A NOTE ABOUT STABLE TRANSITIVITY OF NONCOMPACT EXTENSIONS OF HYPERBOLIC SYSTEMS

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ABSTRACT. Let $f : X \to X$ be the restriction to a hyperbolic basic set of a smooth diffeomorphism. If G is the special Euclidean group $SE(2)$ we show that in the set of C^2 G-extensions of f there exists an open and dense subset of stably transitive transformations. If $G = K \times \mathbb{R}^n$, where K is a compact connected Lie group, we show that an open and dense set of C^2 G-extensions satisfying a certain separation condition are transitive. The separation condition is necessary.

1. INTRODUCTION

This paper is part of a program attempting to classify the obstructions to (stable) topological transitivity in various classes of skew-product transformations with non-compact fiber, which in turn is an active part of the current surge of activity in the study of partially hyperbolic systems. Recall that if X is a topological space, and $f : X \to X$ a continuous map, then f is said to be *transitive* if it has a dense orbit.

Let M be a smooth manifold endowed with a Riemannian metric. Let $f : M \to M$ be a smooth diffeomorphism and $X \subset M$ a compact and f-invariant subset of M.

We say that $f : X \to X$ is *hyperbolic* if there exists a continuous Tf-invariant splitting $E^s \oplus E^u$ of the tangent bundle $T_X M$ and constants $C > 0, 0 < \lambda < 1$, such that for all $n \geq 0$ and $x \in X$ we have:

$$\|(Df^n)_x v\| \leq C\lambda^n \|v\|, \ v \in E^s$$
$$\|(Df^{-n})_x v\| \leq C\lambda^n \|v\|, \ v \in E^u.$$

We say that X is *maximal and isolated* if there exists an open neighborhood U of X such that every compact f-invariant set of U is contained in X.

The set X is a *basic set* for $f : M \to M$ if:

(1) f is hyperbolic on X;
(2) X is maximal and isolated;
(3) $f : X \to X$ is transitive.

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The basic set is nontrivial if it is not a periodic orbit. Throughout, we restrict attention to basic sets that are nontrivial.

Assume that X is a hyperbolic basic set for $f : X \to X$ and let G be an finite-dimensional connected Lie group. Let $\beta : X \to G$ be a continuous map, called a *cocycle*. Define a skew product, or G-extension,

$$f_\beta : X \times G \to X \times G, \quad f_\beta(x, h) = (f(x), \beta(x)h).$$

The G-extension f_β is called *stably transitive* if β lies in the interior (usually in the Hölder or C^r topology, $r \geq 1$) of the subset of extensions that are topologically transitive. The question of interest here is whether noncompact group extensions of a hyperbolic basic set are typically *stably* topologically transitive.

It was already observed in [12] that β taking values in a proper closed sub-semigroup S of G is an obstruction to (stable) transitivity. For example if $G = \mathbb{R}$ and the image of β is included in $\mathbb{R}^+$, or if $G = SL(2, \mathbb{R})$ and the image of β is included in the set of matrices with positive entries, then f_β cannot be transitive. In [11] it is conjectured that this is essentially the only obstruction.

Several classes of groups for which the conjecture has been verified are: G compact [1, 5]; $SE(n)$, $n \geq 4$ even [10]; Euclidean spaces $\mathbb{R}^n$ [13, 5]. In [11] we prove the existence of a transitive extension for the fiber being any connected Lie group. Moreover [11] contains examples of stably transitive extensions with fiber the symplectic groups $Sp(n, \mathbb{R})$, as well as other noncompact groups.

For direct products $G = K \times \mathbb{R}^n$ with K compact, and for the group $G = SE(2)$, the conjecture in [11] was verified when the base is a hyperbolic attractor. In this paper, we verify the conjecture for these groups when X is a general basic set.

Let $G = K \times \mathbb{R}^n$, where K is a compact connected Lie group. A *half-space* in $\mathbb{R}^n$ is a region bounded by a hyperplane passing through the origin. It is clear that f_β cannot be transitive if the $\mathbb{R}^n$-component of β is cohomologous to a cocycle with values in a half-space in $\mathbb{R}^n$. Hence, we define $\mathcal{S}$ to be the open set of C^r cocycles β for which the $\mathbb{R}^n$-component is *not* cohomologous to a cocycle with values in a half-space.

