{B}_\alpha(\widehat{\Delta})}.$$

By estimate (5),

$$\Big\| \sum_{h\in\mathcal{H}_n} J_h \, (\varphi v_n) \circ h \Big\|_{\mathcal{B}_\alpha(\Delta)} \le C \|v\|_{\mathcal{B}_\alpha(\widehat{\Delta})}.$$

Finally,

$$\|\mathcal{L}^n v_n\|_{\mathcal{B}_\alpha(\Delta)} \le \|\varphi^{-1}\|_{\mathcal{B}_\alpha(\Delta)} \Big\| \sum_{h\in\mathcal{H}_n} J_h \, (\varphi v_n) \circ h \Big\|_{\mathcal{B}_\alpha(\Delta)} \le C \, \|\varphi^{-1}\|_{\mathcal{B}_\alpha(\Delta)} \, \|v\|_{\mathcal{B}_\alpha(\Delta)}$$

as required. □

It is elementary that if $f_n : \Delta \to \mathbb{R}$ is a sequence of Hölder functions with $\sup_n |f_n|_{\mathcal{B}_\alpha(\Delta)} < \infty$ and $f_n \to f$ pointwise, then $f \in \mathcal{B}_\alpha(\Delta)$ and $\|f\|_{\mathcal{B}_\alpha(\Delta)} \le \sup_n \|f_n\|_{\mathcal{B}_\alpha(\Delta)}$. Hence Proposition 6 follows from Lemma 9 by setting $f_n = \mathcal{L}^n v_n$ and $f = \bar{v}$.

Hölder disintegration for suspensions. The following generalization to suspensions turns out to be useful in [2]. Let $R : \widehat{\Delta} \to \mathbb{R}^+$ be a measurable roof function that is constant along stable leaves, so that $R : \Delta \to \mathbb{R}^+$ is well-defined. Form the suspensions $\Delta^R = \{(x,u) \in \Delta \times \mathbb{R} : 0 \le u \le R(x)\}$ and $\widehat{\Delta}^R = \{(x,z,u) \in \widehat{\Delta} \times \mathbb{R} : 0 \le u \le R(x)\}$.

For $v : \Delta^R \to \mathbb{R}$, we define $\|v\|_{\mathcal{B}_\alpha(\Delta^R)} = |v|_\infty + |v|_{\mathcal{B}_\alpha(\Delta^R)}$ where

$$|v|_{\mathcal{B}_\alpha(\Delta^R)} = \sup_{\substack{(x,u),(x',u)\in\Delta_0^R\\ x\neq x'}} \frac{|v(x,u) - v(x',u)|}{d(x,x')^\alpha}.$$

Similarly, for $v : \widehat{\Delta}^R \to \mathbb{R}$, we define $\|v\|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)} = |v|_\infty + |v|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)}$ where

$$|v|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)} = \sup_{\substack{(x,z,u),(x',z',u)\in\widehat{\Delta}_0^R\\ (x,z)\neq(x',z')}} \frac{|v(x,z,u) - v(x',z',u)|}{(d(x,x') + \|z - z'\|)^\alpha}.$$

Let $\mathcal{B}_\alpha(\Delta^R)$ and $\mathcal{B}_\alpha(\widehat{\Delta}^R)$ denote the corresponding spaces of continuous observables for which $\|v\|_{\mathcal{B}_\alpha(\Delta^R)}$ and $\|v\|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)}$ respectively are finite.

Suppose that $v : \widehat{\Delta}^R \to \mathbb{R}$. Write $v^u(x,z) = v(x,z,u)$ and note that for fixed $u \ge 0$, the function v^u is defined on the set $\bigcup_{(x,u)\in\Delta^R} \pi^{-1}(x)$. Hence we can define $\eta_x(v^u)$ whenever $(x,u) \in \Delta^R$. In this way, we obtain a function $\bar{v} : \Delta^R \to \mathbb{R}$ given by

$$\bar{v}(x,u) = \eta_x(v^u).$$

Proposition 10. *There exists $C > 0$ such that for any $v \in \mathcal{B}_\alpha(\widehat{\Delta}^R)$, the function $(x,u) \mapsto \bar{v}(x,u) = \eta_x(v^u)$ lies in $\mathcal{B}_\alpha(\Delta^R)$ and $\|\bar{v}\|_{\mathcal{B}_\alpha(\Delta^R)} \le C\,\|v\|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)}$.*

Proof. This is proved in the same way as Proposition 6, but care needs to be taken with the notation since v^u is not well-defined on the whole of $\hat{\Delta}$.

