Good inducing schemes for uniformly hyperbolic flows, and applications to exponential decay of correlations

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Abstract

Given an Axiom A attractor for a $C^{1+\alpha}$ flow ($\alpha > 0$), we construct a countable Markov extension with exponential return times in such a way that the inducing set is a smoothly embedded unstable disk. This avoids technical issues concerning irregularity of boundaries of Markov partition elements and enables an elementary approach to certain questions involving exponential decay of correlations for SRB measures.

1 Introduction

Statistical properties [12, 29, 31] of Anosov and Axiom A diffeomorphisms [3, 33] were developed extensively in the 1970s. Key tools were the construction of finite Markov partitions [10, 32] and the spectral properties of transfer operators [28]. In particular, ergodic invariant probability measures were constructed corresponding to any Hölder potential; moreover, it was shown that hyperbolic basic sets for Axiom A diffeomorphisms are always exponentially mixing up to a finite cycle for such measures, see for example [12, 22, 29].

Still in the 1970s, finite Markov partitions were constructed [11, 26] for Anosov and Axiom A flows. This allows us to model each hyperbolic basic set as a suspension flow over a subshift of finite type, enabling the study of thermodynamic formalism (see e.g. [14]) and statistical properties (see e.g. [17, 27]).

However, rates of mixing for Axiom A flows are still poorly understood. By [24, 30], mixing Axiom A flows can mix arbitrarily slowly. Although there has been

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important progress starting with [16, 18, 20], it remains an open question whether mixing Anosov flows have exponential decay of correlations [14]. Very recently, this question was answered positively [35] in the case of C^∞ three-dimensional flows.

It turns out that using finite Markov partitions for flows raises technical issues due to the irregularity of their boundaries [5, 15, 34]. Even in the discrete-time setting, it is known that the boundaries of elements of a finite Markov partition need not be smooth [13]. In this paper, we propose using the approach of [36] to circumvent such issues at least in the case of SRB measures. In particular, we show that

> *Any attractor for an Axiom A flow can be modelled by a suspension flow over a full branch countable Markov extension where the inducing set is a smoothly embedded unstable disk. The roof function, though unbounded, has exponential tails.*

A precise statement is given in Theorem 2.1 below.

Remark 1.1 The approach of Young towers [36] has proved to be highly effective for studying discrete-time examples like planar dispersing billiards and Hénon-like attractors where suitable Markov partitions are not available. However, as shown in the current paper, there can be advantages (at least in continuous time) to working with countable Markov extensions even when there is a well-developed theory of finite Markov partitions. The extra flexibility of Markov extensions can be used not only to construct the extension but to ensure good regularity properties of the partition elements.

As a consequence of Theorem 2.1, we obtain an elementary proof of the following result:

Theorem 1.2 *Suppose that Λ is an Axiom A attractor with SRB measure μ for a C^{1+} flow ϕ_t with C^{1+} stable holonomies*[1] *and such that the stable and unstable bundles are not jointly integrable. Then for all Hölder observables v, $w : \Lambda \to \mathbb{R}$, there exist constants $c, C > 0$ such that*

$$\Big|\int_\Lambda v\, w \circ \phi_t \, d\mu - \int_\Lambda v \, d\mu \int_\Lambda w \, d\mu\Big| \le C e^{-ct} \quad \textit{for all } t > 0.$$

Remark 1.3 Joint nonintegrability holds for an open and dense set of Axiom A flows and their attractors, see [19] and references therein. It implies mixing and is equivalent to mixing for codimension one Anosov flows. It is conjectured to be equivalent to mixing for Anosov flows [23].

Remark 1.4 (a) In the case when the unstable direction is one-dimensional and the stable holonomies are C^2, this result is due to [9, 8, 4, 5]. In particular, using

[1] C^{1+} means C^r for some $r > 1$.

the fact that stable bunching is a robust sufficient condition for smoothness of stable holonomies together with the robustness of joint nonintegrability, [4] constructed the first robust examples of Axiom A flows with exponential decay of correlations. The smoothness condition on stable holonomies was relaxed from C^2 to C^{1+} in [6] extending the class of examples in [4]. This class of examples is extended further by Theorem 1.2 with the removal of the one-dimensionality restriction on unstable manifolds.

(b) There is no restriction on the dimension of unstable manifolds in [8], and it is not surprising that the smoothness assumption on stable holonomies can also be relaxed as in [6]. However, there is a crucial hypothesis in [8] on the regularity of the inducing set in the unstable direction which is nontrivial in higher dimensions.

Theorem 1.2 is stated in the special case of Anosov flows in [15]. In [15, Appendix] it is argued that at least in the Anosov case the Markov partitions of [26] are sufficiently regular that the methods in [8] can be pushed through. In [5], a sketch is given of how to prove Theorem 1.2 also in the Axiom A case, but the details are not fully worked out.

As mentioned, our approach in this paper completely bypasses such issues since our inducing set is a smoothly embedded unstable disk. Moreover, our method works equally well for Anosov flows and Axiom A attractors. As a consequence, we recover the examples in [15], in particular that codimension one volume-preserving mixing C^{1+} Anosov flows are exponentially mixing in dimension four and higher.

The remainder of the paper is organised as follows. In Section 2, we state precisely and prove our result on good inducing for attractors of Axiom A flows. In Section 3, we prove a result on exponential mixing for a class of skew product Axiom A flows, extending/combining the results in [6, 8]. In Section 4, we complete the proof of Theorem 1.2.

2 Good inducing for attractors of Axiom A flows

Let $\phi_t : M \to M$ be a C^{1+} flow defined on a compact Riemannian manifold (M, d_M), and let $\Lambda \subset M$ be a closed ϕ_t-invariant subset. We assume that Λ is an attracting transitive uniformly hyperbolic set with adapted norm and that Λ is not a single trajectory. In particular, there is a continuous $D\phi_t$-invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ where E^c is the one-dimensional central direction tangent to the flow, and there exists $\lambda \in (0,1)$ such that $|D\phi_t v| \le \lambda^t |v|$ for all $v \in E^s$, $t \ge 1$; $|D\phi_{-t} v| \le \lambda^t |v|$ for all $v \in E^u$, $t \ge 1$. Since the time-s map $\phi_s : \Lambda \to \Lambda$ is ergodic for all but countably many choices of $s \in \mathbb{R}$ [25], we can scale time slightly if necessary so that $\phi_{-1} : \Lambda \to \Lambda$ is transitive. Then there exists $p \in \Lambda$ such that $\bigcup_{i \ge 1} \phi_{-i} p$ is dense in Λ.

We can define (local) stable disks $W^s_\delta(y) = \{z \in W^s(y) : d_M(y,z) < \delta\}$ for $\delta > 0$ sufficiently small for all $y \in \Lambda$. Define local centre-stable disks $W^{cs}_\delta(y) = \bigcup_{|t|<\delta} \phi_t W^s_\delta(y)$. Let Leb and d denote induced Lebesgue measure and induced distance on local unstable manifolds. It is convenient to define local unstable disks $W^u_\delta(y) = \{z \in W^u(y) : d(y,z) < \delta\}$ using the induced distance.

For δ_0 small, define $\mathcal{D} = W^u_{\delta_0}(p)$ and $\widehat{\mathcal{D}} = \bigcup_{x\in\mathcal{D}} W^{cs}_{\delta_0}(x)$. Define $\pi : \widehat{\mathcal{D}} \to \mathcal{D}$ such that $\pi|W^{cs}_{\delta_0}(x) \equiv x$. Whenever $\phi_n y \in \widehat{\mathcal{D}}$, we set $g_n y = \pi(\phi_n y)$.

We are now in a position to give a precise description of our inducing scheme.

Theorem 2.1 *There exists an open unstable disk $Y = W^u_\delta(p) \subset \mathcal{D}$ (for some $\delta \in (0,\delta_0)$) and a discrete return time function $R : Y \to \mathbb{Z}^+ \cup \{\infty\}$ such that*

(i) $\mathrm{Leb}(R > n) = O(\gamma^n)$ *for some $\gamma \in (0,1)$;*

(ii) Each connected component of $\{R = n\}$ is mapped by ϕ_n into $\widehat{\mathcal{D}}$ and mapped homeomorphically by g_n onto Y.

Remark 2.2 Let $\mathcal{P}$ be the partition of Y consisting of connected components of $\{R = n\}$ for $n \ge 1$. (It follows from Theorem 2.1(i) that $\mathcal{P}$ is a partition of Y mod 0.) Define $F : Y \to Y$, $F = g_R = \pi \circ \phi_R$. Note that F is locally the composition of a time-R map ϕ_R (where R is constant on each partition element) with a centre-stable holonomy. Since centre-stable holonomies are Hölder continuous, it follows that F maps partition elements $U \in \mathcal{P}$ homeomorphically onto Y and that $F|_U : U \to Y$ is a bi-Hölder bijection. If moreover, the centre-stable holonomies are C^1, then the partition elements are diffeomorphic to disks.

In the remainder of this section, we prove Theorem 2.1. Our proof is essentially the same as in [36, Section 6] for Axiom A diffeomorphisms, but we closely follow the treatment in [2] which provides many of the details of arguments sketched in [36].

Choice of constants Choose $\delta_0 > 0$ so that the following bounded distortion property holds: there exists $C_1 \ge 1$ so that

$$\frac{|\det D\phi_n(x)|E^u|}{|\det D\phi_n(y)|E^u|} \le C_1 \tag{2.1}$$

for every $n \ge 1$ and all $x, y \in \Lambda$ with $\phi_n x, \phi_n y$ in the same unstable disk such that $d(\phi_j x, \phi_j y) < 4\delta_0$ for all $0 \le j \le n$.

By standard results about stable holonomies, π is absolutely continuous and C^α for some $\alpha \in (0,1)$ when restricted to unstable disks in $\widehat{\mathcal{D}}$. For δ_0 sufficiently small, there exists C_2, $C_3 \ge 1$ such that

$$C_2^{-1} \le \frac{\mathrm{Leb}(\pi(E))}{\mathrm{Leb}(E)} \le C_2 \tag{2.2}$$

for all Lebesgue-measurable subset $E \subset W^u_{\delta_0}(y) \cap \widehat{\mathcal{D}}$ and all $y \in \Lambda$, and

$$d(\pi x, \pi y) \le C_3 d(x,y)^\alpha \tag{2.3}$$

for all $x, y \in \widehat{\mathcal{D}}$ with x, y in the same unstable disk such that $d(x, y) < 4\delta_0$.