Theorem 1.1. *Let X be a hyperbolic basic set for $f : X \to X$ and $r \geq 2$. Suppose that $G = K \times \mathbb{R}^n$. Then there is a C^2-open and C^r-dense set of cocycles $\beta \in \mathcal{S}$ for which $f_\beta : X \times G \to X \times G$ is transitive.*

Recall that $SE(n) = SO(n) \ltimes \mathbb{R}^n$ is the group generated by rotations and translations in $\mathbb{R}^n$. The multiplication in $SE(n)$ is given by $(\theta_1, v_1)(\theta_2, v_2) = (\theta_1\theta_2, \theta_1 v_2 + v_1)$. When the fiber is $SE(n)$, no separation condition is necessary for transitivity.

Theorem 1.2. *Let X be a hyperbolic basic set for $f : X \to X$ and $r \geq 2$. Suppose that $G = SE(2)$. Then there is a C^2-open and C^r-dense set of C^r cocycles $\beta : X \to G$ for which $f_\beta : X \times G \to X \times G$ is transitive.*

Remark. As in [5], we also obtain C^r-open and C^r-dense sets of transitive cocycles for all $r > 0$ (with C^1 interpreted as Lipschitz). If X is an attractor, we proved Hölder open and C^r-dense for all $r > 0$ in [11].

The proof relies on a blend of techniques developed to study extensions with non-compact fibers. Some of these techniques are reviewed in Section 2. In Section 3, we prove Theorem 1.1 in the special case $K = C \times \mathbb{T}^d$ where C is semisimple and $\mathbb{T}^d$ is a torus. In Section 4, we complete the proof of Theorem 1.1. Theorem 1.2 is proved in Section 5.

2. Preliminaries

We review several transitivity criteria needed for the proof of the main results.

Assume that X is a hyperbolic basic set for $f : X \to X$, G a Lie group, and $\beta : X \to G$ a Hölder cocycle. Denote $\beta_n(x) = \beta(f^{n-1}x) \dots \beta(fx)\beta(x)$.

As in [13], for abelian G, we associate to each periodic orbit $f^k x = x$ the weight $\beta_k(x)$ and denote by $\mathcal{H}_\beta = \left\{\beta_k(x) : f^k x = x\right\}$ the collection of all weights assigned to the set of periodic orbits of f.

The following proposition is proved in [13, §6].

Proposition 2.1. *If G is abelian and $\mathcal{H}_\beta$ generates a dense semigroup in G, then f_β is transitive.*

Assume now that G is an arbitrary connected Lie group. As in [11], we associate to each $x \in X$ the set

$$\mathcal{L}_\beta(x) = \{g \in G | \text{ there exist } x_k \in X \text{ and } n_k > 0 \text{ such that } x_k \to x \text{ and } f_\beta^{n_k}(x_k, e) \to (x, g)\}.$$

The set $\mathcal{L}_\beta(x)$ is a closed semigroup in G [11, Lemma 3.1]. The following proposition is a special case of [11, Theorem 3.3].

Proposition 2.2. *Let $G = K \ltimes \mathbb{R}^n$ where K is a compact connected Lie group. If there is a point $x \in X$ such that $\mathcal{L}_\beta(x) = G$, then the extension f_β is transitive.*

We review next techniques of Brin [1, 2]. The group G is a semidirect product $K \ltimes \mathbb{R}^n$ where K is a compact connected Lie group (although, we will use these results for $G = K \times \mathbb{R}^n$ only).

A proof of the following lemma follows from Appendix A of [14].

Lemma 2.3. *Let $f : X \to X$ be a hyperbolic map, $G = K \ltimes \mathbb{R}^n$ and $\beta : X \to G$ a Hölder cocycle. Then f_β admits stable and unstable foliations on $X \times G$, which are Hölder and invariant under right-multiplication by elements of G.*

The following definitions go back to Brin [2]. Denote by $\mathcal{W}^s(\xi)$, respectively $\mathcal{W}^u(\xi)$, the leaves through $\xi \in X \times G$ for the stable and unstable foliations in Lemma 2.3.

Definition 2.4. Let $f : X \to X$ be a hyperbolic map, and $\beta : X \to G$ a Hölder cocycle. A (u, s)-*path* between two points $\xi, \eta \in X \times G$ is a sequence of points $\xi_0 = \xi, \xi_1, \dots, \xi_m =$

$\eta \in X \times G$ such that $\xi_i \in \mathcal{W}^s(\xi_{i-1})$ or $\xi_i \in \mathcal{W}^u(\xi_{i-1})$. We will say that the pair ξ, η is (u,s)-*accessible*, and refer to ξ_i, $0 \le i \le m$, as *junction points*.

The pair of foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ is called *accessible* if each pair of points of $X \times G$ is accessible.