For fixed u, choose a continuous extension $w : \hat{\Delta} \to \mathbb{R}$ of v^u. Then for $(x,u) \in \Delta^R$, we have

$$\bar{v}(x,u) = \eta_x(w) = \lim_{n\to\infty} (\mathcal{L}^n w_n)(x), \quad w_n(x) = w \circ \widehat{F}^n(x,0).$$

But $\mathcal{L}^n w_n = \varphi^{-1} \sum_{h\in\mathcal{H}_n} J_h\,(\varphi w_n) \circ h$, and $w_n \circ h(x) = w \circ \hat{F}^n(hx,0) = w(x, G_n(hx,0)) = v^u(x, G_n(hx,0)) = v(x, G_n(hx,0), u)$. Hence for $(x,u) \in \Delta^R$, we have shown that

$$\bar{v}(x,u) = \lim_{n\to\infty} (M_n v)(x,u), \quad M_n v = \tilde{\varphi}^{-1} \sum_{h\in\mathcal{H}_n} \tilde{J}_h\,\tilde{\varphi} \circ \tilde{h}\,\tilde{v}_n,$$

where

$$\tilde{h}(x,u) = (hx,u), \quad \tilde{J}_h(x,u) = J_h(x), \quad \tilde{\varphi}(x,u) = \varphi(x), \quad \tilde{v}_n(x,u) = v(x, G_n(hx,0), u).$$

It now suffices to prove that $\|M_n v\|_{\mathcal{B}_\alpha(\Delta^R)} \le C\|v\|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)}$.

The main steps can now be sketched as follows. Picking up at the beginning of the proof of Corollary 8, for (x,u), $(x',u) \in \Delta_0^R$,

$$\begin{aligned} |\tilde{v}_n(x,u) - \tilde{v}_n(x',u)| &\le |v|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)}(d(x,x') + \|G_n(hx,0) - G_n(hx',0)\|)^\alpha \\ &\le C|v|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)} d(x,x')^\alpha, \end{aligned}$$

and we deduce that $\|\tilde{v}_n\|_{\mathcal{B}_\alpha(\Delta^R)} \le C\|v\|_{\mathcal{B}_\alpha(\widehat{\Delta}^R)}$.

Next, it follows as before that $\|\tilde\varphi\circ\tilde h\|_{\mathcal{B}_\alpha(\Delta^R)}\le C$ so that $\|(\tilde\varphi\tilde v_n)\circ\tilde h\|_{\mathcal{B}_\alpha(\Delta^R)}\le C\|v\|_{\mathcal{B}_\alpha(\widehat\Delta^R)}$.

Turning to Lemma 9, the estimate $\|\tilde J_h\|_{\mathcal{B}_\alpha(\Delta^R)}\le C\|J_h\|_\alpha$ holds just as before, leading to the desired estimate $\|M_n v\|_{\mathcal{B}_\alpha(\Delta^R)}\le C\|v\|_{\mathcal{B}_\alpha(\widehat\Delta^R)}$. □

5. $\mathcal{C}^1$ REGULARITY

As in the previous section, we assume the set up in Sections 2 and 3. Now we require yet more regularity for the system; namely that N is a compact manifold possibly with boundary, that $G:\Delta\times N\to N$ is $\mathcal{C}^1$ and that $F:\Delta\to\Delta$ is a $\mathcal{C}^2$ uniformly expanding map (as in Definition 5 but with $C^{1+\alpha}$ changed to C^2) with absolutely continuous invariant probability measure ν. As before $\mathcal{H}_n$ denotes the set of inverse branches of F^n, each defined on the open and dense subset Δ_0 of Δ, and $J_h:=|\det(Dh)|$, and we require that there exists $C_\lambda>0$, $\lambda>0$ such that

$$\sup_{x\in\Delta_0}|Dh(x)|\le C_\lambda e^{-\lambda n},\tag{7}$$

for all $h\in\mathcal{H}_n, n\in\mathbb{N}$. Let $\widehat\Delta_0=\Delta_0\times N$.

