Analytic proof of stable local large deviations and application to deterministic dynamical systems

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Abstract
We give a short analytic proof of large local deviations for i.i.d. random variables in the domain of an \( \alpha \)-stable law, \( \alpha \in (0, 1) \cup (1, 2] \). Our proof applies also to the dynamical setting.

1 Introduction

Local large deviation results for i.i.d. random variables in the domain of a stable law have been recently obtained by Caravenna and Doney [7, Theorem 1.1] and refined by Berger [6, Theorem 2.3]. We refer to such results as stable local large deviations (stable LLD).

The aim of this paper is two-fold. First, we provide a new proof of the stable LLD in Theorem 1.1 (we exclude the case \( \alpha = 1 \) but include the case \( \alpha = 2 \) which was previously omitted). Instead of using Fuk-Nagaev inequalities as was done in [6, 7], we give a short analytic proof using Nagaev-type perturbative arguments together with decay of Fourier coefficients. A major advantage of this approach is that it generalises naturally to the dynamical setting. This is the second main aim of this paper where we establish the stable LLD for sequences of nonindependent random variables arising from observables of deterministic dynamical systems.

First we recall the i.i.d. set up in [6, 7]. Let \((X_i)_{i \geq 1}\) be a sequence of i.i.d. random variables with \( \mathbb{E}X_1^2 = \infty \). We suppose that

\[
\mathbb{P}(X_1 > x) = (p + o(1))\ell(x)x^{-\alpha}, \quad \mathbb{P}(X_1 \leq -x) = (q + o(1))\ell(x)x^{-\alpha}
\]

(1.1)
as \( x \to \infty \), where \( \alpha \in (0, 2] \), \( \ell \) is slowly varying and \( p, q \geq 0 \) with \( p + q > 0 \). Equivalently, \( X_1 \) is in the domain of an \( \alpha \)-stable law \( Y_\alpha \) (determined by \( \alpha, p, q \)). Namely, there are sequences \( a_n > 0, b_n \in \mathbb{R} \), such that \( S_n = \sum_{i=1}^{n} X_i \) satisfies

\[
(S_n - b_n)/a_n \to_d Y_\alpha.
\]

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Set $\ell = \ell$ for $\alpha \in (0, 2)$ and $\tilde{\ell}(x) = \int_1^x \frac{\ell(u)}{u} \, du$ for $\alpha = 2$. Then $a_n$ satisfies
\[
\lim_{n \to \infty} \frac{n\tilde{\ell}(a_n)}{a_n^\alpha} = 1.
\]
Also,
\[
b_n = \begin{cases} 
0 & \alpha \in (0, 1) \\
n\mathbb{E}_{X_1 \mathbb{1}_{\{|X_1| \leq a_n\}}} & \alpha = 1 \\
n\mathbb{E}_{X_1} & \alpha \in (1, 2] \end{cases}.
\]

Next, we recall the local limit theorem for stable laws which controls the asymptotics of $\mathbb{P}(S_n \in J)$ for suitable subsets $J \subset \mathbb{R}$. The distribution of a nondegenerate random variable is called lattice if it is concentrated on the set $\{\kappa + h\mathbb{Z}\}$ for some $\kappa \in \mathbb{R}$ and $h > 0$, and is called nonlattice otherwise. In the lattice case, the largest possible $h$ is called the span; we restrict without loss to the case $\kappa = 0$ and $h = 1$, so all lattice distributions in this paper are integer-valued with span 1.

In the nonlattice case, Stone’s stable local limit theorem [16] states that for any $h > 0$
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{a_n}{h} \mathbb{P}\left(S_n \in (x-h, x+h]\right) - g\left(\frac{x-b_n}{a_n}\right) \right| = 0, \tag{1.2}
\]
where $g$ is the density of $Y_\alpha$. (The counterpart for lattice distributions is due to Gnedenko [11, Chap. 9, Sec. 50].)

Stable local large deviations (the main topic of this paper) concerns estimates for $\mathbb{P}(S_n \in J)$ for subsets $J \subset \mathbb{R}$ taking into account the location of $J$:

**Theorem 1.1** Assume (1.1) with $\alpha \in (0, 2]$. Then for every $h > 0$ there is a constant $C > 0$ such that
\[
\mathbb{P}(S_n - b_n \in (x-h, x+h]) \leq C \frac{n \tilde{\ell}(|x|)}{a_n |x|^\alpha} \quad \text{for all } n \geq 1, x \in \mathbb{R}. \tag{1.3}
\]

In particular, in the lattice case (integer-valued with span 1), there is a constant $C > 0$ such that
\[
\mathbb{P}(S_n - [b_n] = N) \leq C \frac{n \tilde{\ell}(|N|)}{a_n |N|^\alpha} \quad \text{for all } n \geq 1, N \in \mathbb{Z}.
\]

**Remark 1.2** Caravenna and Doney [7, Theorem 1.1] proved Theorem 1.1 for $\alpha \in (0, 1) \cup (1, 2)$ (focusing on the nonlattice case) and (amongst other things) this was extended by Berger [6, Theorem 2.3] to the range $\alpha \in (0, 2)$ (focusing on the lattice case). Our analytic proof covers the range $\alpha \in (0, 1) \cup (1, 2]$ so the combined results cover the range $\alpha \in (0, 2]$. In addition, our arguments cover the lattice and nonlattice cases simultaneously (the only difference is addressed in the proof of Lemma 2.1).
As mentioned before, our main contribution is to provide a new proof which generalises easily to the dynamical setting. The estimate \((1.3)\) is easily deduced from the local limit theorem \((1.2)\) for \(|x - b_n| \ll a_n\) and is seen to be sharp for \(|x - b_n| \approx a_n\). Hence the main content of Theorem 1.1 is when \(|x - b_n| \gg a_n\).

**Remark 1.3** We have excluded the problematic case \(\alpha = 1\) which was completely solved by Berger [6]. In fact, our methods apply without modification for \(\alpha = 1\) in the symmetric case \(b_n = 0\). However, in the nonsymmetric case the estimate in Lemma 2.2(i) below fails (see [9, Lemma 5]). Consequently, without refining our methods further we would obtain a suboptimal estimate in Theorem 1.1 for \(\alpha = 1, b_n \neq 0\).

**Remark 1.4** The estimates in Theorem 1.1 are proved under assumption (1.1) which is necessary and sufficient for convergence to the stable law \(Y_\alpha\). For stronger estimates under more restrictive hypotheses, we refer to [6, 8, 12].

Our analytic proof of Theorem 1.1 is given in Section 2. In Section 3, we show that our proof applies to a class of deterministic dynamical systems. In fact, we prove a stronger operator stable LLD in Theorem 3.2 which yields the desired stable LLD in Corollary 3.4.

**Notation** We use “big O” and \(\ll\) notation interchangeably, writing \(a_n = O(b_n)\) or \(a_n \ll b_n\) if there are constants \(C > 0, n_0 \geq 1\) such that \(a_n \leq Cb_n\) for all \(n \geq n_0\). As usual, \(a_n = o(b_n)\) means that \(\lim_{n \to \infty} a_n/b_n = 0\) and \(a_n \sim b_n\) means that \(\lim_{n \to \infty} a_n/b_n = 1\).

## 2 Stable local large deviations in the i.i.d. set up

In this section, we provide an analytic proof of local large deviations for i.i.d. random variables in the domain of a stable law. In particular, we prove Theorem 1.1 for \(\alpha \in (0, 1) \cup (1, 2]\).

Since we exclude the case \(\alpha = 1\), we can suppose without loss that \(b_n = 0\). This is automatic for \(\alpha \in (0, 1)\) while for \(\alpha \in (1, 2]\) we can replace \(X_1\) by \(X_1 - \mathbb{E}X_1\). In other words, we suppose without loss that \(\mathbb{E}X_1 = 0\) for \(\alpha \in (1, 2]\).

### 2.1 Technical lemmas

The proof of the results obtained here exploits classical results on the characteristic function collected in Lemmas 2.1 and 2.2 below.

Fix \(\alpha \in (0, 1) \cup (1, 2]\), \(a_n\) and \(\ell\) as in Section 1. Recall that \(b_n = 0\). For \(s \in \mathbb{R}\), define

\[
\Psi(s) = \mathbb{E}(e^{isX_1}) = \int_{-\infty}^{\infty} e^{isx} dF(x).
\]
Lemma 2.1 There exist constants $\epsilon, c > 0$, such that

$$|\Psi(s)| \leq \exp\{-c|s|^{\alpha}\tilde{\ell}(1/|s|)\} \quad \text{for all } |s| \leq 3\epsilon.$$  

Proof  First, there exists $c_1 > 0$ such that

$$\log|\Psi(s)| = \Re\log\Psi(s) \sim -c_1|s|^{\alpha}\tilde{\ell}(1/|s|) \quad \text{as } s \to 0.$$  

This result can be found in [13, 2] for $\alpha \in (0, 1) \cup (1, 2)$ and the case $\alpha = 2$ is covered in [3]. (The estimate also holds for $\alpha = 1$ by [1].) In the nonlattice case, $|\Psi(s)| < 1$ for all $s \neq 0$ and we can choose $\epsilon = 1$ say. In the lattice case of span $h$, we have that $|\Psi(s)| < 1$ for $0 < |s| < 2\pi/h$, so we could take $\epsilon = \pi/(6h)$.  