Let $d_u = \dim E^u$ and fix $L \geq 3$ so that

$$C_1 C_2^2 \frac{2^{d_u} - 1}{(L-1)^{d_u}} < \frac{1}{4}. \tag{2.4}$$

By the local product structure, there exists $\delta_1 \in (0, \delta_0)$ such that $W^{cs}_{\delta_0}(x) \cap W^u_{\delta_0}(y)$ consists of precisely one point for all $x, y \in \Lambda$ with $d_M(x, y) < 4\delta_1$. Similarly, there exists $\delta \in (0, \delta_1)$ such that $W^{cs}_{\delta_1}(x) \cap W^u_{\delta_1}(y)$ consists of precisely one point for all $x, y \in \Lambda$ with $d_M(x, y) < (L+1)\delta$. Moreover, since local unstable/stable disks have bounded curvature, the intersection point $z \in W^{cs}_{\delta_1}(x) \cap W^u_{\delta_1}(y)$ satisfies $d(z, y) \leq C_4 d_M(x, y)$ where $C_4 \geq 1$ is a constant. Shrink $\delta > 0$ if necessary so that $C_3(3\delta)^\alpha < \frac{1}{2}\delta_0$ and $C_4(L+1)\delta < \delta_0$. Choose $N_1 \geq 1$ such that $\bigcup_{i=1}^{N_1} \phi_{-i} p$ is δ-dense in Λ.

Construction of the partition We consider various small neighbourhoods $\mathcal{D}_c = W^u_{c\delta}(p)$ with $c \in \{1, 2, L-1, L\}$. Define $\widetilde{\mathcal{D}}_c = \bigcup_{x \in \mathcal{D}_c} W^{cs}_{\delta_1}(x)$.

Take $Y = \mathcal{D}_1$. Define a partition $\{I_k : k \geq 1\}$ of $\mathcal{D}_2 \setminus \mathcal{D}_1$,

$$I_k = \big\{ y \in \mathcal{D}_2 : \delta(1 + \lambda^{\alpha k}) \leq d(y, p) < \delta(1 + \lambda^{\alpha(k-1)}) \big\}.$$

Fix $\varepsilon > 0$ small (as stipulated in Propositions 2.4 and 2.5 and Lemma 2.9 below). We define sets Y_n and functions $t_n : Y_n \to \mathbb{N}$, and $R : Y \to \mathbb{Z}^+$ inductively, with $Y_n = \{R > n\}$. Define $Y_0 = Y$ and $t_0 \equiv 0$. Inductively, suppose that $Y_{n-1} = Y \setminus \{R < n\}$ and that $t_{n-1} : Y_{n-1} \to \mathbb{N}$ is given. Write $Y_{n-1} = A_{n-1} \,\dot\cup\, B_{n-1}$ where

$$A_{n-1} = \{t_{n-1} = 0\}, \qquad B_{n-1} = \{t_{n-1} \geq 1\}.$$

Consider the neighbourhood

$$A^{(\varepsilon)}_{n-1} = \big\{ y \in Y_{n-1} : d(\phi_n y, \phi_n A_{n-1}) < \varepsilon \big\}$$

of the set A_{n-1}. Define U^L_{nj}, $j \geq 1$, to be the connected components of $A^{(\varepsilon)}_{n-1} \cap \phi_{-n} \widetilde{\mathcal{D}}_L$ that are mapped inside $\widetilde{\mathcal{D}}_L$ by ϕ_n and mapped homeomorphically onto $\mathcal{D}_L$ by g_n. Let

$$U^c_{nj} = U^L_{nj} \cap g_n^{-1} \mathcal{D}_c \quad \text{for } c = 1, 2, L-1.$$

Define $R|U^1_{nj} = n$ for each U^1_{nj} and take $Y_n = Y_{n-1} \setminus \bigcup_j U^1_{nj}$. Finally, define $t_n : Y_n \to \mathbb{N}$ as

$$t_n(y) = \begin{cases} k, & y \in \bigcup_j U^2_{nj} \text{ and } g_n y \in I_k \text{ for some } k \geq 1 \\ 0, & y \in A_{n-1} \setminus \bigcup_j U^2_{nj} \\ t_{n-1}(y) - 1, & y \in B_{n-1} \setminus \bigcup_j U^2_{nj} \end{cases}$$

and take $A_n = \{t_n = 0\}$, $B_n = \{t_n \geq 1\}$ and $Y_n = A_n \,\dot\cup\, B_n$.

Remark 2.3 By construction, property (ii) of Theorem 2.1 is satisfied. It remains to verify that $\mathrm{Leb}(R > n)$ decays exponentially.

Visualisation of B_n. The set B_n is a disjoint union $B_n = \bigcup_{m=1}^n C_n(m)$ where $C_n(m)$ is a disjoint union of *collars* around each component of $\{R = m\}$. Each collar in $C_n(m)$ is homeomorphic under g_m to $\bigcup_{k\geq n-m+1} I_k$ with outer ring homeomorphic under g_m to I_{n-m+1}, and the union of outer rings is the set $\{t_n = 1\}$. This picture presupposes Proposition 2.4 below which guarantees that each new generation of collars $C_n(n)$ does not intersect the set $\bigcup_{1\leq m\leq n-1} C_{n-1}(m)$ of collars in the previous generation.

Proposition 2.4 *Choose $\varepsilon < (C_3^{-1}\delta)^{1/\alpha}$ sufficiently small that $W_\varepsilon^u(x) \subset \widehat{\mathcal{D}}$ for all $x \in \widetilde{\mathcal{D}}_L$. Then $\bigcup_j U_{nj}^{L-1} \subset A_{n-1}$ for all $n \geq 1$.*

Proof We argue by contradiction. There is nothing to prove for $n = 1$. Let $n \geq 2$ be least such that the result fails and choose j such that U_{nj}^{L-1} intersects B_{n-1}. Then either (i) $U_{nj}^{L-1} \subset B_{n-1}$, or (ii) U_{nj}^{L-1} intersects ∂A_{n-1}.

In case (i), choose $x \in U_{nj}^{L-1}$ (so in particular $\phi_n x \in \widehat{\mathcal{D}}$) with $g_n x = p$. Since $U_{nj}^{L-1} \subset U_{nj}^L \subset A_{n-1}^{(\varepsilon)}$, there exists $y \in A_{n-1}$ with $d(\phi_n x, \phi_n y) < \varepsilon$. In particular, $\phi_n y \in \widehat{\mathcal{D}}$ so $g_n y$ is well-defined. Note that $x \in U_{nj}^{L-1}$ and $y \notin U_{nj}^{L-1}$ since $U_{nj}^{L-1} \subset B_{n-1}$. Hence the geodesic ℓ in $\mathcal{D}$ joining $g_n x$ and $g_n y$ intersects $g_n \partial U_{nj}^{L-1}$. Choose $z \in \partial U_{nj}^{L-1} \cap g_n^{-1}\ell$. Since $g_n = \pi \circ \phi_n$, it follows from (2.3) that

$$\delta < (L-1)\delta = d(g_n x, g_n z) < d(g_n x, g_n y) \leq C_3 d(\phi_n x, \phi_n y)^\alpha < C_3\varepsilon^\alpha < \delta$$

which is a contradiction. This rules out case (i).

In case (ii), choose $x \in U_{nj}^{L-1} \cap \partial A_{n-1}$. We show below that there exists $y \in \partial A_{n-1}^{(\varepsilon)}$ such that $d(\phi_n x, \phi_n y) \leq \varepsilon$. In particular, $g_n x$ and $g_n y$ are well-defined and $d(g_n x, g_n y) \leq C_3\varepsilon^\alpha < \delta$. Since $U_{nj}^L \subset A_{n-1}^{(\varepsilon)}$, we have that $y \notin U_{nj}^L$. It follows that $g_n x \in \mathcal{D}_{L-1}$ while $g_n y \notin \mathcal{D}_L$. Hence $d(g_n x, g_n y) \geq \delta$ which is the desired contradiction.

It remains to verify that there exists $y \in \partial A_{n-1}^{(\varepsilon)}$ such that $d(\phi_n x, \phi_n y) \leq \varepsilon$. Since n is least, B_{n-1} is a disjoint union of collars as described in the visualisation above. Hence there exists a collar $Q \subset C_{n-1}(n-k)$ intersected by U_{nj}^{L-1} for some $1 \leq k < n$ such that x lies in the outer boundary $\partial_o Q$ of Q. Note that $\partial_o Q = \partial A_{n-1} \cap Q$. Let D denote the disk enclosed by $\partial_o Q$ and let

$$S = D \cap \partial(\phi_{-n} B_\varepsilon(\phi_n \partial D)).$$

We claim that $S \neq \emptyset$ and $S \subset Q$. Then S is a $(\dim Y - 1)$-dimensional sphere contained in $\partial A_{n-1}^{(\varepsilon)}$ and there exists $y \in S$ with the desired properties. (The point of the claim is that S lies entirely in Y_{n-1}.)

Note that g_{n-k} maps Q homeomorphically onto the set $J = \bigcup_{i\geq k} I_i$ which is an annulus of radial thickness $\delta\lambda^{\alpha k}$. By (2.3), ϕ_{n-k} maps Q homeomorphically onto a set $\tilde{J} = \pi^{-1}J$ of radial thickness at least $(C_3^{-1}\delta\lambda^{\alpha k})^{1/\alpha} = (C_3^{-1}\delta)^{1/\alpha}\lambda^k$.

Moreover, $\phi_k(\tilde{J} \cap \phi_{n-k} A_{n-1}^{(\varepsilon)}) \subset \phi_n A_{n-1}^{(\varepsilon)}$ is contained in the set of points within d-distance ε of $\phi_n \partial A_{n-1}^{(\varepsilon)}$, so by definition of λ we have that $\tilde{J} \cap \phi_{n-k} A_{n-1}^{(\varepsilon)}$ is contained in the set of points within d-distance $\varepsilon\lambda^k$ of the outer boundary of $\tilde{J}$. Since $\varepsilon < (C_3^{-1}\delta)^{1/\alpha}$, we obtain that $\tilde{J} \cap \phi_{n-k} \partial A_{n-1}^{(\varepsilon)}$ is homeomorphic to a $(\dim Y - 1)$-dimensional sphere contained entirely inside $\tilde{J}$. Hence $S = Q \cap \partial A_{n-1}^{(\varepsilon)}$ is homeomorphic to a $(\dim Y - 1)$-dimensional sphere contained entirely inside Q, as required. ∎

Proposition 2.5 *Choose* $\varepsilon < \big\{C_3^{-1}\delta(\lambda^{-\alpha} - 1)\big\}^{1/\alpha}$. *Then for all* $n \geq 1$,

(a) $A_{n-1}^{(\varepsilon)} \subset \{y \in Y_{n-1} : t_{n-1}(y) \leq 1\}$ *for all* $n \geq 1$.