A (u,s)-*rectangle at* x is a (u,s)-path of four segments that starts and ends in the fiber $\{x\} \times G$ (i.e., it is the lift of a stable-unstable closed rectangle in X starting at x).

Taking into account Lemma 2.3, the following result is a more precise statement of what is proven in [2, Theorem 1]. The claim $m > 0$ is implicit there (note that there is typo in [2, proof of Lemma 3]: $y_1 \in M_1$ should be $z_1 \in M_1$).

Lemma 2.5. *Let $G = K \ltimes \mathbb{R}^n$ where K is a compact connected Lie group, and let $\beta : X \to G$ be a Hölder cocycle over the hyperbolic map $f : X \to X$. Assume that the pair $\xi_1, \xi_2 \in X \times G$ is connected by a (u,s)-path whose junction points are nonwandering.*

Then, given any open neighborhoods $U_k \subset X \times G$ of ξ_k, $k = 1, 2$, there exists $m > 0$ such that $f_\beta^m(U_1) \cap U_2 \neq \emptyset$.

Corollary 2.6. *Let $f : X \to X$ and $\beta : X \to G$ be as in Lemma 2.5. Assume that each point of $X \times G$ is nonwandering. If the pair $(x, hg), (x, g) \in X \times G$ is (u,s)-accessible, then $h \in \mathcal{L}_\beta(x)$.*

Remark. This shows that for such skew products, the holonomy elements generated by (u,s)-rectangles at x (see the group H_C of Brin [2]) are contained in $\mathcal{L}_\beta(x)$.

3. Groups $G = C \times A$ where C is compact semisimple and A is abelian

In this section, we prove Theorem 1.1 for connected Lie groups of the form $G = C \times A$ where C is compact semisimple and A is abelian. Let $p_C : G \to C$ and $p_A : G \to A$ denote the canonical projections.

Writing $A = \mathbb{T}^d \times \mathbb{R}^n$, we denote by $\mathcal{S}$ the open set of C^r cocycles β ($r \ge 2$) for which the $\mathbb{R}^n$-component is *not* cohomologous to a cocycle with values in a half-space.

Lemma 3.1. *Let $L \subset G$ be a closed semi-subgroup. Suppose that $p_A(L) = A$ and $\overline{p_C(L)} = C$. Then $L = G$.*

Proof. We can rewrite G in the form $G = K \times \mathbb{R}^n$ where $K = C \times \mathbb{T}^d$ is compact, and then it follows from the semigroup problem for such groups (see the proof of [11, Theorem 5.10]) that L is a group. (Note that L satisfies the half-space hypothesis in [11, Theorem 5.10] since $p_A(L) = A$).

Since $p_A(L) = A$, it suffices to show that $C \times \{e_A\} \subset L$. Now A is abelian and so $[L, L] = [p_C L, p_C L] \times \{e_A\}$. Therefore $\overline{[p_C L, p_C L]} \times \{e_A\} \subset L$. Since $p_C L$ is dense in C, it is immediate that $[p_C L, p_C L]$ is dense in $[C, C]$. But C is semisimple so $[C, C] = C$. Hence we have shown that $C \times \{e_A\} \subset L$ as required. □

Proposition 3.2. *Theorem 1.1 holds for connected Lie groups $G = C \times A$ where C is compact semisimple and A is abelian.*

Proof. Fix $x_0 \in X$ and consider the closed semigroup $\mathcal{L}_\beta(x_0) \subset G$. We construct a C^2 open and C^r dense set of C^r cocycles $\beta \in \mathcal{S}$ ($r \geq 2$) such that $\mathcal{L}_\beta(x_0)$ satisfies the hypotheses of Lemma 3.1. Then $\mathcal{L}_\beta(x_0) = G$ and the result follows from Proposition 2.2.

By [5, Theorem 1.7], in $\mathcal{S}$ there is an open and dense set $\mathcal{S}_1$ of cocycles $\beta : X \to G$ for which $f_{p_A\beta} : X \times A \to X \times A$ is transitive. Since C is compact, it is immediate that $p_A(\mathcal{L}_\beta(x_0)) = A$. By the proof of [11, Theorem 5.10], $\mathcal{L}_\beta(x_0))$ is a group and hence contains the identity element. It follows that (x_0, g) is nonwandering for all $g \in G$. Since x_0 is arbitrary, all points in $X \times G$ are nonwandering.