Throughout this section we fix $\epsilon$ so that Lemma 2.1 holds.

Lemma 2.2 Let $M > 0$. There exists a constant $C > 0$ such that the following hold for all $|s|, |s + h| \leq M$:

(i) $|\Psi(s + h) - \Psi(s)| \leq C|h|^{\alpha}\tilde{\ell}(1/|h|)$ for $\alpha \in (0, 1)$.

(ii) $|\Psi'(s)| \leq C|s|^{\alpha-1}\tilde{\ell}(1/|s|)$ for $\alpha \in (1, 2]$.

(iii) $|\Psi'(s + h) - \Psi'(s)| \leq C|h|^{\alpha-1}\tilde{\ell}(1/|h|)$ for $\alpha \in (1, 2]$.

Proof  Let $F(x) = \mathbb{P}(X_1 \leq x)$ and $\bar{F}(x) = \mathbb{P}(X_1 > x)$.

(i) See [10, Lemma 3.3.2] or [9, Lemma 5].

(ii) Since $X_1$ has mean zero, $\Psi(s) = 1 + \int_{-\infty}^{\infty}(e^{isx} - 1 - isx) dF(x)$. Hence

$$\Psi'(s) = i\int_{-\infty}^{0} x(e^{isx} - 1) dF(x) - i\int_{0}^{\infty} x(e^{isx} - 1) d\bar{F}(x).$$

Integrating by parts,

$$\Psi'(s) = -i\int_{-\infty}^{0} (e^{isx} - 1) F(x) dx + i\int_{0}^{\infty} (e^{isx} - 1) \bar{F}(x) dx$$

$$+ s\int_{-\infty}^{0} e^{isx} x F(x) dx - s\int_{0}^{\infty} e^{isx} x \bar{F}(x) dx.$$  

The first two integrals are absolutely convergent and the last two integrals are oscillatory. The estimates for $\alpha \in (1, 2)$ are similar to those in [13, 2] for $\Psi$ with $\alpha \in (0, 1)$. The estimates for $\alpha = 2$ are similar to those in [1] for $\Psi$ with $\alpha = 1$.

(iii) We consider $h > 0$, the case $h < 0$ is similar. Using integration by parts and Karamata’s theorem, for $K \geq 1$,

$$\int_{K}^{\infty} x dF(x) = K\bar{F}(K) - 2\int_{K}^{\infty} \bar{F}(x) dx \ll K^{-(\alpha-1)\ell(K)}$$
and similarly $\int_{-\infty}^{-K} x \, dF(x) \ll K^{-(\alpha-1)} \ell(K)$. Hence integration by parts gives

$$|\Psi'(s + h) - \Psi'(s)| \leq \left| \int_{-K}^{K} (e^{hx} - 1)xe^{isx} \, dF(x) \right| + 2\int_{-\infty}^{-K} x \, dF(x) + 2\int_{K}^{\infty} x \, dF(x) \ll \left| h \right| \int_{-K}^{K} x^2 \, dF(x) + K^{-(\alpha-1)} \ell(K)$$

$$\leq |h|K^2 \left( \bar{F}(K) + F(-K) \right) + 2h\left| \int_{-K}^{K} x\bar{F}(x) \, dx \right| + K^{-(\alpha-1)} \ell(K)
\ll |h|K^{2-\alpha} \ell(K) + |h|K^{2-\alpha} \bar{\ell}(K) + K^{1-\alpha} \ell(K).$$

Taking $K = |h|^{-1}$ and recalling that $\ell(K) \ll \bar{\ell}(K)$, we obtain $|\Psi'(s + h) - \Psi'(s)| \ll K^{-(\alpha-1)} \bar{\ell}(K)$.

**Lemma 2.3** Let $L$ be a slowly varying function. For all $c > 0$, $\beta \geq 0$, there exists $C > 0$ such that for all $n \geq 1$,

$$\int_{0}^{\infty} s^{\beta} L(1/s) \exp\{-ncs^{\alpha} \bar{\ell}(1/s)\} \, ds \leq C \frac{L(a_n)}{a_n^{1+\beta}}.$$

In particular,

$$\int_{-3\epsilon}^{-\epsilon} |s|^{\beta} L(1/|s|) |\Psi(s)|^n \, ds \leq C \frac{L(a_n)}{a_n^{1+\beta}}.$$

**Proof** Using the change of variables $s = \sigma/a_n$,

$$\int_{0}^{\infty} s^{\beta} L(1/s) \exp\{-ncs^{\alpha} \bar{\ell}(1/s)\} \, ds \leq \frac{L(a_n)}{a_n^{1+\beta}} \int_{0}^{\infty} \frac{L(a_n/\sigma)}{L(a_n)} \sigma^{\beta} \exp\{-cn\sigma^{\alpha} a_n^{-\alpha} \bar{\ell}(a_n/\sigma)\} \, d\sigma.$$ 

By Potter’s bounds, for any $\delta \in (0, \alpha)$, there exists $c' > 0$ such that

$$\int_{1}^{\infty} \frac{L(a_n/\sigma)}{L(a_n)} \sigma^{\beta} \exp\{-cn\sigma^{\alpha} a_n^{-\alpha} \bar{\ell}(a_n/\sigma)\} \, d\sigma \ll \int_{1}^{\infty} \sigma^{\beta+\delta} \exp\{-c'n\sigma^{\alpha-\delta} a_n^{-\alpha} \bar{\ell}(a_n)\} \, d\sigma.$$

Recalling that $\lim_{n \to \infty} na_n^{-\alpha} \bar{\ell}(a_n) = 1$,

$$\int_{1}^{\infty} \frac{L(a_n/\sigma)}{L(a_n)} \sigma^{\beta} \exp\{-cn\sigma^{\alpha} a_n^{-\alpha} \bar{\ell}(a_n/\sigma)\} \, d\sigma \ll \int_{1}^{\infty} \sigma^{\beta+\delta} e^{-c'\sigma^{\alpha-\delta}} \, d\sigma < \infty.$$ 

A similar argument deals with the integral on $[0, 1]$.

The final statement follows from Lemma 2.1.
2.2 Proof of the stable LLD

In this subsection, we prove Theorem \ref{thm:stable-LLD}. We begin with some comments.

For $\alpha \in (0, 1)$, $h = 1$, the proof below yields the estimate (1.3) directly. For $\alpha \in (1, 2]$, the proof shows that

$$\mathbb{P}(S_n - b_n \in (x - 1, x + 1]) \ll \frac{n}{a_n} \tilde{\ell}(|x|) \left(1 + \frac{a_n}{|x|}\right).$$

This implies the required estimate for $|x - b_n| \gg a_n$. By Remark \ref{rem:stable-LLD}, this is sufficient to prove the result since the range $|x - b_n| \ll a_n$ is covered by the local limit theorem. Similarly, it suffices to prove the result for $|x| > \pi/\epsilon$ (where $\epsilon$ is as in Lemma \ref{lem:stable-LLD}).

In addition, we can suppose without loss that $h = 1$; the result for smaller intervals is immediate and the result for larger intervals can be obtained by taking unions of smaller intervals.

As in \cite{ref1}, we convolve with a suitable function $\gamma$ with compactly supported Fourier transform $\hat{\gamma}$. For our purposes, it suffices to fix a continuous integrable function $\gamma : \mathbb{R} \to [0, \infty)$ with $\gamma \geq 1$ on $[-2, 2]$ such that its Fourier transform $\hat{\gamma}$ is even and $C^2$ with support in $[-\epsilon, \epsilon]$.

Define

$$r(s) = \frac{1}{2\pi} \frac{\sin s}{s} \hat{\gamma}(s).$$

We note that $r$ is $C^2$ and supported in $[-\epsilon, \epsilon]$.