(b) $\phi_{-n} W_\varepsilon^u(\phi_n x) \subset A_{n-1}^{(\varepsilon)}$ *for all* $x \in A_{n-1}$.

Proof (a) Suppose that $t_{n-1}(y) > 1$. Then there exists a collar in $C_{n-1}(n-k)$ containing y. Let Q denote the outer ring of the collar with outer boundary Q_1 and inner boundary Q_2. Then $t_{n-1}|Q \equiv 1$ and $t_{n-1}(y) > 1$, so y lies inside the region bounded by Q_2.

Suppose for contradiction that $y \in A_{n-1}^{(\varepsilon)}$. Then we can choose $x \in A_{n-1}$ with $d(\phi_n x, \phi_n y) < \varepsilon$. Let ℓ be the geodesic in $W_\varepsilon^u(\phi_n x)$ connecting $\phi_n x$ to $\phi_n y$ and define $q_j \in Q_j \cap \phi_{-n}\ell$ for $j = 1, 2$.

Recall that Q is homeomorphic under g_{n-k} to I_k. Moreover, $g_{n-k} q_j$ lie in distinct components of the boundary of I_k, so

$$d(g_{n-k}q_1, g_{n-k}q_2) \geq \delta(\lambda^{\alpha(k-1)} - \lambda^{\alpha k}) = \delta(\lambda^{-\alpha} - 1)\lambda^{\alpha k}.$$

Hence

$$\begin{aligned} d(\phi_n q_1, \phi_n q_2) &\geq \lambda^{-k} d(\phi_{n-k} q_1, \phi_{n-k} q_2) \\ &\geq \lambda^{-k}\big\{C_3^{-1} d(g_{n-k}q_1, g_{n-k}q_2)\big\}^{1/\alpha} \geq \big\{C_3^{-1}\delta(\lambda^{-\alpha} - 1)\big\}^{1/\alpha} > \varepsilon. \end{aligned}$$

But $d(\phi_n q_1, \phi_n q_2) \leq d(\phi_n y, \phi_n x) < \varepsilon$ so we obtain the desired contradiction.

(b) Let $x \in A_{n-1}$ and $y \in \phi_{-n} W_\varepsilon^u(\phi_n x)$. Note that $y \in A_{n-1}^{(\varepsilon)}$ if and only if $y \in Y_{n-1}$. Hence we must show that $y \in Y_{n-1}$. If not, then there exists $k \geq 1$ such that $y \in \{R = n - k\}$. Define $Q \subset C_{n-1}(n-k)$ to be the outer ring of the corresponding collar. Choosing q_1 and q_2 as in part (a) we again obtain a contradiction. ∎

Lemma 2.6 *There exists* $a_1 > 0$ *such that for all* $n \geq 1$,

(a) $\mathrm{Leb}(B_{n-1} \cap A_n) \geq a_1 \,\mathrm{Leb}(B_{n-1})$.

(b) $\mathrm{Leb}(A_{n-1} \cap B_n) \leq \frac{1}{4}\,\mathrm{Leb}(A_{n-1})$.

(c) $\mathrm{Leb}(A_{n-1} \cap \{R = n\}) \leq \frac{1}{4}\,\mathrm{Leb}(A_{n-1})$.

Proof (a) Let $y \in B_{n-1}$. By Proposition 2.4, $y \notin \bigcup_j U_{nj}^{L-1}$ so in particular $y \in Y_n$. Note that $t_n(y) = 0$ if and only if $t_{n-1}(y) = 1$. Hence $B_{n-1} \cap A_n = \{t_{n-1} = 1\}$.

Now let $Q \subset C_{n-1}(n-k) \subset B_{n-1}$ be a collar ($1 \le k \le n$) with outer ring $Q \cap A_n = Q \cap \{t_{n-1} = 1\}$. Then $g_{n-k} = \pi \circ \phi_{n-k}$ maps Q homeomorphically onto $\bigcup_{i \ge k} I_i$ and $Q \cap \{t_{n-1} = 1\}$ homeomorphically onto I_k. Let $d_u = \dim E^u$. By (2.1) and (2.2),

$$\begin{aligned}\frac{\mathrm{Leb}(Q)}{\mathrm{Leb}(Q \cap A_n)} &= \frac{\mathrm{Leb}(Q)}{\mathrm{Leb}(Q \cap \{t_{n-1} = 1\})} \le C_1 \frac{\mathrm{Leb}(\phi_{n-k}Q)}{\mathrm{Leb}(\phi_{n-k}(Q \cap \{t_{n-1} = 1\}))} \\ &\le C_1 C_2^2 \frac{\mathrm{Leb}(\bigcup_{i \ge k} I_i)}{\mathrm{Leb}(I_k)} = C_1 C_2^2 D(d_u, \lambda^\alpha, k)\end{aligned}$$

where $D(d_u, \lambda, k) = \dfrac{(1+\lambda^{k-1})^{d_u} - 1}{(1+\lambda^{k-1})^{d_u} - (1+\lambda^k)^{d_u}}$. Since $\lim_{k \to \infty} D(d_u, \lambda, k) = (1-\lambda)^{-1}$, we obtain that $\mathrm{Leb}(Q) \le C_1 C_2^2 D\, \mathrm{Leb}(Q \cap A_n)$ where $D = \sup_{k \ge 1} D(d_u, \lambda^\alpha, k)$ is a constant depending only on d_u and λ^α. Summing over collars Q, it follows that $\mathrm{Leb}(B_{n-1}) \le C_1 C_2^2 D\, \mathrm{Leb}(B_{n-1} \cap A_n)$.

(b) By Proposition 2.4, $U_{nj}^2 \subset U_{nj}^{L-1} \subset A_{n-1}$ for each j. It follows that $A_{n-1} \cap B_n = \bigcup_j U_{nj}^2 \setminus U_{nj}^1$. By (2.1), (2.2) and (2.4),

$$\frac{\mathrm{Leb}(U_{nj}^2 \setminus U_{nj}^1)}{\mathrm{Leb}(U_{nj}^{L-1})} \le C_1 C_2^2 \frac{\mathrm{Leb}(\mathcal{D}_2 \setminus \mathcal{D}_1)}{\mathrm{Leb}(\mathcal{D}_{L-1})} = C_1 C_2^2 \frac{2^{d_u} - 1}{(L-1)^{d_u}} < \frac{1}{4}.$$

Hence

$$\frac{\mathrm{Leb}(A_{n-1} \cap B_n)}{\mathrm{Leb}(A_{n-1})} \le \frac{\sum_j \mathrm{Leb}(U_{nj}^2 \setminus U_{nj}^1)}{\sum_j \mathrm{Leb}(U_{nj}^{L-1})} < \frac{1}{4}.$$

(c) Proceeding as in part (b) with $U_{nj}^2 \setminus U_{nj}^1$ replaced by U_{nj}^1, leads to the estimate

$$\frac{\mathrm{Leb}(A_{n-1} \cap \{R = n\})}{\mathrm{Leb}(A_{n-1})} \le \frac{\sum_j \mathrm{Leb}(U_{nj}^1)}{\sum_j \mathrm{Leb}(U_{nj}^{L-1})} \le \frac{C_1 C_2^2}{(L-1)^{d_u}} < \frac{1}{4}. \quad \blacksquare$$

Corollary 2.7 *For all $n \ge 1$,*

(a) $\mathrm{Leb}(A_{n-1} \cap A_n) \ge \frac{1}{2} \mathrm{Leb}(A_{n-1})$.

(b) $\mathrm{Leb}(B_{n-1} \cap B_n) \le (1 - a_1) \mathrm{Leb}(B_{n-1})$.

(c) $\mathrm{Leb}(B_n) \le \frac{1}{4} \mathrm{Leb}(A_{n-1}) + (1 - a_1) \mathrm{Leb}(B_{n-1})$.

(d) $\mathrm{Leb}(A_n) \ge \frac{1}{2} \mathrm{Leb}(A_{n-1}) + a_1 \mathrm{Leb}(B_{n-1})$.

Proof Recall that $A_{n-1} \subset Y_{n-1} = Y_n \,\dot\cup\, \{R = n\} = A_n \,\dot\cup\, B_n \,\dot\cup\, \{R = n\}$. Hence by Lemma 2.6(b,c),

$$\begin{aligned}\mathrm{Leb}(A_{n-1}) &= \mathrm{Leb}(A_{n-1} \cap A_n) + \mathrm{Leb}(A_{n-1} \cap B_n) + \mathrm{Leb}(A_{n-1} \cap \{R = n\}) \\ &\le \mathrm{Leb}(A_{n-1} \cap A_n) + \tfrac{1}{2}\,\mathrm{Leb}(A_{n-1}),\end{aligned}$$

proving (a). Similarly, by Lemma 2.6(a),

$$\begin{aligned}\mathrm{Leb}(B_{n-1}) &= \mathrm{Leb}(B_{n-1} \cap A_n) + \mathrm{Leb}(B_{n-1} \cap B_n) + \mathrm{Leb}(B_{n-1} \cap \{R = n\}) \\ &\ge a_1\,\mathrm{Leb}(B_{n-1}) + \mathrm{Leb}(B_{n-1} \cap B_n),\end{aligned}$$

proving (b).

Next, recall that $B_n = B_n \cap Y_{n-1} = B_n \cap \big(A_{n-1} \,\dot\cup\, B_{n-1}\big)$. Hence part (c) follows from Lemma 2.6(b) and part (b). Similarly, $A_n = A_n \cap \big(A_{n-1} \,\dot\cup\, B_{n-1}\big)$ and part (d) follows from Lemma 2.6(a) and part (a). ∎

Corollary 2.8 *There exists $a_0 > 0$ such that $\mathrm{Leb}(B_n) \le a_0\,\mathrm{Leb}(A_n)$ for all $n \ge 0$.*

Proof Let $a_0 = \dfrac{2 + a_1}{2a_1}$. We prove the result by induction. The case $n = 0$ is trivial since $B_0 = \emptyset$. For the induction step from $n-1$ to n, we consider separately the cases $\mathrm{Leb}(B_{n-1}) > \frac{1}{2a_1}\,\mathrm{Leb}(A_{n-1})$ and $\mathrm{Leb}(B_{n-1}) \le \frac{1}{2a_1}\,\mathrm{Leb}(A_{n-1})$.