In the following we use the notation $Dv=(D_uv,D_sv)$ and $DG=(D_uG,D_sG)$. We require in addition the following uniform exponential contraction in the stable direction: the constants $C_\lambda>0$, $\lambda>0$ can be chosen so that also

$$\|D_sG_n(x,z)\|\le C_\lambda e^{-\lambda n},\tag{8}$$

for all $n\in\mathbb{N}$, $(x,z)\in\widehat\Delta_0$.

Define $\mathcal{C}^1(\Delta)$ to be the space of continuous functions $v:\Delta\to\mathbb{R}$ that are continuously differentiable on Δ_0 with bounded derivative. This is a Banach space under the norm $\|v\|_{\mathcal{C}^1}=\sup_{x\in\Delta}|v(x)|+\sup_{x\in\Delta_0}|Dv(x)|$. The space $\mathcal{C}^1(\widehat\Delta)$ is defined similarly.

Under these assumptions we prove:

Proposition 11. *The disintegration $\{\eta_x\}_{x\in\Delta}$ is smooth in the following sense: there exists $C>0$ such that, for any $v\in\mathcal{C}^1(\widehat\Delta)$, the function $x\mapsto\bar v(x):=\eta_x(v)$ lies in $\mathcal{C}^1(\Delta)$ and*

$$\|D\bar v(x)\|\le C\sup_{z\in N}|v(x,z)|+C\sup_{z\in N}\|Dv(x,z)\|\,,$$

for all $x\in\Delta_0$.

This fills a gap in [4] since there are inaccuracies in terms (30) and (31) therein.

Remark 12. The estimate of Proposition 11 corresponds to property (3) of [5, Definition 2.5]. Here we have an additional term, but the application of the estimate in [5, §8] is unaffected.

The remainder of this section is devoted to the proof of Proposition 11.

For all $n\in\mathbb{N}$ and $v\in\mathcal{C}^0(\widehat\Delta)$, let

$$M_nv(x):=(\mathcal{L}^nv_n)(x)=\sum_{h\in\mathcal{H}_n}\left(\tfrac{\varphi\circ h}{\varphi}\cdot J_h\right)(x)\cdot v(x,G_n(hx,0)).$$

By Proposition 3, M_nv converges in $\mathcal{C}^0(\Delta)$ and $M_nv(x)\to\eta_x(v)$.

We first show that M_nv is Cauchy in $\mathcal{C}^1(\Delta)$. Recall that $M_nv-M_{n+m}v=\mathcal{L}^{n+m}K_{n,m}$ where $K_{n,m}=\mathcal{U}^mv_n-v_{n+m}$. Hence

$$M_nv-M_{n+m}v=\sum_{\ell\in\mathcal{H}_{n+m}}\left(\tfrac{\varphi\circ\ell}{\varphi}\cdot J_\ell\right)\cdot K_{n,m}\circ\ell.\tag{9}$$

We note that $K_{n,m}(\ell x)=v(x,G_n(F^m\circ\ell(x),0))-v(x,G_{n+m}(\ell x,0))$.

The compact manifold N can be smoothly embedded as a submanifold of a vector space $\mathbb{R}^d$. We fix such an embedding, so the quantity $D_uG_n(hx,z) - D_uG_n(hx,z')$ below is well-defined.

Lemma 13. *(a) There is a constant $C > 0$ such that for all $m, n \in \mathbb{N}$ with $m \le n$, and all $h \in \mathcal{H}_n$, $(x,z) \in \widehat{\Delta}_0$,*

$$\|D_uG_m(hx,z)\ Dh(x)\| \le Ce^{-\lambda(n-m)}. \tag{10}$$

(b) There is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $h \in \mathcal{H}_n$, $(x,z), (x,z') \in \widehat{\Delta}_0$,

$$\|[D_uG_n(hx,z) - D_uG_n(hx,z')]\ Dh(x)\| \le Ce^{-\lambda n}\|z - z'\|. \tag{11}$$