It follows from results on generators for compact semisimple Lie groups [9, 4] that there is an open and dense set of pairs of elements $(c_1, c_2) \in C \times C$ that generate C as a closed (semi)group. From [6, Lemma 2.9.1] we obtain an open and dense subset $\mathcal{S}_2$ of $\beta \in \mathcal{S}_1$ for which the (u, s)-rectangles of $p_C\beta$ at x_0 determine a pair of generators of C. Therefore, by Corollary 2.6, $p_C\mathcal{L}_\beta(x_0)$ contains a generator set of C for all extensions $\beta \in \mathcal{S}_2$. It follows that $\overline{p_C(\mathcal{L}_\beta(x_0))} = C$ for all these cocycles. This completes the proof. $\square$

4. Groups $G = K \times \mathbb{R}^n$ where K is compact

In this section, we complete the proof of Theorem 1.1. The following result is analogous to Field *et al.* [5, Proposition 6.1].

Lemma 4.1. *Let G be a group of the form $G = K \times \mathbb{R}^n$ where K is a compact connected Lie group. Let X be a hyperbolic basic set for $f : X \to X$ and $\beta : X \to G$ a Hölder cocycle. Let $Y \subset X$ be a closed f-invariant subset. If the restriction $(f_\beta)|_{Y \times G} : Y \times G \to Y \times G$ is transitive, then $f_\beta : X \times G \to X \times G$ is transitive.*

Proof. Let $y_0 \in Y$. We can define a closed semigroup $\mathcal{L}_\beta(y_0) \subset G$ as in Section 2, but we can do it using all trajectories in X or just those in Y. Denote the two closed semigroups $\mathcal{L}^X_\beta(y_0)$ and $\mathcal{L}^Y_\beta(y_0)$ respectively. Clearly, $\mathcal{L}^Y_\beta(y_0) \subset \mathcal{L}^X_\beta(y_0)$. Now transitivity of $Y \times G$ implies that $\mathcal{L}^Y_\beta(y_0) = G$. Hence $\mathcal{L}^X_\beta(y_0) = G$, implying transitivity of $X \times G$ by Proposition 2.2. $\square$

Proof of Theorem 1.1 It is always possible to embed a horseshoe inside a hyperbolic basic set [8, Theorem 6.5.5, Exercise 6.5.1]). By Lemma 4.1, it suffices to prove the result for the horseshoe. Hence, we may suppose from the beginning that X is totally disconnected.

By the structure theorem for compact Lie groups ([3, Ch. V, Theorem 8.1]) there is a finite cover $p : C \times \mathbb{T}^d \to K$ where C is compact semisimple and $\mathbb{T}^d$ is a torus. This extends by the identity to a finite cover $p : \widehat{G} \to G$ where $\widehat{G} = C \times \mathbb{T}^d \times \mathbb{R}^n$ is a group of the form studied in Section 3.

Let $\widehat{\mathcal{S}}$ consist of C^r cocycles $\hat{\beta} : X \to \widehat{G}$ for which the $\mathbb{R}^n$-component of $\hat{\beta}$ is not cohomologous to a cocycle with values in a half-space. Recall that the cocycles we are considering are C^r functions on a neighborhood of X in the ambient manifold, but we are only interested in their restrictions to X. Since X is totally disconnected, each C^r cocycle $\beta : X \to G$ has a C^r lift $\beta : X \to \widehat{G}$. Clearly $p^{-1}(\mathcal{S}) = \widehat{\mathcal{S}}$.

By Proposition 3.2, it follows that there is an open and dense set of transitive cocycles $\hat{\beta} : X \to \widehat{G}$ in $\widehat{\mathcal{S}}$. Clearly, if $\hat{\beta}$ defines a transitive $\widehat{G}$-extension, then $\beta = p\hat{\beta}$ defines a transitive G-extension.

Given $\beta \in \mathcal{S}$, let $\hat{\beta} \in \widehat{\mathcal{S}}$ be a lift of it to $\widehat{G}$. After an arbitrarily small perturbation, we may suppose that $\hat{\beta}$ defines a stably transitive $\widehat{G}$-extension. Moreover, cocycles β' that are Hölder close to β can be lifted to cocycles $\hat{\beta}'$ that are Hölder close to $\hat{\beta}$. Since $\hat{\beta}$ is stably transitive, the cocycles $\hat{\beta}'$ are transitive and hence the cocycles β' are transitive. Hence β is stably transitive as required. □

5. $SE(2)$-extensions

In this section, we prove Theorem 1.2.

Lemma 5.1. *There is an open and dense set of pairs of elements $(g_1, g_2) \in SE(2)^2$ such that the closed semigroup H generated by g_1 and g_2 is a group of the form $H = K \ltimes \mathbb{R}^2$ for a closed subgroup $K \subset SO(2)$.*

Proof. Let $g_1, g_2 \in SE(2)$ be two rotations with distinct centers of rotation, such that g_1 has order at least seven. The set of such pairs is open and dense in $SE(2)^2$.