Suppose first that $\mathrm{Leb}(B_{n-1}) > \frac{1}{2a_1}\,\mathrm{Leb}(A_{n-1})$. By Corollary 2.7(c),

$$\mathrm{Leb}(B_n) < \big\{\tfrac{1}{2}a_1 + (1 - a_1)\big\}\,\mathrm{Leb}(B_{n-1}) = (1 - \tfrac{1}{2}a_1)\,\mathrm{Leb}(B_{n-1}) < \mathrm{Leb}(B_{n-1}).$$

By Corollary 2.7(d),

$$\mathrm{Leb}(A_n) > (\tfrac{1}{2} + a_1\tfrac{1}{2a_1})\,\mathrm{Leb}(A_{n-1}) = \mathrm{Leb}(A_{n-1}).$$

Hence by the induction hypothesis,

$$\mathrm{Leb}(B_n) < \mathrm{Leb}(B_{n-1}) \le a_0\,\mathrm{Leb}(A_{n-1}) < a_0\,\mathrm{Leb}(A_n),$$

establishing the result at time n.

Finally, suppose that $\mathrm{Leb}(B_{n-1}) \le \frac{1}{2a_1}\,\mathrm{Leb}(A_{n-1})$. By Corollary 2.7(a,c),

$$\begin{aligned}\mathrm{Leb}(B_n) &\le \tfrac{1}{4}\,\mathrm{Leb}(A_{n-1}) + \mathrm{Leb}(B_{n-1}) \le (\tfrac{1}{4} + \tfrac{1}{2a_1})\,\mathrm{Leb}(A_{n-1}) \\ &\le (\tfrac{1}{2} + \tfrac{1}{a_1})\,\mathrm{Leb}(A_n) = a_0\,\mathrm{Leb}(A_n),\end{aligned}$$

completing the proof. ∎

Lemma 2.9 *Let $\varepsilon \in (0, \frac{1}{2}\delta_0)$ be small as in Propositions 2.4 and 2.5. There exist $c_1 > 0$ and $N \ge 1$ such that*

$$\mathrm{Leb}\Big(\bigcup_{i=0}^{N}\{R = n + i\}\Big) \ge c_1\,\mathrm{Leb}(A_{n-1}) \quad \textit{for all } n \ge 1.$$

Proof Fix $\lambda \in (0,1)$, $L > 1$, $0 < \delta < \delta_1 < \delta_0$ and $N_1 \geq 1$ as defined from the outset. Recall that $C_3(3\delta)^\alpha < \frac{1}{2}\delta_0$ and $C_4(L+1)\delta < \delta_0$. Choose $N_2 \geq 1$ such that $\lambda^{N_2} < \varepsilon/\delta_0$ and take $N = N_1 + N_2$.

We claim that

(*) For all $z \in \Lambda$, there exists $i \in \{1, \ldots, N_1\}$ such that $\pi(\phi_{i+N_2} W^u_\varepsilon(z) \cap \widetilde{\mathcal{D}}_L) \supset \mathcal{D}_L$.

Fix $z \in \Lambda$. By the definition of N_1, there exists $1 \leq i \leq N_1$ such that $d_M(\phi_{-i}p, \phi_{N_2}z) < \delta$. Let $y \in \mathcal{D}_L$. Then

$$d_M(\phi_{-i}y, \phi_{N_2}z) \leq d(\phi_{-i}y, \phi_{-i}p) + d_M(\phi_{-i}p, \phi_{N_2}z) \leq d(y,p) + d_M(\phi_{-i}p, \phi_{N_2}z) < (L+1)\delta.$$

Using the local product structure and choice of δ, we can define $x \in W^{cs}_{\delta_1}(\phi_{-i}y) \cap W^u_{\delta_1}(\phi_{N_2}z)$. Then $\phi_i x \in W^{cs}_{\delta_1}(y) \subset \widetilde{\mathcal{D}}_L$ and $g_i x = \pi\phi_i x = y$. Also,

$$d(x, \phi_{N_2}z) \leq C_4 d_M(\phi_{-i}y, \phi_{N_2}z) < C_4(L+1)\delta < \delta_0.$$

By the definition of N_2,

$$\phi_i x \in \phi_i W^u_{\delta_0}(\phi_{N_2}z) \subset \phi_{i+N_2} W^u_\varepsilon(z).$$

Hence we obtain that $y = \pi\phi_i x \in \pi(\phi_{i+N_2} W^u_\varepsilon(z) \cap \widehat{\mathcal{D}}_L)$ proving (*).

Next, we claim that

(**) For all $z \in \phi_n A_{n-1}$, $n \geq 1$, there exist $i \in \{0, \ldots, N\}$ and j such that $U^1_{n+i,j} \subset \phi_{-n} W^u_{\delta_0}(z)$.

To prove (**), define $V_\varepsilon = \phi_{-n} W^u_\varepsilon(z)$. By Proposition 2.5(b), $V_\varepsilon \subset A^{(\varepsilon)}_{n-1}$. We now consider two possible cases.

Suppose first that $V_\varepsilon \subset A_{n+i}$ for all $0 \leq i \leq N$. By claim (*), there exists $1 \leq i \leq N = N_1 + N_2$ such that

$$\pi(\phi_{n+i} V_\varepsilon \cap \widetilde{\mathcal{D}}_L) = \pi(\phi_i W^u_\varepsilon(z) \cap \widetilde{\mathcal{D}}_L) \supset \mathcal{D}_L,$$

while $V_\varepsilon \subset A_{n+i-1}$ by assumption. This means that $V_\varepsilon \supset U^L_{n+i,j}$ for some j. Hence

$$U^1_{n+i,j} \subset U^L_{n+i,j} \subset V_\varepsilon \subset \phi_{-n} W^u_{\delta_0}(z),$$

and we are done.

In this way, we reduce to the second case where there exists $0 \leq i \leq N$ least such that $V_\varepsilon \not\subset A_{n+i}$. Since i is least, $V_\varepsilon \subset A^{(\varepsilon)}_{n+i-1}$. (The ε is required in case $i = 0$.) By Proposition 2.5(a), $V_\varepsilon \subset \{t_{n+i-1} \leq 1\}$. Hence

$$\begin{aligned} V_\varepsilon \setminus A_{n+i} &= (V_\varepsilon \cap B_{n+i}) \cup (V_\varepsilon \cap \{R = n+i\}) \\ &\subset \{t_{n+i-1} \leq 1,\, t_{n+i} \geq 1\} \cup \{R = n+i\} \subset \bigcup_j U^2_{n+i,j}. \end{aligned}$$

Since $V_\varepsilon \setminus A_{n+i} \neq \emptyset$, this means that there exists j so that V_ε intersects $U^2_{n+i,j}$. Hence we can choose $a_2 \in W^u_\varepsilon(z) \cap \phi_n U^2_{n+i,j}$.

Recall that $\phi_{n+i} U^m_{n+i,j} \subset \widetilde{\mathcal{D}}_m$ and $g_{n+i} U^m_{n+i,j} = \mathcal{D}_m$ for $m = 1, 2$. In particular, $b_2 = \phi_i a_2 \in \widetilde{\mathcal{D}}_2$ and $c_2 = g_i a_2 \in \mathcal{D}_2$.

Let $c_1 \in \mathcal{D}_1$. Then $d_M(c_1, b_2) \le d_M(c_1, c_2) + d_M(c_2, b_2) < 3\delta + \delta_1 < 4\delta_1$. Hence, using the local product structure and definition of δ_1, we can define $b_1 \in W^{cs}_{\delta_0}(c_1) \cap W^u_{\delta_0}(b_2)$ and $a_1 = \phi_{-i} b_1$. Note that

$$\phi_i a_r = b_r, \qquad \pi b_r = c_r, \quad r = 1, 2.$$

Hence

$$d(a_1, a_2) \le d(b_1, b_2) \le C_3 d(c_1, c_2)^\alpha < C_3 (3\delta)^\alpha < \tfrac{1}{2}\delta_0,$$

and so $d(a_1, z) \le d(a_1, a_2) + d(a_2, z) < \frac{1}{2}\delta_0 + \varepsilon < \delta_0$. It follows that $a_1 \in W^u_{\delta_0}(z)$ and thereby that $c_1 \in g_i(W^u_{\delta_0}(z) \cap \phi_{-i}\widehat{\mathcal{D}}_1)$. This proves that $\mathcal{D}_1 \subset g_i(W^u_{\delta_0}(z) \cap \phi_{-i}\widehat{\mathcal{D}}_1)$. Hence $U^1_{n+i,j} \subset g_{-(n+i)}\mathcal{D}_1 \subset \phi_{-n} W^u_{\delta_0}(z)$ verifying claim (**).

We are now in a position to complete the proof of the lemma. Let $n \ge 1$, and let $Z \subset \phi_n A_{n-1}$ be a maximal set of points such that the balls $W^u_{\delta_0/2}(z)$ are disjoint for $z \in Z$. If $x \in \phi_n A_{n-1}$, then $W^u_{\delta_0/2}(x)$ intersects at least one $W^u_{\delta_0/2}(z)$, $z \in Z$, by maximality of the set Z. Hence $\phi_n A_{n-1} \subset \bigcup_{z\in Z} W^u_{\delta_0}(z)$. It follows that

$$A_{n-1} \subset \bigcup_{z \in Z} \phi_{-n} W^u_{\delta_0}(z).$$

Let $z \in Z$ and let $U_z = U^1_{n+i,j}$ be as in claim (**). In particular, $g_{n+i} U_z = \mathcal{D}_1 = W^u_\delta(p)$. Also, $\operatorname{Leb}(\phi_{n+i} U_z) \le |D\phi_1|^{im}_\infty \operatorname{Leb}(\phi_n U_z)$ where $m = \dim E^u$. Hence, by (2.2),

$$\frac{1}{\operatorname{Leb}(\phi_n U_z)} \le |D\phi_1|^{Nm}_\infty \frac{1}{\operatorname{Leb}(\phi_{n+i} U_z)} \le C_3 |D\phi_1|^{Nm}_\infty \frac{1}{\operatorname{Leb}(W^u_\delta(p))}.$$

By (2.1),

$$\frac{\operatorname{Leb}(\phi_{-n} W^u_{\delta_0}(z))}{\operatorname{Leb}(U_z)} \le C_1 \frac{\operatorname{Leb}(W^u_{\delta_0}(z))}{\operatorname{Leb}(\phi_n U_z)} \le K,$$

where $K = C_1 C_3 |D\phi_1|^{Nm}_\infty \dfrac{\sup_{y\in Y} \operatorname{Leb}(W^u_{\delta_0}(y))}{\operatorname{Leb}(W^u_\delta(p))}$.

Finally, the sets U_z are connected components of $\bigcup_{0\le i\le N}\{R = n+i\}$ lying in distinct disjoint sets $\phi_{-n} W^u_{\delta_0}(z)$. Hence

$$\operatorname{Leb}(A_{n-1}) \le \sum_{z\in Z} \operatorname{Leb}(\phi_{-n} W^u_{\delta_0}(z)) \le K \sum_{z \in Z} \operatorname{Leb}(U_z) \le K \operatorname{Leb}\Big(\bigcup_{0\le i\le N} \{R = n+i\}\Big),$$

as required. ∎

We can now complete the proof of Theorem 2.1.