Proof. (a) Choose $n_0 \ge 1$ sufficiently large that $(1 + C_\lambda)e^{-\lambda n_0} \le 1$. Let $C = C_\lambda \max_{j \le n_0} \|G_j\|_{\mathcal{C}^1}$. Then for $m \le n_0$, and all $n \in \mathbb{N}$, $h \in \mathcal{H}_n$, it follows from (7) that

$$\|D_uG_m(hx,z)\ Dh(x)\| \le \|G_m\|_{\mathcal{C}^1} C_\lambda e^{-\lambda n} \le Ce^{-\lambda(n-m)}.$$

It remains to consider the case $m \ge n_0$. We proceed by induction. Let $h \in \mathcal{H}_n$, $n \in \mathbb{N}$. Since $G_m(x,z) = G_{n_0}(F^{m-n_0}x, G_{m-n_0}(x,z))$, we have $G_m(hx,z) = G_{n_0}(\ell x, G_{m-n_0}(hx,z))$ where $\ell := F^{m-n_0} \circ h \in \mathcal{H}_{n-m+n_0}$. Hence

$$\begin{aligned} D_uG_m(hx,z)\ Dh(x) &= D_uG_{n_0}(\ell x, G_{m-n_0}(hx,z))\ D\ell(x) \\ &\quad + D_sG_{n_0}(\ell x, G_{m-n_0}(hx,z))\ D_uG_{m-n_0}(hx,z)\ Dh(x). \end{aligned} \tag{12}$$

Using the estimate (7) and the induction hypothesis,

$$\begin{aligned} \|D_uG_m(hx,z)\ Dh(x)\| &\le \|G_{n_0}\|_{\mathcal{C}^1} C_\lambda e^{-\lambda(n-m+n_0)} + C_\lambda e^{-\lambda n_0} Ce^{-\lambda(n-m+n_0)} \\ &\le C(1 + C_\lambda)e^{-\lambda n_0} e^{-\lambda(n-m)} \le Ce^{-\lambda(n-m)}, \end{aligned}$$

completing the proof of part (a).
(b) Choose $n_0 \ge 1$ sufficiently large that $e^{-\lambda n_0} C_\lambda^2 < 1$. Let $C_0 = \max_{j \le n_0} \|G_j\|_{\mathcal{C}^2}$ and let C_1 be the constant in part (a). Choose $C \ge C_0 C_\lambda$ such that

$$C_0C_\lambda^2 + C_0C_1C_\lambda + e^{-\lambda n_0}C_\lambda^2 C \le C.$$

For $n \le n_0$, $h \in \mathcal{H}_n$, it follows from (7) that

$$\|[D_uG_n(hx,z) - D_uG_n(hx,z')]\ Dh(x)\| \le C_0\|z - z'\| C_\lambda e^{-\lambda n} \le Ce^{-\lambda n}\|z - z'\|,$$

so it remains to consider the case $n \ge n_0$.

Starting from (12), we have for all $m \ge n_0$,

$$[D_uG_m(hx,z) - D_uG_m(hx,z')]\ Dh(x) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= [D_uG_{n_0}(\ell x, G_{m-n_0}(hx,z)) - D_uG_{n_0}(\ell x, G_{m-n_0}(hx,z'))]\ D\ell(x), \\ I_2 &= [D_sG_{n_0}(\ell x, G_{m-n_0}(hx,z)) - D_sG_{n_0}(\ell x, G_{m-n_0}(hx,z'))]\ D_uG_{m-n_0}(hx,z)\ Dh(x), \\ I_3 &= D_sG_{n_0}(\ell x, G_{m-n_0}(hx,z'))\ [D_uG_{m-n_0}(hx,z) - D_uG_{m-n_0}(hx,z')]\ Dh(x). \end{aligned}$$

Using estimates (7), (8) and part (a),

$$\begin{aligned} \|I_1\| &\le C_0\|G_{m-n_0}(hx,z) - G_{m-n_0}(hx,z')\| C_\lambda e^{-\lambda(n-m+n_0)} \le C_0C_\lambda^2 e^{-\lambda n}\|z - z'\|, \\ \|I_2\| &\le C_0\|G_{m-n_0}(hx,z) - G_{m-n_0}(hx,z')\| C_1 e^{-\lambda(n-m+n_0)} \le C_0C_1C_\lambda e^{-\lambda n}\|z - z'\|. \end{aligned}$$