**Lemma 2.4** For $n \geq 1$, $x \in \mathbb{R}$,

$$\mathbb{P}(S_n \in (x - 1, x + 1]) \leq \int_{-\infty}^{\infty} e^{-isx} r(s) \Psi(s)^n \, ds.$$

**Proof** By the Fourier inversion formula,

$$\gamma(y) = \frac{1}{2\pi} \int_{-\epsilon}^{x} e^{isy} \hat{\gamma}(s) \, ds = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} e^{-isy} \hat{\gamma}(s) \, ds. \quad (2.1)$$

Hence by Fubini,

$$\int_{x-1}^{x+1} \int_{-\infty}^{\infty} \gamma(y - y') dF_n(y') \, dy = \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \left(\int_{x-1}^{x+1} e^{-isy} \, dy\right) \hat{\gamma}(s) \left(\int_{-\infty}^{\infty} e^{isy} dF_n(y')\right) \, ds$$

$$= 2 \int_{-\epsilon}^{\epsilon} e^{-isx} r(s) \Psi(s)^n \, ds. \quad (2.2)$$

\[\text{It is easily verified that such a } \gamma \text{ exists. Start with an even } C^\infty \text{ function } \hat{\gamma} : \mathbb{R} \to [0, \infty) \text{ supported in } [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ with inverse Fourier transform } \gamma. \text{ Then } \gamma \text{ is real-valued and } C^\infty. \text{ Taking } \hat{\gamma} \neq 0 \text{ ensures that } \gamma(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\gamma}(\xi) \, d\xi > 0. \text{ Replacing } \gamma(x) \text{ by } \gamma(ax) \text{ with } a \text{ sufficiently small ensures that } \gamma > 0 \text{ on } [-2, 2]. \text{ (Such a scaling shrinks the support of } \hat{\gamma}, \text{ so the new } \hat{\gamma} \text{ remains supported in } [-\frac{\pi}{2}, \frac{\pi}{2}].) \text{ Next, replace } \gamma \text{ by } c\gamma \text{ for } c \text{ sufficiently large, ensuring that } \gamma \geq 1 \text{ on } [-2, 2]. \text{ Finally, replace } \gamma \text{ by } \gamma^2 \text{ and } \hat{\gamma} \text{ by } \hat{\gamma} \hat{\gamma} \text{ to ensure that } \gamma \geq 0.\]
Next, since $\gamma \geq 0$ and $\gamma|_{[-2,2]} \geq 1$,

$$\int_{-\infty}^{\infty} \gamma(y - y') dF_n(y') \geq \int_{y-2}^{y+2} dF_n(y') = \mathbb{P}(S_n \in (y - 2, y + 2]).$$

Hence

$$\int_{x-1}^{x+1} \int_{-\infty}^{\infty} \gamma(y - y') dF_n(y') dy \geq \int_{x-1}^{x+1} \mathbb{P}(S_n \in (y - 2, y + 2]) dy \geq 2\mathbb{P}(S_n \in (x - 1, x + 1]).$$

Combining this with (2.2) yields the desired result.

**Proof of Theorem 1.1** Recall that we have reduced to the cases $b_n = 0$, $h = 1$. Also we can suppose that $|x| \geq a_n$ and in particular $|x| \geq \pi/\epsilon$. We suppose that $x \geq \pi/\epsilon$, the case $x \leq -\pi/\epsilon$ being similar.

By Lemma 2.4 it suffices to estimate $I_{n,x} = \int_{-\infty}^{\infty} e^{-isx} r(s) \Psi(s)^n ds$.

**The case** $\alpha \in (0,1)$. We exploit the modulus of continuity of $\Psi$ (see, for instance, [14, Chapter 1]). Note that $I_{n,x} = -\int_{-\infty}^{\infty} e^{-isx} r(s - \frac{\pi}{x}) \Psi(s - \frac{\pi}{x})^n ds$. Hence

$$|I_{n,x}| = \frac{1}{2} \left| \int_{-\infty}^{\infty} e^{-isx} (r(s)\Psi(s)^n - r(s - \frac{\pi}{x})\Psi(s - \frac{\pi}{x})^n) ds \right| \leq I_1 + I_2$$  (2.3)

where

$$I_1 = \int_{-\infty}^{\infty} |r(s) - r(s - \frac{\pi}{x})| ||\Psi(s)||^n ds, \quad I_2 = \int_{-\infty}^{\infty} |r(s - \frac{\pi}{x})||\Psi(s)^n - \Psi(s - \frac{\pi}{x})^n| ds.$$

Since $r$ is supported in $[-\epsilon, \epsilon]$ and $x \geq \pi/\epsilon$, the integrands in $I_1$ and $I_2$ are supported in $[-2\epsilon, 2\epsilon]$. Using also that $r$ is bounded and Lipschitz,

$$I_1 \ll x^{-1} \int_{-2\epsilon}^{2\epsilon} |\Psi(s)|^n ds, \quad I_2 \ll \int_{-2\epsilon}^{2\epsilon} |\Psi(s)^n - \Psi(s - \frac{\pi}{x})^n| ds.$$

By Lemma 2.3, $I_1 \ll \frac{1}{a_n} \frac{1}{x^\alpha}$.

Next recall the inequality

$$|u^n - v^n| \leq n|u - v|(|u|^{n-1} + |v|^{n-1}),$$  (2.4)

which holds for all $u, v \in \mathbb{C}$, $n \geq 1$. Using this and Lemma 2.2(i),

$$|\Psi(s)^n - \Psi(s - \frac{\pi}{x})^n| \ll nx^{-\alpha} \ell(x) \left(|\Psi(s)|^{n-1} + |\Psi(s - \frac{\pi}{x})|^{n-1}\right).$$

Hence by Lemma 2.3, $I_2 \ll nx^{-\alpha} \ell(x) \int_{-3\epsilon}^{3\epsilon} |\Psi(s)|^{n-1} ds \ll \frac{n}{a_n} \frac{\ell(x)}{x^\alpha}$. 

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The case $\alpha \in (1, 2]$. Let $D_n = \Psi^{n-1}\Psi'$. Integrating by parts, $I_{n,x} = E_1 + E_2$ where

$$E_1 = \frac{1}{ix} \int_{-\infty}^{\infty} e^{-isx} r(s) \Psi(s)^n \, ds, \quad E_2 = \frac{n}{ix} \int_{-\infty}^{\infty} e^{-isx} r(s) D_n(s) \, ds.$$  

Integrating by parts once more, and using that $r$ is $C^2$ and supported in $[-\epsilon, \epsilon]$, 

$$|E_1| \leq \frac{1}{x^2} \int_{-\epsilon}^{\epsilon} |r''(s)||\Psi(s)|^n \, ds + \frac{n}{x^2} \int_{-\epsilon}^{\epsilon} |r'(s)||D_n(s)| \, ds$$

$$\ll \frac{1}{x^2} \int_{-\epsilon}^{\epsilon} |\Psi(s)|^n \, ds + \frac{n}{x^2} \int_{-\epsilon}^{\epsilon} |\Psi(s)|^{n-1} \, ds.$$  

(Here, we used also that $\Psi'$ is bounded on $[-\epsilon, \epsilon]$ by Lemma 2.2(ii).) By Lemma 2.3

$$|E_1| \ll \frac{n}{a_n x^2} \ll \frac{n}{a_n x^\alpha}.$$  

Next, we exploit the modulus of continuity of $rD_n$, writing

$$|E_2| \leq \frac{n}{x} \int_{-\infty}^{\infty} |r(s) - r(s - \frac{\pi}{x})| |D_n(s)| \, ds + \frac{n}{x} \int_{-\infty}^{\infty} |r(s - \frac{\pi}{x})| |D_n(s) - D_n(s - \frac{\pi}{x})| \, ds$$

$$\ll \frac{n}{x^2} \int_{-2\epsilon}^{2\epsilon} |\Psi(s)|^{n-1} \, ds + \frac{n}{x} \int_{-2\epsilon}^{2\epsilon} |D_n(s) - D_n(s - \frac{\pi}{x})| \, ds.$$  

Again, $\frac{n}{x} \int_{-2\epsilon}^{2\epsilon} |\Psi(s)|^{n-1} \, ds \ll \frac{n}{a_n x^2} \ll \frac{n}{a_n x^\alpha}$, so it remains to estimate

$$J = \frac{n}{x} \int_{-2\epsilon}^{2\epsilon} |D_n(s) - D_n(s - \frac{\pi}{x})| \, ds.$$  

Relabel $\{s, s - \frac{\pi}{x}\} = \{s_1, s_2\}$ where $|\Psi(s_1)| \leq |\Psi(s_2)|$. Then $J = J_1 + J_2$ where $J_i = \frac{n}{x} \int K_i$ for $i = 1, 2$, and

$$K_1 = |\Psi(s_1)|^{n-1} |\Psi'(s_1) - \Psi'(s_2)|, \quad K_2 = |\Psi(s_1)|^{n-1} - |\Psi(s_2)|^{n-1} |\Psi'(s_2)|.$$  

By Lemma 2.2(iii), $K_1 \ll x^{1-\alpha} \frac{\tilde{\ell}(x)}{x^\alpha} |\Psi(s_1)|^{n-1}$. Hence by Lemma 2.3

$$J_1 \ll \frac{n}{a_n x^\alpha} \int_{-3\epsilon}^{3\epsilon} |\Psi(s)|^{n-1} \, ds \ll \frac{n}{a_n x^\alpha}.$$  

Next, by (2.4),

$$K_2 \ll n |\Psi'(s_2)| |\Psi'(s)| |\Psi(s_2)|^{n-2}.$$  

By the mean value theorem for vector-valued functions and Lemma 2.2(ii), there exists $s^* \in (s - \frac{\pi}{x}, s)$ such that

$$K_2 \ll nx^{-1} |\Psi'(s_2)||\Psi'(s^*)||\Psi(s_2)|^{n-2} \leq K_3 + K_4$$  

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where
\[ K_3 = n x^{-1} |\psi'(s_2)||\psi'(s_2) - \psi'(s^*)| |\psi(s_2)|^{n-2}, \quad K_4 = n x^{-1} |\psi'(s_2)|^2 |\psi(s_2)|^{n-2}. \]

Correspondingly, we have \( J_2 \ll J_3 + J_4 = n x^{-1} \left( \int K_3 + \int K_4 \right) . \)