Corollary 2.10 $\mathrm{Leb}(R > n) = O(\gamma^n)$ *for some* $\gamma \in (0, 1)$.

Proof By Corollary 2.8 and Lemma 2.9,

$$\begin{aligned}\mathrm{Leb}(R \geq n) &= \mathrm{Leb}(A_{n-1}) + \mathrm{Leb}(B_{n-1}) \\ &\leq (1 + a_0)\,\mathrm{Leb}(A_{n-1}) \leq d_2\,\mathrm{Leb}\Big(\bigcup_{i=0}^{N}\{R = n + i\}\Big)\end{aligned}$$

where $d_2 = c_1^{-1}(1 + a_0)$. It follows that

$$\begin{aligned}d_2^{-1}\,\mathrm{Leb}(R \geq n) &\leq \mathrm{Leb}(R = n) + \cdots + \mathrm{Leb}(R = n + N) \\ &= \mathrm{Leb}(R \geq n) - \mathrm{Leb}(R > n + N).\end{aligned}$$

Hence

$$\mathrm{Leb}(R > n + N) \leq (1 - d_2^{-1})\,\mathrm{Leb}(R \geq n).$$

In particular, $\mathrm{Leb}(R > kN) \leq \gamma^{kN}$ with $\gamma = (1 - d_2^{-1})^{1/N}$ and the result follows. ∎

3 Exponential decay of correlations for flows

In this section, we consider exponential decay of correlations for a class of uniformly hyperbolic skew product flows satisfying a uniform nonintegrability condition, generalising from C^2 flows as treated in [8] to $C^{1+\alpha}$ flows. In doing so, we remove the restriction in [9, 6] that unstable manifolds are one-dimensional.

The arguments are a straightforward combination of those in [6, 8]. We follow closely the presentation in [6], with the focus on incorporating the ideas from [8] where required.

Quotienting by stable leaves leads to a class of semiflows considered in Subsection 3.1. The flows are considered in Subsection 3.2.

The current section is completely independent from Section 2, so overlaps in notation will not cause any confusion.

3.1 $C^{1+\alpha}$ uniformly expanding semiflows

Fix $\alpha \in (0, 1)$. Let $Y \subset \mathbb{R}^m$ be an open ball[2] in Euclidean space with Euclidean distance d. We suppose that $\operatorname{diam} Y = 1$. Let Leb denote Lebesgue measure on Y. Let $\mathcal{P}$ be a countable partition mod 0 of Y consisting of open sets.

Suppose that $F : \bigcup_{U\in\mathcal{P}} U \to Y$ is $C^{1+\alpha}$ on each $U \in \mathcal{P}$ and maps U diffeomorphically onto Y. Let $\mathcal{H} = \{h : U \to Y : U \in \mathcal{P}\}$ denote the family of inverse branches, and let $\mathcal{H}_n$ denote the inverse branches for F^n. We say that F is a $C^{1+\alpha}$ *uniformly expanding map* if there exist constants $C_1 \geq 1$, $\rho_0 \in (0, 1)$ such that

[2]More generally, we could consider a John domain as in [8] but the current setting suffices for our purposes.

(i) $|Dh|_\infty \le C_1\rho_0^n$ for all $h \in \mathcal{H}_n$, $n \ge 1$;

(ii) $|\log|\det Dh||_\alpha \le C_1$ for all $h \in \mathcal{H}$;

where $|\psi|_\alpha = \sup_{y\neq y'} |\psi(y) - \psi(y')|/d(y,y')^\alpha$. Under these assumptions, it is standard [1] that there exists a unique F-invariant absolutely continuous measure μ. The density $d\mu/d\,\mathrm{Leb}$ is C^α, bounded above and below, and μ is ergodic and mixing.

We consider roof functions $r : \bigcup_{U\in\mathcal{P}} U \to \mathbb{R}^+$ that are C^1 on partition elements U with $\inf r > 0$. Define the suspension $Y^r = \{(y,u) \in Y \times \mathbb{R} : 0 \le u \le r(y)\}/\sim$ where $(y, r(y)) \sim (Fy, 0)$. The suspension semiflow $F_t : Y^r \to Y^r$ is given by $F_t(y,u) = (y, u+t)$ computed modulo identifications, with ergodic invariant probability measure $\mu^r = (\mu \times \text{Lebesgue})/\bar{r}$ where $\bar{r} = \int_Y r\,d\mu$. We say that F_t is a $C^{1+\alpha}$ *uniformly expanding semiflow* if F is a $C^{1+\alpha}$ uniformly expanding map and we can choose C_1 from condition (i) and $\varepsilon > 0$ such that

(iii) $|D(r \circ h)|_\infty \le C_1$ for all $h \in \mathcal{H}$;

(iv) $\sum_{h\in\mathcal{H}} e^{\varepsilon|r\circ h|_\infty} |\det Dh|_\infty < \infty$.

Let $r_n = \sum_{j=0}^{n-1} r \circ F^j$ and define

$$\psi_{h_1,h_2} = r_n \circ h_1 - r_n \circ h_2 : Y \to \mathbb{R},$$

for $h_1, h_2 \in \mathcal{H}_n$. We require the following *uniform nonintegrability* condition [8, Equation (6.6)]:

(UNI) There exists $E > 0$ and $h_1, h_2 \in \mathcal{H}_{n_0}$, for some sufficiently large $n_0 \ge 1$, with the following property: There exists a continuous unit vector field $\ell : \mathbb{R}^m \to \mathbb{R}^m$ such that $|D\psi_{h_1,h_2}(y) \cdot \ell(y)| \ge E$ for all $y \in Y$.

(The requirement "sufficiently large" can be made explicit as in [6, Equations (2.1) to (2.3)].) From now on, n_0, h_1 and h_2 are fixed.

Define $F_\alpha(Y^r)$ to consist of L^∞ functions $v : Y^r \to \mathbb{R}$ such that $\|v\|_\alpha = |v|_\infty + |v|_\alpha < \infty$ where

$$|v|_\alpha = \sup_{(y,u)\neq(y',u)} \frac{|v(y,u) - v(y',u)|}{d(y,y')^\alpha}.$$

Define $F_{\alpha,k}(Y^r)$ to consist of functions with $\|v\|_{\alpha,k} = \sum_{j=0}^k \|\partial_t^j v\|_\alpha < \infty$ where ∂_t denotes differentiation along the semiflow direction.

We can now state the main result in this section. Given $v \in L^1(Y^r)$, $w \in L^\infty(Y^r)$, define the correlation function

$$\rho_{v,w}(t) = \int v\, w \circ F_t\, d\mu^r - \int v\, d\mu^r \int w\, d\mu^r.$$

Theorem 3.1 *Suppose that $F_t : Y^r \to Y^r$ is a $C^{1+\alpha}$ uniformly expanding semiflow satisfying (UNI). Then there exist constants $c, C > 0$ such that*

$$|\rho_{v,w}(t)| \le Ce^{-ct}\|v\|_{\alpha,1}\|w\|_{\alpha,1},$$

for all $t > 0$ and all v, $w \in F_{\alpha,1}(Y^r)$ (alternatively all $v \in F_{\alpha,2}(Y^r)$, $w \in L^\infty(Y^r)$).

In the remainder of this subsection, we prove Theorem 3.1.

For $s \in \mathbb{C}$, let P_s denote the (non-normalised) transfer operator

$$P_s = \sum_{h\in\mathcal{H}} A_{s,h}, \qquad A_{s,h}v = e^{-sr\circ h}|\det Dh|\, v\circ h.$$

For $v : Y \to \mathbb{C}$, define $\|v\|_\alpha = \max\{|v|_\infty, |v|_\alpha\}$ where $|v|_\alpha = \sup_{y\neq y'}|v(y) - v(y')|/d(y,y')^\alpha$. Let $C^\alpha(Y)$ denote the space of functions $v : Y \to \mathbb{C}$ with $\|v\|_\alpha < \infty$. We introduce the family of equivalent norms

$$\|v\|_b = \max\{|v|_\infty, |v|_\alpha/(1 + |b|^\alpha)\}, \quad b \in \mathbb{R}.$$

Proposition 3.2 *Write $s = \sigma + ib$. There exists $\varepsilon \in (0, 1)$ such that the family $s \mapsto P_s$ of operators on $C^\alpha(Y)$ is continuous on $\{\sigma > -\varepsilon\}$. Moreover, $\sup_{|\sigma|<\varepsilon}\|P_s\|_b < \infty$.*

Proof The first five lines of the proof of [6, Proposition 2.5] should be changed to the following:

Using the inequality $1 - t \le -\log t$ valid for $t > 0$, we obtain for $a > b > 0$ that $a - b = a(1 - \frac{b}{a}) \le -a\log\frac{b}{a} = a(\log a - \log b)$. Hence $\big||\det Dh(x)| - |\det Dh(y)|\big| \le |\det Dh|_\infty\big(\log|\det Dh(x)| - \log|\det Dh(y)|\big)$ and so by (ii),

$$\big||\det Dh(x)| - |\det Dh(y)|\big| \le C_1|\det Dh|_\infty\, d(x,y)^\alpha \quad \text{for all } h \in \mathcal{H},\ x,y \in Y. \tag{3.1}$$

The proof now proceeds exactly as for [6, Proposition 2.5] (with R, h' and $|x - y|$ changed to r, $\det Dh$ and $d(x,y)$). ∎

The unperturbed operator P_0 has a simple leading eigenvalue $\lambda_0 = 1$ with strictly positive C^α eigenfunction f_0. By Proposition 3.2, there exists $\varepsilon \in (0, 1)$ such that P_σ has a continuous family of simple eigenvalues λ_σ for $|\sigma| < \varepsilon$ with associated C^α eigenfunctions f_σ. For $s = \sigma + ib$ with $|\sigma| \le \varepsilon$, we define the normalised transfer operators

$$L_s v = (\lambda_\sigma f_\sigma)^{-1}P_s(f_\sigma v) = (\lambda_\sigma f_\sigma)^{-1}\sum_{h\in\mathcal{H}} A_{s,h}(f_\sigma v).$$

In particular, $L_\sigma 1 = 1$ and $|L_s|_\infty \le 1$.

Set $C_2 = C_1^2/(1-\rho)$, $\rho = \rho_0^\alpha$. Then

(ii$_1$) $|\log|\det Dh||_\alpha \le C_2$ for all $h \in \mathcal{H}_n$, $n \ge 1$,

(iii$_1$) $|D(r_n \circ h)|_\infty \le C_2$ for all $h \in \mathcal{H}_n$, $n \ge 1$.