Writing $h = k \circ \ell$ where $k \in \mathcal{H}_{m-n_0}$, $\ell \in \mathcal{H}_{n-m+n_0}$,

$$\begin{aligned} &[D_uG_{m-n_0}(hx,z) - D_uG_{m-n_0}(hx,z')]\ Dh(x) \\ &\qquad = [D_uG_{m-n_0}(k(\ell x),z) - D_uG_{m-n_0}(k(\ell x),z')]\ Dk(\ell x)\ D\ell(x). \end{aligned}$$

It follows inductively that

$$\begin{aligned}\|[D_uG_{m-n_0}(hx,z)-D_uG_{m-n_0}(hx,z')]\ Dh(x)\| &\le Ce^{-\lambda(m-n_0)}C_\lambda e^{-\lambda(n-m+n_0)}\,\|z-z'\| \\ &= CC_\lambda e^{-\lambda n}\,\|z-z'\|\,.\end{aligned}$$

Hence

$$\|I_3\| \le C_\lambda^2 e^{-\lambda n_0} Ce^{-\lambda n}\,\|z-z'\|\,.$$

It follows from the choice of n_0 and C that $\|[D_uG_m(hx,z)-D_uG_m(hx,z')]\ Dh(x)\| \le Ce^{-\lambda n}\,\|z-z'\|$ for all $m \ge n_0$, and the proof of (b) is complete. □

Lemma 14. *Suppose $v \in \mathcal{C}^1(\widehat{\Delta})$ and that $K_{n,m}$ is defined as above. Then*

$$\sup_{m\in\mathbb{N}}\sup_{\ell\in\mathcal{H}_{n+m}}\sup_{x\in\Delta_0}\|DK_{n,m}(\ell x)D\ell(x)\| \to 0, \quad \textit{as } n\to\infty.$$

Proof. Recall that $K_{n,m}(\ell x) = v(x, G_n(F^m\circ\ell(x),0)) - v(x, G_{n+m}(\ell x,0))$. Since $G_{n+m}(\ell x,0) = G_n(F^m\circ\ell(x), G_m(\ell x,0))$,

$$K_{n,m}(\ell x) = v(x,G_n(hx,0)) - v(x,G_n(hx,G_m(\ell x,0))),$$

where for convenience we write $h = F^m\circ\ell$. Differentiating we obtain

$$DK_{n,m}(\ell x)D\ell(x) = J_1 + J_2 - J_3, \tag{13}$$

where writing $z = G_m(\ell x,0)$,

$$\begin{aligned} J_1 &= D_uv(x,G_n(hx,0)) - D_uv(x,G_n(hx,z)),\\ J_2 &= D_sv(x,G_n(hx,0))\ D_uG_n(hx,0)\ Dh(x)\\ &\qquad - D_sv(x,G_n(hx,z))\ D_uG_n(hx,z)\ Dh(x),\\ J_3 &= D_sv(x,G_n(hx,z))\ D_sG_n(hx,z)\ D_uG_m(\ell x,0)\ D\ell(x).\end{aligned}$$

By (8),

$$\|G_n(hx,0)-G_n(hx,z)\| \le C_\lambda e^{-\lambda n}\operatorname{diam}(\pi^{-1}(x)).$$

Therefore, by the uniform continuity of Dv, we have that $\|J_1\|\to 0$ uniformly in x, ℓ and m as $n\to\infty$.

Next, $J_2 = J_2' + J_2''$ where

$$\begin{aligned} J_2' &= [D_sv(x,G_n(hx,0)) - D_sv(x,G_n(hx,z))]\ D_uG_n(hx,0)\ Dh(x),\\ J_2'' &= D_sv(x,G_n(hx,z))\ [D_uG_n(hx,0) - D_uG_n(hx,z)]\ Dh(x).\end{aligned}$$

The same argument used for J_1 shows that

$$\sup_{x,\ell,m}\|D_sv(x,G_n(hx,0)) - D_sv(x,G_n(hx,z))\| \to 0$$

as $n\to\infty$. Combining this with Lemma 13(a) we obtain that $\|J_2'\|\to 0$ uniformly in x, ℓ and m as $n\to\infty$.