By Lemma 2.2(ii),(iii), \( K_3 \ll n x^{-\alpha} \tilde{\ell}(x)|s_2|^{\alpha-1} \tilde{\ell}(1/|s_2|)|\psi(s_2)|^{n-2} . \) Hence, by Lemma 2.3

\[ J_3 \ll n^2 x^{-(\alpha+1)} \tilde{\ell}(x) \int_{-3\epsilon}^{3\epsilon} |s|^{\alpha-1} \tilde{\ell}(1/|s|)|\psi(s)|^{n-2} \, ds \]
\[ \ll n^2 x^{-(\alpha+1)} \tilde{\ell}(x) a_n^{-\alpha} \tilde{\ell}(a_n) \sim n x^{-(\alpha+1)} \tilde{\ell}(x) = \frac{n}{a_n} \frac{\tilde{\ell}(x)}{x^\alpha} x. \]

Finally, by Lemma 2.2(ii), \( K_4 \ll n x^{-1} |s_2|^{2(\alpha-1)} \tilde{\ell}(1/|s_2|)^2 |\psi(s_2)|^{n-2} . \) Hence, by Lemma 2.3

\[ J_4 \ll n^2 x^{-2} \int_{-3\epsilon}^{3\epsilon} |s|^{2(\alpha-1)} \tilde{\ell}(1/|s|)^2 |\psi(s)|^{n-2} \, ds \]
\[ \ll n^2 x^{-2} a_n^{-1} a_n^{-2(\alpha-1)} \tilde{\ell}(a_n)^2 \sim n x^{-2} a_n^{-1} a_n^{2-\alpha} \tilde{\ell}(a_n) = \frac{n}{a_n} \frac{\tilde{\ell}(x)}{x^\alpha} \frac{a_n^{2-\alpha} \tilde{\ell}(a_n)}{x^{2-\alpha} \tilde{\ell}(x)}. \]

Combining the estimates for \( J_1, J_3, J_4 \) we obtain
\[ J \ll \frac{n}{a_n} \frac{\tilde{\ell}(x)}{x^\alpha} \left( 1 + \frac{a_n}{x} + \frac{a_n^{2-\alpha} \tilde{\ell}(a_n)}{x^{2-\alpha} \tilde{\ell}(x)} \right). \]

By Potter’s bounds, \( \frac{a_n^{2-\alpha} \tilde{\ell}(a_n)}{x^{2-\alpha} \tilde{\ell}(x)} \ll 1 + \frac{a_n}{x} \) for \( \alpha \in (1, 2] \). Hence we have shown that \( |I_{n,x}| \ll \frac{n}{a_n} \frac{\tilde{\ell}(x)}{x^\alpha} \left( 1 + \frac{a_n}{x} \right) \) as required.

\section{Stable LLD for dynamical systems}

In this section we show that the previous results can be generalized to a class of deterministic dynamical systems. The main result of this section is Theorem 3.2. In Subsections 3.4 and 3.5, we verify that Theorem 3.2 applies to Gibbs-Markov maps and AFU maps.

\subsection{Dynamical systems set up}

Let \( f : \Lambda \to \Lambda \) be a measure-preserving map on a probability space \( (\Lambda, \mu) \). Let \( v : \Lambda \to \mathbb{R} \) be a measurable observation with \( \int_{\Lambda} v^2 \, d\mu = \infty \). We fix \( \alpha \in (0, 1) \cup (1, 2] \) throughout and assume...
(H1) $\mu(v > x) = (p + o(1))\ell(x)x^{-\alpha}$, $\mu(v \leq -x) = (q + o(1))\ell(x)x^{-\alpha}$,
as $x \to \infty$, where $\ell$ is slowly varying and $p, q \geq 0$ with $p + q > 0$. Define $\tilde{\ell}$, $a_n$ and $b_n$ as in Section [1].

Let $R : L^1 \to L^1$ be the transfer operator for $f$ defined via the formula

$$
\int \Lambda R \phi \psi \, d\mu = \int \Lambda \phi \psi \circ f \, d\mu.
$$

Given $s \in \mathbb{R}$, define the perturbed operator $R(s) : L^1 \to L^1$ by $R(s)\phi = R(e^{isv}\phi)$.

We assume that there is a Banach space $B \subset L^\infty$ containing constant functions, with norm $\| \cdot \|$ satisfying $|\phi|_\infty \leq \|\phi\|$ for $\phi \in B$, such that

(H2) There exist $\epsilon > 0$, $C > 0$ such that for all $s \in [-\epsilon, \epsilon]$, $h \in [0, \epsilon]$,

(i) $\|R(s)\| \leq C$ for $\alpha \in (0, 2]$ and $\|R'(s)\| \leq C$ for $\alpha \in (1, 2]$.

(ii) $\|R(s+h) - R(s)\| \leq Ch^\alpha\ell(1/h)$ for $\alpha \in (0, 1)$ and $\|R(s+h) - R(s)\| \leq Ch$

for $\alpha \in (1, 2]$.

(iii) $\|R'(s+h) - R'(s)\| \leq Ch^{\alpha-1}\tilde{\ell}(1/h)$ for $\alpha \in (1, 2]$.

Since $R(0) = R$ and $B$ contains constant functions, 1 is an eigenvalue of $R(0)$. We assume:

(H3) The eigenvalue 1 is simple, and the remainder of the spectrum of $R(0) : B \to B$

is contained in a disk of radius less than 1.

Hypotheses (H1)–(H3) suffice for the methods used in this paper. However, certain conclusions are stated more cleanly when we assume in addition:

(H4) For $v$ nonlattice, the spectral radius of $R(s) : B \to B$ is less than 1 for $s \neq 0$.

For $v$ lattice of span 1, the spectral radius is less than 1 for $0 < |s| \leq \pi$.

Under hypotheses (H1)–(H3), distributional convergence $(v_n - b_n)/a_n \to_d Y_\alpha$ to an $\alpha$-stable random variable $Y_\alpha$ holds by [2]. Moreover, under (H1)–(H4) the corresponding local limit theorem also holds:

**Proposition 3.1** Let $\alpha \in (0, 1) \cup (1, 2]$ and assume (H1)–(H4). Let $g$ be the density of $Y_\alpha$. Suppose that $A, B \subset \Lambda$ are measurable.

If $v$ is nonlattice, then for any $h > 0$,

$$
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{a_n}{h} \mu\left( \frac{x - b_n}{a_n} \right) \right| = 0.
$$

If $v$ is lattice (integer-valued with span 1), then

$$
\lim_{n \to \infty} \sup_{N \in \mathbb{N}} \left| a_n \mu\left( \frac{N - [b_n]}{a_n} \right) \right| = 0.
$$
Versions of Proposition 3.1 are proved in [2, 3] but without uniformity over \( x \in \mathbb{R} \) and \( N \in \mathbb{N} \). The only reference we know for a uniform dynamical local limit theorem like that in Proposition 3.1 is [15, Theorem 2.7]. However, the result in [15] is stated only in the nonlattice case, for 1-sided stable laws with \( \alpha \in (0, 1) \), and under slightly different hypotheses. For completeness, in Appendix A we indicate the general proof of Proposition 3.1.

We can now state the main result in the dynamical setting: namely an operator stable LLD. Let \( R_{n,J} \) denote the operator \( R_n(1_{\{v_n - b_n \in J\}} \phi) \) on \( L^\infty \).

**Theorem 3.2** Let \( \alpha \in (0, 1) \cup (1, 2] \) and assume (H1)–(H3). There is a family of positive bounded linear operators \( A_{n,x} \), \( n \geq 1 \), \( x \in \mathbb{R} \), defined on \( L^\infty \) and \( \mathcal{B} \), and there exist constants \( C > 0 \), \( x_0 > 0 \), such that

- \( R_{n,(x-1,x+1]} \leq A_{n,x} \) as operators on \( L^\infty \) for all \( n \geq 1 \), \( x \in \mathbb{R} \);
- For \( \alpha \in (0, 1) \): \( \|A_{n,x}\| \leq C \frac{n}{a_n} \tilde{\ell}(|x|) \|\phi\|_1 \) for all \( n \geq 1 \), \( |x| \geq x_0 \);
- For \( \alpha \in (1, 2] \): \( \|A_{n,x}\| \leq C \frac{n}{a_n} \tilde{\ell}(|x|) \left(1 + \frac{a_n}{|x|}\right) \) for all \( n \geq 1 \), \( x \in \mathbb{R} \).

A consequence of Theorem 3.2 is the usual stable LLD.

**Corollary 3.3** Let \( \alpha \in (0, 1) \cup (1, 2] \) and assume (H1)–(H4). Then for every \( h > 0 \), there exists \( C > 0 \) such that

\[
\left| \int_\Lambda \phi \psi \circ f^n 1_{\{v_n - b_n \in (x-h,x+h]\}} d\mu \right| \leq \|\phi\|_1 \|\psi\|_1 \frac{n}{a_n} \tilde{\ell}(|x|) \|\phi\|_1 \|\psi\|_1 \\
\text{for all } n \geq 1, \ x \in \mathbb{R}, \ \phi \in \mathcal{B} \text{ with } \phi \geq 0 \text{ and } \psi \in L^1.
\]

**Proof** The range \( |x| \ll a_n \) follows from the local limit theorem in Proposition 3.1. Hence in applying Theorem 3.2 we can disregard the factor \( \left(1 + \frac{a_n}{|x|}\right) \). Moreover, we can suppose without loss that \( x \) lies in the range \( |x| \geq x_0 \) covered by Theorem 3.2. As in Section 2.2, we can suppose without loss that \( h = 1 \).