Write

$$L_s^n v = \lambda_\sigma^{-n} f_\sigma^{-1} \sum_{h\in\mathcal{H}_n} A_{s,h,n}(f_\sigma v), \quad A_{s,h,n}v = e^{-sr_n\circ h}|\det Dh| v\circ h.$$

Lemma 3.3 (Lasota-Yorke inequality) *There is a constant $C_3 > 1$ such that*

$$|L_s^n v|_\alpha \le C_3(1+|b|^\alpha)|v|_\infty + C_3\rho^n|v|_\alpha \le C_3(1+|b|^\alpha)\{|v|_\infty + \rho^n\|v\|_b\},$$

for all $s=\sigma+ib$, $|\sigma|<\varepsilon$, and all $n\ge 1$, $v\in C^\alpha(Y)$.

Proof It follows from (ii_1) that

$$\big||\det Dh(x)| - |\det Dh(y)|\big| \le C_2|\det Dh|_\infty\, d(x,y)^\alpha \le C_2 e^{C_2}|\det Dh(z)|\, d(x,y)^\alpha$$

for all $h\in\mathcal{H}_n$, $n\ge 1$, $x,y,z\in Y$. The proof now proceeds exactly as for [6, Lemma 2.7]. ∎

Corollary 3.4 $\|L_s^n\|_b \le 2C_3$ *for all $s=\sigma+ib$, $|\sigma|<\varepsilon$, and all $n\ge 1$.*

Proof This is unchanged from [6, Corollary 2.8]. ∎

Given $b\in\mathbb{R}$, we define the cone

$$\mathcal{C}_b = \Big\{ (u,v) : u,v\in C^\alpha(Y),\ u>0,\ 0\le |v|\le u,\ |\log u|_\alpha \le C_4|b|^\alpha, \\ |v(x)-v(y)| \le C_4|b|^\alpha u(y) d(x,y)^\alpha \quad \text{for all } x,y\in Y \Big\}.$$

Throughout $B_\delta(y) = \{x\in\mathbb{R}^m : d(x,y)<\delta\}$.

Lemma 3.5 (Cancellation Lemma) *Assume that the (UNI) condition is satisfied (with associated constants $E>0$ and $n_0\ge 1$). Let $h_1,h_2\in\mathcal{H}_{n_0}$ be the branches from (UNI).*

There exists $0<\delta<\Delta=4\pi/E$ such that for all $s=\sigma+ib$, $|\sigma|<\varepsilon$, $|b|\ge 1$, and all $(u,v)\in\mathcal{C}_b$ we have the following:

For every $y'\in Y$ with $B_{(\delta+\Delta)/|b|}(y')\subset Y$, there exists $y''\in B_{\Delta/|b|}(y')$ such that one of the following inequalities holds on $B_{\delta/|b|}(y'')$:

Case h_1: $|A_{s,h_1,n_0}(f_\sigma v) + A_{s,h_2,n_0}(f_\sigma v)| \le \frac{3}{4}A_{\sigma,h_1,n_0}(f_\sigma u) + A_{\sigma,h_2,n_0}(f_\sigma u)$,

Case h_2: $|A_{s,h_1,n_0}(f_\sigma v) + A_{s,h_2,n_0}(f_\sigma v)| \le A_{\sigma,h_1,n_0}(f_\sigma u) + \frac{3}{4}A_{\sigma,h_2,n_0}(f_\sigma u)$.

Proof Let $\theta = V - b\psi_{h_1,h_2}$ where $\psi_{h_1,h_2} = r_{n_0}\circ h_1 - r_{n_0}\circ h_2$ and $V=\arg(v\circ h_1) - \arg(v\circ h_2)$.

We follow the following steps from [6, Lemma 2.9]:

(1) Reduce to the situation where $|v(h_m y')| > \frac{1}{2}u(h_m y')$ for both $m = 1$ and $m = 2$.
(2) Establish the estimate $|V(y) - V(y')| \le \pi/6$ for all $y \in B_{(\delta+\Delta)/|b|}(y')$.
(3) Construct $y'' \in B_{\Delta/|b|}(y')$ such that

$$b(\psi_{h_1,h_2}(y'') - \psi_{h_1,h_2}(y')) = \theta(y') - \pi \mod 2\pi.$$

(4) Deduce that $|\theta(y) - \pi| \le 2\pi/3$ for all $y \in B_{(\delta+\Delta)/|b|}(y')$.
(5) Conclude the desired result.

Only step (3) requires any change from the argument in [6, Lemma 2.9]. We provide here the modified argument. Approximate the continuous unit vector field $\ell : \mathbb{R}^m \to \mathbb{R}^m$ in (UNI) by a smooth vector field $\ell : \mathbb{R}^m \to \mathbb{R}^m$ with $|\ell(x)| \le 1$ for all $x \in \mathbb{R}^m$. By condition (iii$_1$), the approximation can be chosen close enough that

$$|D\psi_{h_1,h_2}(y) \cdot \ell(y)| \ge \tfrac{1}{2}E \quad \text{for all } y \in Y. \tag{3.2}$$

Let $g : [0, \Delta/|b|] \to \mathbb{R}^m$ be the solution to the initial value problem

$$\tfrac{dg}{dt} = \ell \circ g, \quad g(0) = y'$$

and set $y_t = g(t)$. Note that $d(y_t, y') \le \int_0^t |\ell(g(s))|\,ds \le \Delta/|b|$, so $y_t \in B_{\Delta/|b|}(y')$ for all $t \in [0, \Delta/|b|]$. By the mean value theorem applied to $\psi_{h_1,h_2} \circ g : [0, \Delta/|b|] \to \mathbb{R}$ and (3.2),

$$|\psi_{h_1,h_2}(y_t) - \psi_{h_1,h_2}(y')| \ge t \inf_{s \in [0,\Delta/|b|]} |D\psi_{h_1,h_2}(y_s) \cdot \ell(y_s)| \ge \tfrac{1}{2}Et = (2\pi/\Delta)t$$

for all $t \in [0, \Delta/|b|]$. It follows that $b(\psi_{h_1,h_2}(y_t) - \psi_{h_1,h_2}(y'))$ fills out an interval around 0 of length at least 2π as t varies in $[0, \Delta/|b|]$. In particular, we can choose $y'' \in B_{\Delta/|b|}(y')$ such that (3) holds. ∎

Let $\{y'_1, \dots, y'_k\} \subset Y$ be a maximal set of points such that the open balls $B_{(\delta+\Delta)/|b|}(y'_i)$ are disjoint and contained in Y.

Let $(u, v) \in \mathcal{C}_b$. For each $i = 1, \dots, k$, there exists a ball $B_i = B_{\delta/|b|}(y''_i)$ on which the conclusion of Lemma 3.5 holds. Write $\mathrm{type}(B_i) = h_m$ if we are in case h_m. Let $\widehat{B}_i = B_{\frac{1}{2}\delta/|b|}(y''_i)$

There exists a universal constant $C > 0$ and a C^1 function $\omega_i : Y \to [0, 1]$ such that $\omega_i \equiv 1$ on $\widehat{B}_i$, $\omega_i \equiv 0$ on $Y \setminus B_i$, and $\|\omega_i\|_{C^1} \le C|b|/\delta$. Define $\omega : Y \to [0, 1]$,

$$\omega(y) = \begin{cases} \sum_{\mathrm{type}(B_i)=h_m} \omega_i(F^{n_0} y), & y \in \mathrm{range}\, h_m,\ m = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\|\omega\|_{C^1} \le C'|b|$ where $C' = C\delta$ is independent of $(u, v) \in \mathcal{C}_b$ and $s \in \mathbb{C}$, and we can assume that $C' > 4$. Then $\chi = 1 - \omega/C' : Y \to [\frac{3}{4}, 1]$ satisfies $|D\chi| \le |b|$. Moreover, if $\mathrm{type}(B_i) = h_m$ then $\chi \equiv \eta$ on $\mathrm{range}\, h_m$ where $\eta = 1 - 1/C' \in (0, 1)$.

Corollary 3.6 *Let δ, Δ be as in Lemma 3.5. Let $|b| \geq 1$, $(u, v) \in \mathcal{C}_b$. Let $\chi = \chi(b, u, v)$ be the C^1 function described above (using the branches $h_1, h_2 \in \mathcal{H}_{n_0}$ from (UNI)). Then $|L_s^{n_0} v| \leq L_\sigma^{n_0}(\chi u)$ for all $s = \sigma + ib$, $|\sigma| < \varepsilon$.*

Proof This is immediate from Lemma 3.5 and the definition of χ. ∎

Define the disjoint union $\widehat{B} = \bigcup \widehat{B}_i$.

Proposition 3.7 *Let $K > 0$. There exists $c_1 > 0$ such that $\int_{\widehat{B}} w \, d\mu \geq c_1 \int_Y w \, d\mu$ for all C^α function $w : Y \to (0, \infty)$ with $|\log w|_\alpha \leq K|b|^\alpha$, for all $|b| \geq 16\pi/E$.*

Proof Let $y \in Y$. Since $(\delta + \Delta)/|b| \leq 2\Delta/|b| = 8\pi/(E|b|) \leq \frac{1}{2}$, there exists $z \in Y$ with $B_{(\delta+\Delta)/|b|}(z) \subset Y$ such that $d(z, y) < (\delta + \Delta)/|b|$. By maximality of the set of points $\{y_1', \ldots, y_k'\}$, there exists y_i' such that $B_{(\delta+\Delta)/|b|}(z)$ intersects $B_{(\delta+\Delta)/|b|}(y_i')$. Hence $Y \subset \bigcup_{i=1}^k B_i^*$ where $B_i^* = B_{3(\delta+\Delta)/|b|}(y_i')$. Since the density $d\mu/d\,\mathrm{Leb}$ is bounded above and below, there is a constant $c_2 > 0$ such that $\mu(\widehat{B}_i) \geq c_2 \mu(B_i^*)$ for each i.