Since g_1, g_2 are compact, the closed semigroup and group they generate coincide. Denote this by H. Up to a conjugation, we may assume that $g_1 = (\theta_1, 0)$ and $g_2 = (\theta_2, v_2)$, where $\theta_1, \theta_2 \in SO(2)$ and $v_2 \in \mathbb{R}^2 \setminus \{0\}$. Then

$$g_3 = g_1 g_2 g_1^{-1} g_2^{-1} = (\theta_1, 0)(\theta_2, v_2)(\theta_1^{-1}, 0)(\theta_2^{-1}, -\theta_2^{-1} v_2) = (e, \theta_1 v_2 - v_2) \in H \cap (\mathbb{R}^2 \setminus \{0\}).$$

Thus the group H contains g_1, a rotation of order at least seven, and g_3, a non-trivial translation. Denote by H_0 the closed group generated by g_1, g_3. Then $H_0 = K_1 \ltimes L$ where $K_1 \subset SO(2)$ is the closed group generated by g_1, $|K_1| \geq 7$, and $L \subset \mathbb{R}^2$ is a closed K_1-invariant subgroup. We claim that $L = \mathbb{R}^2$. This implies that H is a semidirect product $K \ltimes \mathbb{R}^2$.

To prove the claim, note that by the crystallographic restriction [7, pages 70-81], L is not a lattice. If L is not $\mathbb{R}^2$, then it is isomorphic to $\mathbb{R} \times \mathbb{Z}$, and therefore the $\mathbb{R}$-direction must be preserved by K_1. This is impossible. □

Proof of Theorem 1.2 Let $p : SE(2) \to SO(2)$ be the canonical projection.

Using [11, Lemma 4.1] and the proof of [11, Theorem 1.7(b)], for an open dense set $\mathcal{U}_1$ of cocycles $\beta : X \to SE(2)$ we can generate as holonomies a pair of elements $g_1, g_2 \in \mathcal{L}_\beta(x_0)$ satisfying the property of Lemma 5.1. Therefore, $\mathbb{R}^2 \subset \mathcal{L}_\beta(x_0)$ for all cocycles in $\mathcal{U}_1$.

It follows from [5, Theorem 1.3] that, for a dense and open set $\mathcal{U}_2$ of β, the $SO(2)$ extension given by $p\beta$ is transitive.

We will show that $\mathcal{L}_\beta(x_0) = SE(2)$ for cocycles in $\mathcal{U}_1 \cap \mathcal{U}_2$. Since $\mathbb{R}^2 \subset \mathcal{L}_\beta(x_0)$, we only have to prove that $SO(2) \subset \mathcal{L}_\beta(x_0)$.

Let $\theta \in SO(2)$. By the transitivity of the $p\beta$ extension, there is a sequence of orbits $\mathcal{P}_n = \{x_n, \ldots, f^{p_n}(x_n)\}$ for which $x_n \to x_0$, $f^{p_n}(x_n) \to x_0$, and $p\beta_{p_n}(x_n) \to \theta$. Write $\beta_{p_n}(x_n) =$

$(\theta_n, v_n) \in SO(2) \ltimes \mathbb{R}^2$. Since $\mathbb{R}^2 \subset \mathcal{L}_\beta(x_0)$, one can find a sequence $\mathcal{R}_n = \{y_n, \dots, f^{r_n}(y_n)\}$ of orbits such that $y_n \to x_0$, $f^{r_n}(y_n) \to x_0$, and $d(\beta_{r_n}(y_n), (e, -v_n)) \to 0$. [Here d is the right-invariant metric on $SE(2)$ considered in [11, §2.2].] Denote by $\{z_n, \dots, f^{p_n+r_n}(z_n)\}$ the orbit that shadows the pseudo-orbit $\{\mathcal{R}_n, \mathcal{P}_n\}$. Using [11, Lemma 2.2] one obtains that $d(\beta_{p_n+r_n}(z_n), \beta_{r_n}(y_n)\beta_{p_n}(x_n)) \leq d(\beta_{r_n}(f^{p_n}z_n), \beta_{r_n}(y_n)) + d(\beta_{p_n}(z_n), \beta_{p_n}(x_n))$ is small. Therefore, $\beta_{p_n+r_n}(z_n) \to (\theta, 0)$, which shows that $(\theta, 0) \in \mathcal{L}_\beta(x_0)$. □

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