Now,

\[
\int_\Lambda \phi \psi \circ f^n 1_{\{v_n - b_n \in J\}} d\mu = \int_\Lambda \psi R^n(1_{\{v_n - b_n \in J\}} \phi) d\mu = \int_\Lambda \psi R_{n,J} \phi d\mu
\]

so

\[
\left| \int_\Lambda \phi \psi \circ f^n 1_{\{v_n - b_n \in (x-1,x+1]\}} d\mu \right| \leq \|R_{n,(x-1,x+1]} \phi\|_\infty \|\psi\|_1 \leq \|A_{n,x}\| \|\phi\|_1 \ll \frac{n}{a_n} \tilde{\ell}(|x|) \|\phi\|_1 \|\psi\|_1
\]

by Theorem 3.2.
Corollary 3.4 Let $\alpha \in (0,1) \cup (1,2]$ and assume (H1)–(H4). Then for every $h > 0$, there exists $C > 0$ such that

$$
\mu(z \in A \cap f^{-n}B : v_n(z) - b_n \in (x-h, x+h]) \leq C \|1_A\| \mu(B) \frac{n \tilde{\ell}(|x|)}{a_n |x|^\alpha}
$$

for all $n \geq 1$, $x \in \mathbb{R}$ and all measurable $A, B \subset \Lambda$ with $1_A \in \mathcal{B}$.

In particular, if $v$ is lattice (integer-valued with span 1), then for every $h > 0$, there exists $C > 0$ such that

$$
\mu(z \in A \cap f^{-n}B : v_n(z) - \lfloor b_n \rfloor = N) \leq C \|1_A\| \mu(B) \frac{n \tilde{\ell}(|N|)}{a_n |N|^\alpha}
$$

for all $n \geq 1$, $N \in \mathbb{Z}$ and all measurable $A, B \subset \Lambda$ with $1_A \in \mathcal{B}$.

Proof This is an immediate consequence of Corollary 3.3 with $\phi = 1_A$ and $\psi = 1_B$.

The proof of Theorem 3.2 takes up the remainder of this section. We suppose from now on that (H1)–(H3) hold. As in Section 2, we can suppose without loss that $b_n = 0$. Equivalently, for $\alpha \in (1,2]$ we can suppose without loss that $\int_\Lambda v d\mu = 0$.

3.2 Technical lemmas

In this subsection, we assume (H1)–(H3). By (H2) and (H3), there exists $\epsilon > 0$ and a continuous family $\lambda(s)$ of simple eigenvalues of $R(s)$ for $|s| \leq 3\epsilon$ with $\lambda(0) = 1$. The associated spectral projections $P(s)$, $|s| \leq 3\epsilon$, form a continuous family of bounded linear operators on $\mathcal{B}$. Moreover, there is a continuous family of linear operators $Q(s)$ on $\mathcal{B}$ and constants $C > 0$, $\delta_0 \in (0,1)$ such that

$$
R(s) = \lambda(s)P(s) + Q(s) \quad \text{for } |s| \leq 3\epsilon.
$$

$$
\|Q(s)^n\| \leq C \delta_0^n \quad \text{for } |s| \leq 3\epsilon, \ n \geq 1.
$$

(3.1) (3.2)

Hence we can shrink $\epsilon$ so that

$$
\|R(s)^n\| \leq C|\lambda(s)|^n \quad \text{for } |s| \leq 3\epsilon, \ n \geq 1.
$$

(3.3)

Let $\zeta(s) = \frac{P(s)1}{\int P(s)1 d\mu}$ be the normalized eigenvector corresponding to $\lambda(s)$.

Lemma 3.5 There exists $\epsilon > 0$ such that the properties of $R(s)$ listed in (H2) are inherited by $P(s)$, $Q(s)$, $\lambda(s)$ and $\zeta(s)$ for all $|s|, |s+h| \leq 3\epsilon$.

Proof This is a standard consequence of perturbation theory for smooth families of operators.

The next result is the analogue of Lemmas 2.1 and 2.2(ii) for $\lambda(s)$.  

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Lemma 3.6 There exist constants $\epsilon$, $c$, $C > 0$ such that the following hold for all $|s|, |s + h| \leq 3\epsilon$,

(i) $|\lambda(s)| \leq \exp\{-c|s|^{\alpha} \tilde{\ell}(1/|s|)\}$ for $\alpha \in (0, 1) \cup (1, 2]$.

(ii) $|\lambda'(s)| \leq C|s|^{\alpha - 1} \tilde{\ell}(1/|s|)$ for $\alpha \in (1, 2]$.

Proof Write

$$
\lambda(s) = \int R(s)\zeta(s)\,d\mu = \int R(s)1\,d\mu + \int (R(s) - R(0))(\zeta(s) - \zeta(0))\,d\mu
$$

$$
= \int \exp isv\,d\mu + \int (R(s) - R(0))(\zeta(s) - \zeta(0))\,d\mu = \Psi(s) + V(s).
$$

The estimates for $\Psi$ in Lemmas 2.1 and 2.2 are unchanged (since the distribution of $v$ is given by (H1)) so it suffices to verify that the contributions from $V$ are negligible.

For $\alpha \in (0, 1)$, we choose $\alpha' \in (\frac{1}{2}\alpha, \alpha)$. Then $\|R(s) - R(0)\| \ll |s|^{\alpha'}$ by (H2) and $\|\zeta(s) - \zeta(0)\| \ll |s|^{\alpha'}$ by Lemma 3.5. Since $\mathcal{B} \subset L^\infty$,

$$
|V(s)| \ll \|R(s) - R(0)\|\|\zeta(s) - \zeta(0)\| \ll |s|^{2\alpha'} = o(|s|^\alpha \tilde{\ell}(1/|s|)).
$$

Similarly, $|V(s)| \ll |s|^2 = o(|s|^\alpha \tilde{\ell}(1/|s|))$ when $\alpha \in (1, 2]$. This completes the proof of part (i).

Next, for $\alpha \in (1, 2]$,

$$
V'(s) = \int \lambda'(s)\zeta(s)\,d\mu + \int \lambda'(s)(\zeta(s) - \zeta(0))\,d\mu
$$

so $|V'(s)| \ll |s| = o(|s|^{\alpha - 1} \tilde{\ell}(1/|s|))$ as required for part (ii).

From now on, $\epsilon > 0$ is fixed in accordance with the above properties.

Corollary 3.7 Let $L$ be a slowly varying function. For all $c > 0$, $\beta \geq 0$, there exists $C > 0$ such that for all $n \geq 1$,

$$
\int_{-3\epsilon}^{3\epsilon} |s|^\beta L(1/|s|)|\lambda(s)|^n\,ds \leq C \frac{L(a_n)}{a_n^{1+\beta}}.
$$

Proof This follows from Lemmas 2.3 and 3.6(i).

We require the following estimates on the derivatives of $R(s)^n$ and $Q(s)^n$.

Lemma 3.8 Let $\alpha \in (1, 2]$ and fix $\delta_1 \in (\delta_0, 1)$. Then there exists $C > 0$ such that for all $|s|, |s + h| \leq 3\epsilon$,

$$
\|(R(s)^n)'\| \leq Cn|\lambda(s)|^{n-1} \quad \text{and} \quad \|(Q(s + h)^n)' - (Q(s)^n)\| \leq C\delta_1^n|h|^{\alpha - 1} \tilde{\ell}(1/|h|).
$$
Proof. We start from \((R(s)^n)' = \sum_{j=0}^{n-1} R(s)^j R'(s) R(s)^n-j-1\). By (3.3) and (H2)(i)

\[
\|(R(s)^n)\| \ll \sum_{j=0}^{n-1} |\lambda(s)^j| |\lambda(s)|^{n-j-1} = n|\lambda(s)|^{n-1}.
\]

Next, fix \(\delta_2 \in (\delta_0, \delta_1)\). By (3.2) and Lemma 3.5

\[
\|Q(s+h)^n - Q(s)^n\| \leq \sum_{j=0}^{n-1} \|Q(s+h)^j\| \|Q(s+h) - Q(s)\| \|Q(s)^{n-j-1}\|
\ll |h| \sum_{j=0}^{n-1} \delta_0^n \ll \delta_2^n |h|.
\]

(3.4)

Let \(j, k \geq 0\) with \(j+k = n-1\). Then

\[
(Q^j Q^k)'(s+h) - (Q^j Q^k)'(s) = (Q(s+h)^j - Q(s)^j) Q'(s+h) Q(s+h)^k
+ Q(s)^j (Q'(s+h) - Q'(s)) Q(s+h)^k
+ Q(s)^j Q'(s) (Q(s+h)^k - Q(s)^k)
\]

so by (3.2), (3.4) and Lemma 3.5

\[
\|(Q^j Q^k)'(s+h) - (Q^j Q^k)'(s)\| \ll \delta_2^{n-1} |h| + \delta_2^{n-1} |h|^{n-\tilde{\ell}(1/|h|)} \ll \delta_2^n |h|^{n-\tilde{\ell}(1/|h|)}.
\]

Substituting into \((Q(s)^n)' = \sum_{j=0}^{n-1} (Q^j Q^j)'(s)\) we obtain \(\|(Q(s+h)^n)' - (Q(s)^n)\| \ll n\delta_2^n |h|^{n-\tilde{\ell}(1/|h|)} \ll \delta_1^n |h|^{n-\tilde{\ell}(1/|h|)}\) as required. 