Let $x \in \widehat{B}_i$, $y \in B_i^*$. Then $d(x, y) \leq 4(\delta + \Delta)/|b|$ and so $|w(x)/w(y)| \leq e^{K'}$ where $K' = \{4(\delta + \Delta)\}^\alpha K$. It follows that

$$\int_{\widehat{B}_i} w \, d\mu \geq \mu(\widehat{B}_i) \inf_{\widehat{B}_i} w \geq c_2 e^{-K'} \mu(B_i^*) \sup_{B_i^*} w \geq c_1 \int_{B_i^*} w \, d\mu,$$

where $c_1 = c_2 e^{-K'}$. Since the sets $\widehat{B}_i \subset Y$ are disjoint,

$$\int_{\widehat{B}} w \, d\mu = \sum_i \int_{\widehat{B}_i} w \, d\mu \geq c_1 \sum_i \int_{B_i^*} w \, d\mu \geq c_1 \int_Y w \, d\mu$$

as required. ∎

Lemma 3.8 (Invariance of cone) *There is a constant C_4 depending only on C_1, C_2, $|f_0^{-1}|_\infty$ and $|f_0|_\alpha$ such that the following holds:*

For all $(u, v) \in \mathcal{C}_b$, we have that

$$\big(L_\sigma^{n_0}(\chi u) \, , \, L_s^{n_0} v \big) \in \mathcal{C}_b,$$

for all $s = \sigma + ib$, $|\sigma| < \varepsilon$, $|b| \geq 1$. (Here, $\chi = \chi(b, u, v)$ is from Corollary 3.6.)

Proof This is unchanged from [6, Lemma 2.12]. ∎

Lemma 3.9 (L^2 contraction) *There exist $\varepsilon, \beta \in (0, 1)$ such that*

$$\int_Y |L_s^{mn_0} v|^2 \, d\mu \leq \beta^m |v|_\infty^2$$

for all $m \geq 1$, $s = \sigma + ib$, $|\sigma| < \varepsilon$, $|b| \geq \max\{16\pi/E, 1\}$, and all $v \in C^\alpha(Y)$ satisfying $|v|_\alpha \leq C_4|b|^\alpha |v|_\infty$.

Proof Define $u_0 \equiv 1, v_0 = v/|v|_\infty$ and inductively,

$$u_{m+1} = L_\sigma^{n_0}(\chi_m u_m), \qquad v_{m+1} = L_s^{n_0}(v_m),$$

where $\chi_m = \chi(b, u_m, v_m)$. It is immediate from the definitions that $(u_0, v_0) \in \mathcal{C}_b$, and it follows from Lemma 3.8 that $(u_m, v_m) \in \mathcal{C}_b$ for all m. Hence inductively the χ_m are well-defined as in Corollary 3.6.

We proceed as in [6, Lemma 2.13] in the following steps.

(1) It suffices to show that there exists $\beta \in (0,1)$ such that $\int_Y u_{m+1}^2 \, d\mu \le \beta \int u_m^2 \, d\mu$ for all m.

(2) Define $w = L_0^{n_0}(u_m^2)$. Then

$$u_{m+1}^2(y) \le \begin{cases} \xi(\sigma)\eta_1 w(y) & y \in \widehat{B} \\ \xi(\sigma) w(y) & y \in Y \setminus \widehat{B} \end{cases}$$

where $\xi(\sigma)$ can be made as close to 1 as desired by shrinking ε. Here, $\eta_1 \in (0,1)$ is a constant independent of v, m, s, y.

(3) The function $w : Y \to \mathbb{R}$ satisfies the hypotheses of Proposition 3.7; consequently $\int_{\widehat{B}} w \, d\mu \ge c_1 \int_{Y \setminus \widehat{B}} w \, d\mu$. This leads to the desired conclusion. ∎

Lemma 3.10 (C^α contraction) *Let $E' = \max\{16\pi/E, 2\}$. There exists $\varepsilon \in (0,1)$, $\gamma \in (0,1)$ and $A > 0$ such that $\|P_s^n\|_b \le \gamma^n$ for all $s = \sigma + ib$, $|\sigma| < \varepsilon$, $|b| \ge E'$, $n \ge A \log |b|$.*

Proof This is unchanged from [6, Proposition 2.14, Corollary 2.15 and Theorem 2.16]. ∎

Proof of Theorem 3.1 This is identical to [6, Section 2.7]. We note that there is a typo in the statement of [6, Lemma 2.23] where $|b| \le D'$ should be $|b| \ge D'$ (twice). Also, for the second statement of [6, Proposition 2.18] it would be more natural to argue that

$$\begin{aligned} \int_Y e^{\varepsilon r} \, d\,\mathrm{Leb} &= \sum_{h \in \mathcal{H}} \int_{h(Y)} e^{\varepsilon r} \, d\,\mathrm{Leb} \\ &= \sum_{h \in \mathcal{H}} \int_Y e^{\varepsilon r \circ h} |\det Dh| \, d\,\mathrm{Leb} \le \mathrm{Leb}(Y) \sum_{h \in \mathcal{H}} e^{\varepsilon |r \circ h|_\infty} |\det Dh|_\infty \end{aligned}$$

which is finite by condition (iv). Hence $\int_Y e^{\varepsilon r} \, d\mu < \infty$ by boundedness of $d\mu/d\,\mathrm{Leb}$. ∎

3.2 $C^{1+\alpha}$ uniformly hyperbolic skew product flows

Let $X = Y \times Z$ where Y is an open ball of diameter 1 with Euclidean metric d_Y and (Z, d_Z) is a compact Riemannian manifold. Define the metric $d((y,z),(y',z')) = d_Y(y,y') + d_Z(z,z')$ on X. Let $f(y,z) = (Fy, G(y,z))$ where $F : Y \to Y$, $G : X \to Z$ are $C^{1+\alpha}$.

We say that $f : X \to X$ is a $C^{1+\alpha}$ *uniformly hyperbolic skew product* if $F : Y \to Y$ is a $C^{1+\alpha}$ uniformly expanding map satisfying conditions (i) and (ii) as in Section 3.1, with absolutely continuous invariant probability measure μ, and moreover

(v) There exist constants $C > 0$, $\gamma_0 \in (0,1)$ such that $d(f^n(y,z), f^n(y,z')) \le C\gamma_0^n d(z,z')$ for all $y \in Y$, $z, z' \in Z$.

Let $\pi^s : X \to Y$ be the projection $\pi^s(y,z) = y$. This defines a semiconjugacy between f and F and there is a unique f-invariant ergodic probability measure μ_X on X such that $\pi^s_*\mu_X = \mu$.

Suppose that $r : \bigcup_{U\in\mathcal{P}} U \to \mathbb{R}^+$ is C^1 on partition elements U with $\inf r > 0$. Define $r : X \to \mathbb{R}^+$ by setting $r(y,z) = r(y)$. Define the suspension $X^r = \{(x,u) \in X \times \mathbb{R} : 0 \le u \le r(x)\}/\sim$ where $(x, r(x)) \sim (fx, 0)$. The suspension flow $f_t : X^r \to X^r$ is given by $f_t(x,u) = (x, u+t)$ computed modulo identifications, with ergodic invariant probability measure $\mu_X^r = (\mu_X \times \text{Lebesgue})/\bar{r}$.

We say that f_t is a $C^{1+\alpha}$ *uniformly hyperbolic skew product flow* provided $f : X \to X$ is a $C^{1+\alpha}$ uniformly hyperbolic skew product as above, and $r : Y \to \mathbb{R}^+$ satisfies conditions (iii) and (iv) as in Section 3.1. If $F : Y \to Y$ and $r : Y \to \mathbb{R}^+$ satisfy condition (UNI) from Section 3.1, then we say that the skew product flow f_t satisfies (UNI).

Define $F_\alpha(X^r)$ to consist of L^∞ functions $v : X^r \to \mathbb{R}$ such that $\|v\|_\alpha = |v|_\infty + |v|_\alpha < \infty$ where

$$|v|_\alpha = \sup_{(y,z,u)\neq(y',z',u)} \frac{|v(y,z,u) - v(y',z',u)|}{d((y,z),(y',z'))^\alpha}.$$

Define $F_{\alpha,k}(X^r)$ to consist of functions with $\|v\|_{\alpha,k} = \sum_{j=0}^k \|\partial_t^j v\|_\alpha < \infty$ where ∂_t denotes differentiation along the flow direction.

We can now state the main result in this section. Given $v \in L^1(X^r)$, $w \in L^\infty(X^r)$, define the correlation function

$$\rho_{v,w}(t) = \int v\, w \circ f_t \, d\mu_X^r - \int v \, d\mu_X^r \int w \, d\mu_X^r.$$

Theorem 3.11 *Assume that $f_t : X \to X$ is a $C^{1+\alpha}$ hyperbolic skew product flow satisfying the (UNI) condition. Then there exist constants $c, C > 0$ such that*

$$|\rho_{v,w}(t)| \le Ce^{-ct}\|v\|_{\alpha,1}\|w\|_{\alpha,1},$$

for all $t > 0$ and all $v, w \in F_{\alpha,1}(X^r)$ (alternatively all $v \in F_{\alpha,2}(X^r)$, $w \in F_\alpha(X^r)$).

Proof This is unchanged from [6, Section 4]. ∎

4 Proof of Theorem 1.2

We return to the situation of Section 2, so $\Lambda \subset M$ is a uniformly hyperbolic attractor for a $C^{1+\alpha}$ flow, $\alpha \in (0,1)$, defined on a compact Riemannian manifold. Define the open unstable disk $Y = W^u_\delta(p)$ with discrete return time $R : Y \to \mathbb{Z}^+$ and induced map $F = \pi \circ \phi_R : Y \to Y$ as in Theorem 2.1.

Under smoothness assumptions on holonomies, we verify the conditions on the suspension flow f_t in Section 3 and obtain Theorem 1.2 as an easy consequence.

Proposition 4.1 *Suppose that the centre-stable holonomies are $C^{1+\alpha}$. (In particular, $\pi : \widehat{\mathcal{D}} \to \mathcal{D}$ is $C^{1+\alpha}$.) Then (after shrinking δ_0 in Section 2 if necessary) F is a $C^{1+\alpha}$ uniformly expanding map.*

Proof As in Remark 2.2, it is immediate that $F|_U : U \to Y$ is a $C^{1+\alpha}$ diffeomorphism for all $U \in \mathcal{P}$. Let $h : Y \to U$ be an inverse branch with $R|_U = n$, and define $\pi_U = \pi|_{\phi_n(U)} : \phi_n(U) \to \mathcal{D}$. Then

$$\lambda^{-1}|v| \le \lambda^{-n}|v| \le |D\phi_n(x)v| \le |(D\pi_U)^{-1}|_\infty |DF(x)v|$$

for all $x \in U$, $v \in T_xY$. Hence $|Dh|_\infty \le \rho_0$ where $\rho_0 = \lambda \sup_U |(D\pi_U)^{-1}|_\infty$. Shrinking δ_0, we can ensure that $\rho_0 < 1$. In particular, condition (i) in Section 3.1 holds (with $C_1 = 1$). Condition (ii) is the standard distortion estimate. ∎

In the remainder of this section, we suppose moreover that the stable holonomies are $C^{1+\alpha}$. Shrink $\delta_0 \in (0,1)$ as in Proposition 4.1 and shrink $\delta_1 \in (0,\delta_0)$ so that $\phi_t(W^s_{\delta_1}(y)) \subset W^s_{\delta_0}(\phi_t y)$ for all $t > 0$, $y \in \Lambda$. Recall that $\mathcal{D} = W^u_{\delta_0}(p)$ and

$$\widehat{\mathcal{D}} = \bigcup_{y \in \mathcal{D}} W^{cs}_{\delta_0}(y) = \bigcup_{|t|<\delta_0} \phi_t\Big(\bigcup_{y \in \mathcal{D}} W^s_{\delta_0}(y)\Big).$$

The projection $\pi^s : \bigcup_{y\in\mathcal{D}} W^s_{\delta_0}(y) \to \mathcal{D}$ given by $\pi^s|W^s_{\delta_0}(y) \equiv y$ is $C^{1+\alpha}$. Moreover, $\pi = \pi^s \circ \phi_{r_0}$ where $\phi_{r_0} : \widehat{\mathcal{D}} \to \bigcup_{y\in\mathcal{D}} W^s_{\delta_0}(y)$ and $r_0 : \widehat{\mathcal{D}} \to (-\delta_0,\delta_0)$ is $C^{1+\alpha}$. Define $r = R + r_0$ on Y. The choice $\delta_0 < 1$ ensures that $\inf r \ge 1 - \delta_0 > 0$. Define the corresponding semiflow $F_t : Y^r \to Y^r$.