The first factor of J_2'' is bounded by $\|v\|_{\mathcal{C}^1}$, so it follows from Lemma 13(b) that $\|J_2''\|\to 0$ uniformly in x, ℓ and m as $n\to\infty$.

Turning to J_3, the three factors are bounded by $\|v\|_{\mathcal{C}^1}$, $C_\lambda e^{-\lambda n}$ and $Ce^{-\lambda(n+m-m)} = Ce^{-\lambda n}$ respectively, where we have used (8) and Lemma 13(a). Hence $\|J_3\|\to 0$ uniformly in x, ℓ and m as $n\to\infty$. The combination of these estimates completes the proof of the lemma. □

A standard consequence of the assumptions used in this section is that there exists $C_d > 0$ such that

$$\sum_{h\in\mathcal{H}_n}\|DJ_h(x)\| \le C_d \qquad \text{for all } n\in\mathbb{N},\ x\in\Delta_0. \tag{14}$$

Observe that, differentiating (9),

$$D(M_n v - M_{n+m} v) = \sum_{\ell \in \mathcal{H}_{n+m}} D\Big(\frac{\varphi \circ \ell}{\varphi} \cdot J_h\Big) \cdot K_{n,m} \circ \ell + \Big(\frac{\varphi \circ \ell}{\varphi} \cdot J_h\Big) \cdot \Big(DK_{n,m}\Big) \circ \ell \; D\ell.$$

Using (14) and Lemma 14, together with the previously proven fact (3) that $\sup_{x,m} \|K_{n,m}(x)\| \to 0$ as $n \to \infty$ and that $\sum_{\ell \in \mathcal{H}_n} \Big(\frac{\varphi \circ \ell}{\varphi} \cdot J_h\Big)(x) = 1$ proves that

$$\sup_{x \in \Delta_0} \sup_{m \in \mathbb{N}} \|D(M_n v - M_{n+m} v)(x)\| \to 0$$

as $n \to \infty$ and hence the sequence $M_n v$ is Cauchy in $\mathcal{C}^1(\Delta)$. This proves the first claim of Proposition 11, namely that $\bar{v} \in \mathcal{C}^1(\Delta)$. Moreover, we have shown that $M_n v \to \bar{v}$ in $\mathcal{C}^1(\Delta)$ so it remains to show that there exists $C > 0$ such that $\|M_n v\|_{\mathcal{C}^1} \le C \|v\|_{\mathcal{C}^1}$. It is clear that $\|M_n v\|_{\mathcal{C}^0} \le \|v\|_{\mathcal{C}^0}$, so it remains to prove:

Lemma 15. *There exists $C > 0$ such that, for all $v \in \mathcal{C}^1(\widehat{\Delta})$, $n \in \mathbb{N}$, $x \in \Delta_0$,*

$$\|DM_n v(x)\| \le C \sup_{z \in N} |v(x,z)| + C \sup_{z \in N} \|Dv(x,z)\|.$$

Proof. Write $M_n v(x) = \sum_{h \in \mathcal{H}_n} (\frac{\varphi \circ h}{\varphi} \cdot J_h)(x) \cdot B_n(x)$, where $B_n(x) = v(x, G_n(hx, 0))$. First we estimate $\|DB_n(x)\|$. Differentiating we obtain

$$DB_n(x) = D_u v(x, G_n(hx,0)) + D_s v(x, G_n(hx,0)) \; D_u G_n(hx,0) \; Dh(x)$$

and so by Lemma 13,

$$\|DB_n(x)\| \le \sup_{z \in N} \|Dv(x,z)\| \, (1 + \|D_u G_n(hx,0) \; Dh(x)\|) \le C \sup_{z \in N} \|Dv(x,z)\|.$$

Clearly, $|B_n(x)| \le \sup_{z \in N} |v(x,z)|$. Using also the estimate from (14) completes the proof of the lemma. ☐

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