Corollary 3.9 (i) Let \(\alpha \in (0, 1) \cup (1, 2)\). There exists \(C > 0\) such that for all \(|h| \leq \epsilon\),

\[
\int_{-2\epsilon}^{2\epsilon} \|R(s)^j\| \|R(s+h)^k\| ds \leq C a_n^{-1} \quad \text{for all } j, k \geq 0, n \geq 1 \text{ with } j+k = n.
\]

(ii) Let \(\alpha \in (1, 2)\). There exists \(C > 0\) such that

\[
\int_{-2\epsilon}^{2\epsilon} \|(R(s)^n)\| ds \leq C a_n^{-1} \quad \text{for all } n \geq 1.
\]

Proof (i) By (3.3),

\[
\|R(s)^j\| \|R(s+h)^k\| \ll |\lambda(s)^j| |\lambda(s+h)|^k \ll |\lambda(s)|^n + |\lambda(s+h)|^n.
\]

Also, by Lemma 3.8 \(\|(R(s)^n)\| \ll n|\lambda(s)|^{n-1}\). Hence both parts follow from Corollary 3.7. 

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3.3 Proof of the operator stable LLD

In this subsection, we prove Theorem 3.2. Define \( r : \mathbb{R} \to \mathbb{R} \) as in Section 2. Recall that \( r \) is \( C^2 \) and supported in \([-\epsilon, \epsilon]\).

**Lemma 3.10** \( 1_{\{v_n \in (x-1,x+1)\}} \leq \int_\epsilon e^{-isx}r(s) e^{isv_n} ds \) for \( n \geq 1 \), \( x \in \mathbb{R} \).

**Proof** Define \( \gamma : \mathbb{R} \to [0, \infty) \) as in Section 2. Since \( \gamma \geq 0 \) and \( \gamma|_{[-2,2]} \geq 1 \),

\[
1_{\{v_n \in (x-1,x+1)\}} = \frac{1}{2} \int_\epsilon^{x+1} 1_{\{v_n \in (x-1,x+1)\}} dy
\leq \frac{1}{2} \int_\epsilon^{x+1} 1_{\{v_n \in (y-2,y+2)\}} dy \leq \frac{1}{2} \int_\epsilon^{x+1} \gamma(y - v_n) dy.
\]

Using the Fourier inversion formula (2.1),

\[
1_{\{v_n \in (x-1,x+1)\}} \leq \frac{1}{4\pi} \int_\epsilon^\infty \left( \int_{x-1}^{x+1} e^{-isy} dy \right) \hat{\gamma}(s) e^{isv_n} ds = \int_{-\epsilon}^\epsilon e^{-isx}r(s) e^{isv_n} ds
\]

by Fubini.

**Proof of Theorem 3.2** We take \( x_0 = \pi/\epsilon \). As usual, we suppose that \( x \geq \pi/\epsilon \), the case \( x \leq -\pi/\epsilon \) being similar. Recall that we have taken \( b_n = 0 \) without loss.

By Lemma 3.10, for \( \phi \in L^\infty \) nonnegative,

\[
R_{n,(x-1,x+1)} \phi = R^n(1_{\{v_n \in (x-1,x+1)\}} \phi) \leq \int_{-\epsilon}^\epsilon e^{-isx}r(s) R^n(e^{isv_n} \phi) ds = A_{n,x} \phi.
\]

where

\[
A_{n,x} = \int_{-\epsilon}^\epsilon e^{-isx}r(s) R^n ds.
\]

Hence it suffices to estimate \( \|A_{n,x}\| \).

**The case** \( \alpha \in (0,1) \). The same modulus of continuity argument as in the i.i.d. case (cf. [2.3]) yields \( \|A_{n,x}\| \leq I_1 + I_2 \) where

\[
I_1 = \int_{-\infty}^{\infty} |r(s) - r(s - \frac{\pi}{x})| \|R(s)^n\| ds \ll x^{-1} \int_{-2\epsilon}^{2\epsilon} \|R(s)^n\| ds,
\]

\[
I_2 = \int_{-\infty}^{\infty} |r(s - \frac{\pi}{x})| \|R(s)^n - R(s - \frac{\pi}{x})^n\| ds \ll \int_{-2\epsilon}^{2\epsilon} \|R(s)^n - R(s - \frac{\pi}{x})^n\| ds.
\]

By Corollary 3.9(1), \( I_1 \ll x^{-1} a_0^{-1} \ll \frac{n}{a_n} \frac{f(x)}{x^\alpha} \).

Next,

\[
\|R(s_1)^n - R(s_2)^n\| \leq \sum_{j=0}^{n-1} \|R(s_1)^j\| \|R(s_1) - R(s_2)\| \|R(s_2)^{n-j-1}\|,
\]

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so by (H2)(ii) and Corollary 3.9(i),

\[ I_2 \ll \frac{\ell(x)}{x^\alpha} \sum_{j=0}^{n-1} \int_{-2\epsilon}^{2\epsilon} \| R(s) \| \| R(s - \frac{\pi}{x}) \| ds \ll \frac{n \ell(x)}{a_n x^\alpha}. \]

**The case** \( \alpha \in (1, 2] \). Integrating by parts, \( A_{n,x} = E_1 + E_2 \) where

\[ E_1 = \frac{1}{ix} \int_{-\infty}^{\infty} e^{-isx} r'(s) R(s)^n ds, \quad E_2 = \frac{1}{ix} \int_{-\infty}^{\infty} e^{-isx} r(s) (R(s)^n)' ds. \]

Integrating by parts once more and using that \( r \) is \( C^2 \) and supported in \([ -\epsilon, \epsilon ]\),

\[ \| E_1 \| \leq \frac{1}{x^2} \int_{-\infty}^{\infty} |r''(s)| \| R(s)^n \| ds + \frac{1}{x^2} \int_{-\infty}^{\infty} |r'(s)| \| (R(s)^n)' \| ds \]

\[ \ll \frac{1}{x^2} \int_{-\epsilon}^{\epsilon} \| R(s)^n \| ds + \frac{1}{x^2} \int_{-\epsilon}^{\epsilon} \| (R(s)^n)' \| ds. \]

By Corollary 3.9

\[ \| E_1 \| \ll \frac{1}{a_n x^2} + \frac{n}{a_n x^2} \ll \frac{n \ell(x)}{a_n x^\alpha}. \]

Next, we exploit the modulus of continuity of \( r(R^n)' \), writing

\[ \| E_2 \| \ll \frac{1}{x} \int_{-\infty}^{\infty} |r(s) - r(s - \frac{\pi}{x})| \| (R(s)^n)' \| ds \]

\[ + \frac{1}{x} \int_{-\infty}^{\infty} |r(s - \frac{\pi}{x})| \| (R(s)^n)' - (R(s - \frac{\pi}{x})^n)' \| ds \]

\[ \ll \frac{1}{x^2} \int_{-2\epsilon}^{2\epsilon} \| (R(s)^n)' \| ds + \frac{1}{x} \int_{-2\epsilon}^{2\epsilon} \| (R(s)^n)' - (R(s - \frac{\pi}{x})^n)' \| ds. \]

Again \( \frac{1}{x^2} \int_{-2\epsilon}^{2\epsilon} \| (R(s)^n)' \| ds \ll \frac{n}{a_n} \frac{1}{x^2} \ll \frac{n \ell(x)}{a_n x^\alpha} \) so it remains to estimate

\[ J = \frac{1}{x} \int_{-2\epsilon}^{2\epsilon} \| (R(s)^n)' - (R(s - \frac{\pi}{x})^n)' \| ds. \]

By (3.1),

\[ (R(s)^n)' = n\lambda(s)^{n-1}\lambda'(s)P(s) + \lambda(s)^nP'(s) + (Q(s)^n)'. \]

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Relabel \(\{s, s - \frac{n}{x}\} = \{s_1, s_2\}\) where \(|\lambda(s_1)| \leq |\lambda(s_2)|\). Then \(J \leq \frac{1}{x}(F_1 + \cdots + F_6)\) where

\[
F_1 = \frac{n}{x} \int_{-2\epsilon}^{2\epsilon} |\lambda(s_1)^{n-1} - \lambda(s_2)^{n-1}| |\lambda'(s_2)| \|P(s_2)\| \, ds,
\]

\[
F_2 = \frac{n}{x} \int_{-2\epsilon}^{2\epsilon} |\lambda(s_2)^{n-1}| |\lambda'(s_1) - \lambda'(s_2)| \|P(s_2)\| \, ds,
\]

\[
F_3 = \frac{n}{x} \int_{-2\epsilon}^{2\epsilon} |\lambda(s_2)^{n-1}| |\lambda'(s_2)| \|P(s_1) - P(s_2)\| \, ds,
\]

\[
F_4 = \frac{1}{x} \int_{-2\epsilon}^{2\epsilon} |\lambda(s_1)^n - \lambda(s_2)^n| \|P'(s_2)\| \, ds,
\]

\[
F_5 = \frac{1}{x} \int_{-2\epsilon}^{2\epsilon} |\lambda(s_2)^n| \|P'(s_1) - P'(s_2)\| \, ds,
\]

\[
F_6 = \frac{1}{x} \int_{-2\epsilon}^{2\epsilon} \||(Q^n)'(s_1) - (Q^n)'(s_2)\| \, ds.
\]