Proposition 4.2 *$F_t : Y^r \to Y^r$ is a $C^{1+\alpha}$ uniformly expanding semiflow.*

Proof By Proposition 4.1, F is a $C^{1+\alpha}$ uniformly expanding map. In particular, conditions (i) and (ii) are satisfied.

Notice that $F = \pi^s \circ \phi_r$ where $r = R + r_0$ is $C^{1+\alpha}$ on partition elements $U \in \mathcal{P}$. Since $Dr = Dr_0$ on partition elements, it is immediate that $\sup_{h\in\mathcal{H}} |D(r \circ h)|_\infty \le |Dr_0|_\infty \sup_{h\in\mathcal{H}} |Dh|_\infty \le \rho_0 |Dr_0|_\infty < \infty$ verifying condition (iii) on r. Recall that $\mathrm{Leb}(R > n) = O(\gamma^n)$ for some $\gamma \in (0,1)$, so we can choose $\varepsilon > 0$ such that $\int_Y e^{\varepsilon R}\, d\,\mathrm{Leb} < \infty$. Condition (ii) ensures that $|\det Dh|_\infty \le$

$(\mathrm{Leb}\, Y)^{-1} e^{C_1}\, \mathrm{Leb}(\mathrm{range}\, h)$ for all $h \in \mathcal{H}$. Hence $\sum_{h\in\mathcal{H}} e^{\varepsilon|r\circ h|_\infty} |\det Dh|_\infty \ll \sum_{h\in\mathcal{H}} e^{\varepsilon|R\circ h|_\infty}\, \mathrm{Leb}(\mathrm{range}\, h) = \int_Y e^{\varepsilon R}\, d\,\mathrm{Leb} < \infty$ verifying condition (iv) on r. ∎

We now make a $C^{1+\alpha}$ change of coordinates so that $\widehat{\mathcal{D}}$ is identified with $\mathcal{D} \times W^s_{\delta_0}(p) \times (-\delta_0, \delta_0)$ where $\{y\} \times W^s_{\delta_0}(p)$ is identified with $W^s_{\delta_0}(y)$ for all $y \in \mathcal{D}$ and $(-\delta_0, \delta_0)$ is the flow direction. Let $X = Y \times Z$ where $Z = W^s_{\delta_0}(p)$ and define $r : X \to (0, \infty)$ by $r(y, z) = r(y)$. Also, define $f = \phi_r : X \to X$ and the corresponding suspension flow $f_t : X^r \to X^r$

Proposition 4.3 *$f_t : X^r \to X^r$ is a $C^{1+\alpha}$ uniformly hyperbolic skew product flow.*

Proof Note that $\pi^s(X) = Y$ and $\pi^s(y, z) = y$. Also, $f(y, z) = (Fy, G(y, z))$ where $G : X \to Z$ is $C^{1+\alpha}$. Since Z corresponds to the exponential contracting stable foliation, condition (v) in Section 3.2 is satisfied. Hence $f : X \to X$ is a $C^{1+\alpha}$ uniformly hyperbolic skew product and the corresponding suspension flow $f_t : X^r \to X^r$ is a $C^{1+\alpha}$ uniformly hyperbolic skew product flow. ∎

Next we recall the standard argument that joint nonintegrability implies (UNI) in the current situation. (Similar arguments are given for instance in [7, Section 3] and [21, Section 5.3].)

Joint nonintegrability is defined in terms of the temporal distortion function. To define this intrinsically (independently of the inducing scheme) we have to introduce the first return time $\tau : X \to \mathbb{R}^+$ and the Poincaré map $g : X \to X$ given by

$$\tau(x) = \inf\{t > 0 : \phi_t(x) \in X\}, \qquad g(x) = \phi_{\tau(x)}(x).$$

Note that τ is constant along stable leaves by the choice of X.

For $x_1, x_2 \in X$, define the local product $[x_1, x_2]$ to be the unique intersection point of $W^u(x_1) \cap W^s(x_2)$. The *temporal distortion function* D is defined to be

$$D(x_1, x_2) = \sum_{j=-\infty}^{\infty} \big\{\tau(g^j x_1) - \tau(g^j[x_1, x_2]) - \tau(g^j[x_2, x_1]) + \tau(g^j x_2)\big\}$$

at points $x_1, x_2 \in X$. The stable and unstable bundles are jointly integrable if and only if $D \equiv 0$.

Lemma 4.4 *Joint nonintegrability of the stable and unstable bundles implies (UNI).*

Proof For points $x, x' \in X$ with $x' \in W^u(x)$, we define

$$D_0(x, x') = \sum_{j=1}^{\infty} \big\{\tau(g^{-j}x) - \tau(g^{-j}x')\big\}.$$

Since τ is constant along stable leaves,

$$\begin{aligned}D(x_1, x_2) &= \sum_{j=1}^{\infty} \big\{\tau(g^{-j}x_1) - \tau(g^{-j}[x_1, x_2]) - \tau(g^{-j}[x_2, x_1]) + \tau(g^{-j}x_2)\big\} \\ &= D_0(x_1, [x_1, x_2]) + D_0(x_2, [x_2, x_1]).\end{aligned}$$

Next, we find a more convenient expression for D_0 in terms of r and f. Note that for any $x \in X$, there exists $N(x) \in \mathbb{Z}^+$ (the number of returns to X up to time $r(x)$) such that

$$r(x) = \sum_{\ell=0}^{N(x)-1} \tau(g^\ell x), \qquad f(x) = g^{N(x)}x.$$

Corresponding to the partition $\mathcal{P}$ of Y, we define the collection $\widetilde{\mathcal{P}} = \{\bar{U} \times \bar{Z} : U \in \mathcal{P}\}$ of closed subsets of X. Suppose that $x, x' \in V_0$, $V_0 \in \widetilde{\mathcal{P}}$, with $x' \in W^u(x)$. The induced map $f : X \to X$ need not be invertible since it is not the first return to X. However, we may construct suitable inverse branches z_j, z_j' of x, x' as follows. Set $z_0 = x$, $z_0' = x'$. Since f is transitive and continuous on closures of partition elements, there exists $V_1 \in \widetilde{\mathcal{P}}$ and $z_1 \in V_1$ such that $fz_1 = z_0$. Since F is full-branch, $f(W^u(z_1) \cap V_1) \supset W^u(z_0)$, so there exists $z_1' \in W^u(z_1) \cap V_1$ such that $fz_1' = z_0'$. Inductively, we obtain $V_n \in \widetilde{\mathcal{P}}$ and $z_j, z_j' \in V_n$ with $z_j' \in W^u(z_j)$ such that $fz_j = z_{j-1}$ and $fz_j' = z_{j-1}'$.

By construction, $z_{j-1} = fz_j = g^{N(z_j)}z_j$. Hence $z_j = g^{-(N(z_1)+\cdots+N(z_j))}x$ and

$$r(z_j) = \sum_{\ell=0}^{N(z_j)-1} \tau(g^\ell g^{-(N(z_1)+\cdots+N(z_j))}x) = \sum_{\ell=N(z_1)+\cdots+N(z_{j-1})+1}^{N(z_1)+\cdots+N(z_j)} \tau(g^{-\ell}x).$$

A similar expression holds for $r(z_j')$. Hence

$$D_0(x, x') = \sum_{j=1}^{\infty} \big\{r(z_j) - r(z_j')\big\}.$$

We are now in a position to complete the proof of the lemma, showing that if (UNI) fails, then $D \equiv 0$. To do this, we make use of [8, Proposition 7.4] (specifically the equivalence of their conditions 1 and 3). Namely, the failure of the (UNI) condition in Section 3.1 means that we can write $r = \xi \circ F - \xi + \zeta$ on Y where $\xi : Y \to \mathbb{R}$ is continuous (even C^1) and ζ is constant on partition elements $U \in \mathcal{P}$. Extending ξ and ζ trivially to $X = Y \times Z$, we obtain that $r = \xi \circ f - \xi + \zeta$ on X where $\xi : X \to \mathbb{R}$ is continuous and constant on stable leaves, and ζ is constant on elements $V \in \widetilde{\mathcal{P}}$. In particular,

$$\sum_{j=1}^{n} r(z_j) = \sum_{j=1}^{n} \big\{\xi(z_{j-1} - \xi(z_j) + \zeta(z_j)\big\} = \xi(x) - \xi(z_n) + \sum_{j=1}^{n} \zeta(z_j).$$

For $x, x' \in V_0$, $V_0 \in \widetilde{\mathcal{P}}$, with $x' \in W^u(x)$, it follows that

$$\sum_{j=1}^{n} \{r(z_j) - r(z'_j)\} = \xi(x) - \xi(x') - \xi(z_n) + \xi(z'_n).$$

Taking the limit as $n \to \infty$, we obtain that $D_0(x, x') = \xi(x) - \xi(x')$. Hence $D(x_1, x_2) = \xi(x_1) - \xi([x_1, x_2]) - \xi([x_2, x_1]) + \xi(x_2)$. Since ξ is constant on stable leaves, $D(x_1, x_2) = 0$ as required. ∎

Proof of Theorem 1.2 By Proposition 4.3 and Lemma 4.4, f_t is a $C^{1+\alpha}$ uniformly hyperbolic flow satisfying (UNI). The result for $C^{1+\alpha}$ observables follows from Theorem 3.11. As in [18], the result follows from a standard interpolation argument (see also [6, Corollary 2.3]). ∎

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