The hardest term \(F_1\) is estimated in the same way as \(K_2\) in the proof of Theorem 1.1, so we write the calculation without the justifications:

\[
F_1 \ll \frac{n^2}{x} \int_{-2\epsilon}^{2\epsilon} |\lambda(s_1) - \lambda(s_2)| |\lambda(s_2)|^{n-2} |\lambda'(s_2)| \, ds
\]

\[
\ll \frac{n^2}{x^2} \int_{-2\epsilon}^{2\epsilon} |\lambda'(s^*)| |\lambda'(s_2)| |\lambda(s_2)|^{n-2} \, ds
\]

\[
\ll \frac{n^2}{x^2} \int_{-2\epsilon}^{2\epsilon} |\lambda'(s^*) - \lambda'(s_2)| \|\lambda'(s_2)\| |\lambda(s_2)|^{n-2} \, ds + \frac{n^2}{x^2} \int_{-2\epsilon}^{2\epsilon} |\lambda'(s_2)|^2 |\lambda(s_2)|^{n-2} \, ds
\]

\[
\ll \frac{n^2}{x^2} \int_{-3\epsilon}^{3\epsilon} \int_{-3\epsilon}^{3\epsilon} |s|^{\alpha-1} |\lambda(s)|^{n-2} \, ds + \frac{n^2}{x^2} \int_{-3\epsilon}^{3\epsilon} |s|^{2(\alpha-1)} |\lambda(s)|^{n-2} \, ds
\]

\[
\ll \frac{n}{a_n} \frac{\tilde{\ell}(x)}{x^\alpha} \left( a_n + a_n^{2-\alpha} \tilde{\ell}(a_n) \right) \ll \frac{n}{a_n} \frac{\tilde{\ell}(x)}{x^\alpha} \left( 1 + \frac{a_n}{x} \right).
\]

This is the only term that requires Lemma 3.6(ii). The terms \(F_2, \ldots, F_5\) require only the rougher estimates in Lemma 3.5 combined with Corollary 3.7 and we obtain

\[
F_2 \ll \frac{n}{a_n} \frac{\tilde{\ell}(x)}{x^\alpha}, \quad F_3, F_4 \ll \frac{n}{a_n} \frac{1}{x^2}, \quad F_5 \ll \frac{1}{a_n} \frac{\tilde{\ell}(x)}{x^\alpha}.
\]

Finally, by Lemma 3.8 \(F_6 \ll \delta_1 \frac{n \tilde{\ell}(x)}{x^\alpha}\) which ends the proof.

### 3.4 Gibbs–Markov maps

Let \((\Lambda, \mu)\) be a probability space with an at most countable measurable partition \(\{\Lambda_j\}\), and let \(f: \Lambda \to \Lambda\) be an ergodic measure-preserving transformation. Define
$s(z, z')$ to be the least integer $n \geq 0$ such that $f^n z$ and $f^n z'$ lie in distinct partition elements. It is assumed that $s(z, z') = \infty$ if and only if $z = z'$; then $d_\theta(z, z') = \theta^s(z, z')$ is a metric for $\theta \in (0, 1)$.

Let $g = \frac{d\mu}{d\nu_j}: \Lambda \to \mathbb{R}$. We say that $f$ is a \textit{Gibbs-Markov map} if

- $f\Lambda_j$ is a union of partition elements and $f|_{\Lambda_j} : \Lambda_j \to f\Lambda_j$ is a measurable bijection for each $j \geq 1$;
- $\inf_j \mu(f\Lambda_j) > 0$;
- There are constants $C > 0$, $\theta \in (0, 1)$ such that $|\log g(z) - \log g(z')| \leq C d_\theta(z, z')$ for all $z, z' \in \Lambda_j$, $j \geq 1$.

Standard references for Gibbs-Markov maps include [2, 5].

Given $\phi : \Lambda \to \mathbb{R}$, let

$$D_j \phi = \sup_{z, z' \in \Lambda_j, z \neq z'} |\phi(z) - \phi(z')|/d_\theta(z, z'), \quad |\phi|_\theta = \sup_{j \geq 1} D_j \phi.$$  

We define the Banach space $\mathcal{F}_\theta \subset L^\infty$ to consist of functions $\phi : \Lambda \to \mathbb{R}$ such that $|\phi|_\theta < \infty$ with norm $\|\phi\|_\theta = |\phi|_\infty + |\phi|_\theta < \infty$.

**Proposition 3.11** Assume $f$ is a mixing Gibbs-Markov map and let $v : \Lambda \to \mathbb{R}$ with $\int_\Lambda v^2 \, d\mu = \infty$ and $|v|_\theta < \infty$. Fix $\alpha \in (0, 1) \cup (1, 2]$ and assume that the tails of $v$ satisfy (H1).

Then conditions (H1)–(H3) are satisfied with Banach space $\mathcal{B} = \mathcal{F}_\theta$.

**Proof**  Condition (H1) is satisfied by assumption and condition (H3) is well-known for mixing Gibbs-Markov maps [2, 5]. It remains to verify that (H2) holds. In fact, for any $M > 0$ the conditions in (H2) hold for all $s \in [-M, M]$, $h \in [0, 1]$. We verify this for (H2)(iii) All the other calculations are simpler and hence omitted.

Now $(R'(s + h) - R'(s))\phi = iR(\phi \psi)$ where $\psi = ve^{i\nu e^{ihv} - 1}$. A standard calculation shows that

$$\|R(\phi \psi)\|_\theta \ll \sum_j \mu(\Lambda_j) (\sup_j |\phi \psi| + D_j (\phi \psi)) \leq \|\phi\|_\theta \sum_j \mu(\Lambda_j) \left(2\sup_j |\psi| + D_j \psi\right).$$

where $\sup_j = \sup_{\Lambda_j}$ and $\inf_j \mu_{\Lambda_j}$. Hence

$$\|R'(s + h) - R'(s)\|_\theta \ll \sum_j \mu(\Lambda_j) \left\{\sup_j |v(e^{ihv} - 1)| + D_j (ve^{i\nu e^{ihv} - 1})\right\},$$

Also, $D_je^{i\nu v} \leq |s||v|_\theta \ll |s|$, so $\|R'(s + h) - R'(s)\|_\theta \ll S_1 + S_2 + S_3 + S_4$, where

- $S_1 = \sum_j \mu(\Lambda_j) \sup_j |v(e^{ihv} - 1)|$,
- $S_2 = \sum_j \mu(\Lambda_j) \sup_j |v(e^{ihv} - 1)| D_j e^{i\nu} \ll |s|S_1 \leq MS_1$,
- $S_3 = \sum_j \mu(\Lambda_j) \sup_j |v| D_j e^{ihv} \leq h \sum_j \mu(\Lambda_j) \sup_j |v|$,
- $S_4 = \sum_j \mu(\Lambda_j) \sup_j |e^{ihv} - 1| D_j v \ll S_3.$

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Next, $\sup_j |v| - \inf_j |v| \leq |v|_\theta \ll 1$. Hence

$$\sum_j \mu(\Lambda_j) \sup_j |v| \ll \sum_j \mu(\Lambda_j)(1 + \inf_j |v|) \leq 1 + \int_\Lambda |v| \, d\mu,$$

and we obtain $S_3, S_4 \ll h$.

Finally,

$$\sup_j |v(e^{ihv} - 1)| \ll (1 + \inf_j |v|)(h + \inf_j |e^{ihv} - 1|) \ll h(1 + \inf_j |v|) + \inf_j |v(e^{ihv} - 1)|,$$

and so

$$S_1 \ll \sum_j \mu(\Lambda_j)(h(1 + \inf_j |v|) + \inf_j |v(e^{ihv} - 1)|)$$

$$\leq h \left(1 + \int_\Lambda |v| \, d\mu\right) + \int_\Lambda |v(e^{ihv} - 1)| \, d\mu.$$

But $\int_\Lambda |v(e^{ihv} - 1)| \, d\mu = \int_{-\infty}^{\infty} |x||e^{ihx} - 1| \, dF(x)$ where $F(x) = \mu(v \leq x)$. Since the tails of $v$ are given by (H1), it follows exactly as in the proof of Lemma 2.2(iii) that

$$\int_{-\infty}^{\infty} |x||e^{ihx} - 1| \, dF(x) \ll h^{\alpha-1}\tilde{\ell}(1/h).$$

Hence $S_1, S_2 \ll h + h^{\alpha-1}\tilde{\ell}(1/h)$.

Altogether, $\|R'(s + h) - R'(s)\|_{\theta} \ll h + h^{\alpha-1}\tilde{\ell}(1/h) \ll h^{\alpha-1}\tilde{\ell}(1/h)$ as required.

**Remark 3.12** Further assumptions on $f$ and $v$ are required for hypothesis (H4) to be satisfied. In the lattice case (integer-valued with span 1), it suffices that $f$ is full-branch ($f\Lambda_j = \Lambda$ for all $j \geq 1$); see [2, Theorem 3.1 and Corollary 3.2] for more general conditions on $f$. In the nonlattice case, it suffices that $f$ is full-branch and that there exist two fixed points $z_1, z_2 \in \Lambda$ such that $v(z_1)/v(z_2)$ is irrational.

### 3.5 AFU maps

Let $\Lambda = [0, 1]$ with measurable partition $\{I\}$ consisting of open intervals. A map $f : \Lambda \to \Lambda$ is called AFU if $f|_I$ is $C^2$ and strictly monotone for each $I$, and

(A) (Adler’s condition) $f''/(f')^2$ is bounded on $\bigcup I$.

(F) (finite images) The set of images $\{fI\}$ is finite.

(U) (uniform expansion) There exists $\rho > 1$ such that $|f'| \geq \rho$ on $\bigcup I$.

A standard reference for such maps is [17] (see also [3]). Since AFU maps are not necessarily Markov, the Hölder spaces $F$ are not preserved by the transfer operator of $f$ and it is standard to consider the space of bounded variation functions. Accordingly, we define the Banach space $B = BV \subset L^\infty$ to consist of functions $\phi : \Lambda \to \mathbb{R}$ such that $\text{Var} \phi < \infty$ with norm $\|\phi\| = |\phi|_\infty + \text{Var} \phi$. Here

$$\text{Var} \phi = \sup_{0 = z_0 < \cdots < z_k = 1} \sum_{i=1}^k |\phi(z_i) - \phi(z_{i-1})|$$
denotes the variation of \( \psi \) on \( \Lambda \). Also, we let \( \text{Var}_I \psi \) denote the variation of \( \psi \) on \( I \).

We suppose that \( f : \Lambda \to \Lambda \) is topologically mixing. Then there is a unique absolutely continuous \( f \)-invariant probability measure \( \mu \), and \( \mu \) is mixing.

**Proposition 3.13** Assume \( f \) is a topologically mixing AFU map and let \( v : \Lambda \to \mathbb{R} \) with \( \int_{\Lambda} v^2 \, d\mu = \infty \) and \( \sup_I \text{Var}_I v < \infty \). Fix \( \alpha \in (0, 1) \cup (1, 2] \) and assume that the tails of \( v \) satisfy (H1).

Then conditions (H1)–(H3) are satisfied with Banach space \( B = BV \).

**Proof** The proof essentially goes word for word as the proof of Proposition 3.11 with minor changes. Condition (H1) is satisfied by assumption and condition (H3) is well-known for mixing AFU maps. It remains to verify that (H2) holds. Fix \( M > 0 \).

As before, we verify (H2)(iii) for all \( s \in [-M, M], h \in [0, 1] \); the other calculations being simpler.

Again, \( (R'(s + h) - R'(s)) \phi = iR(\phi \psi) \) where \( \psi = ve^{i\theta v} - 1 \), and a standard calculation shows that
\[
\|R(\psi)\| \ll \|\sum_I \mu(I)(\sup_I |\psi| + \text{Var}_I \psi)\|.
\]

Also, \( \text{Var}_I e^{i\theta v} \leq |s| \text{Var}_I v \ll |s| \), so
\[
\|R'(s + h) - R'(s)\| \leq S_1 + S_2 + S_3 + S_4
\]
\[
S_1 = \sum_I \mu(I) \sup_I |v(e^{ihv} - 1)|
\]
\[
S_2 = \sum_I \mu(I) \sup_I |v(e^{ihv} - 1)| \text{Var}_I v^{i\theta} \ll |s| S_1 \leq MS_1,
\]
\[
S_3 = \sum_I \mu(I) \sup_I |v| \text{Var}_I e^{ihv} \leq h \sum_I \mu(I) \sup_I |v|,
\]
\[
S_4 = \sum_I \mu(I) \sup_I |e^{ihv} - 1| \text{Var}_I v \ll S_3.
\]

The calculation continues exactly as in Proposition 3.11 and we omit the remaining details. \( \blacksquare \)

### Appendix

#### Uniform local limit theorem

In this appendix, we sketch the proof of Proposition 3.1. Since the nonlattice case is covered (for one-sided stable laws with \( \alpha \in (0, 1) \)) in [15, Theorem 2.7]) we focus here on the lattice case. As should be clear from the proof, there is no difference in the arguments for one-sided and two-sided, and likewise for \( \alpha \in (0, 1) \) and \( \alpha \in (1, 2] \).

In general \( b_n \neq 0 \), so we redefine \( R(\theta) \phi = R(e^{i\theta} \tilde{v}) \phi \) where \( \tilde{v} = v \) for \( \alpha \in (0, 1) \) and \( \tilde{v} = v - \int_{\Lambda} v \, d\mu \) for \( \alpha \in (1, 2] \). In particular, \( \nu_n = \tilde{v}_n + b_n \).

Hypotheses (H1)–(H3) ensure as before that (3.1), (3.2) and Lemma 3.6(i) hold for some \( \epsilon, c, C > 0, \delta_0 \in (0, 1) \). Moreover,
\[
\lim_{n \to \infty} \lambda(\theta/a_n)^n = E(e^{i\theta Y}) \quad \text{for all } \theta \in \mathbb{R}. \tag{A.1}
\]

\(^2\)There are inaccuracies in the formulas for \( g \) and \( \lambda \) in [13] but the proof is unchanged with the correct formulas; all that matters is that (A.1) holds, which is the case by [2, 3].
(This is the main step in the proof of the distributional limit theorems in [2, 3].) In addition, there exists $\gamma > 0$ such that

$$\|P(\theta) - P(0)\| \leq C|\theta|^\gamma \text{ for } |\theta| \leq \epsilon. \quad (A.2)$$

(This is contained in Lemma 3.5) Finally, assuming (H4), for any $M > 0$, there exists $C > 0$, $\delta_0 \in (0, 1)$ such that

$$\|R(\theta)^n\| \leq C\delta_0^n \text{ for } \epsilon \leq |\theta| \leq M. \quad (A.3)$$

As we now show, the uniform local limit theorem follows from the 6 properties listed above.

**Proof of Proposition 3.1**  
Note that

$$\int_A 1_{A \cap f^{-n}B} e^{i\theta v_n} d\mu = e^{i\theta b_n} \int_A 1_B R^n(e^{i\theta n} 1_A) d\mu = e^{i\theta b_n} \int_B R(\theta)^n 1_A d\mu.$$ 

Hence

$$a_n \mu (z \in A \cap f^{-n}B : v_n(z) - [b_n] = N) = \frac{a_n}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta(N + [b_n] - b_n)} \int_B R(\theta)^n 1_A d\mu d\theta = I_1 + I_2 + I_3$$

where using (3.1),

$$I_1 = \frac{a_n}{2\pi} \int_{-\epsilon}^{\epsilon} e^{-i\theta(N + [b_n] - b_n)} \int_B \lambda(\theta)^n P(0) 1_A d\mu d\theta,$$

$$I_2 = \frac{a_n}{2\pi} \int_{-\epsilon}^{\epsilon} e^{-i\theta(N + [b_n] - b_n)} \int_B \lambda(\theta)^n (P(\theta) - P(0)) 1_A d\mu d\theta,$$

$$I_3 = \frac{a_n}{2\pi} \int_{|\theta| \leq \pi} e^{-i\theta(N + [b_n] - b_n)} \int_B R(\theta)^n 1_A d\mu d\theta.$$

By (A.3), $|I_3| \leq a_n \|1_A\| \mu(B) \int_{|\theta| \leq \pi} \|R(\theta)^n\| d\theta \ll a_n \delta_0^n$. Also,

$$|I_2| \leq a_n \|1_A\| \mu(B) \int_{-\epsilon}^{\epsilon} |\lambda(\theta)|^n \|P(\theta) - P(0)\| d\theta \ll a_n \int_0^\epsilon \exp\left\{-nc\theta^\alpha \ell(1/\theta)\right\} \theta^\gamma d\theta \ll a_n^{-\gamma}$$

by (A.2) and Lemmas 3.6(i) and 2.3. Hence $I_2 \to 0$ and $I_3 \to 0$ uniformly in $N$.

Finally, recall that $g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\theta x} E(e^{i\theta Y}) d\theta$, so $I_1 - \mu(A) \mu(B) g((N + [b_n] - b_n)/a_n) = \mu(A) \mu(B) \frac{1}{2\pi} (J_1 + J_2)$ where

$$J_1 = \int_{-\alpha_n}^{\alpha_n} e^{-i\theta(N + [b_n] - b_n)/a_n} \left\{\lambda(\theta/a_n)^n - E(e^{i\theta Y})\right\} d\theta,$$

$$J_2 = -\int_{|\theta| \geq \alpha_n} e^{-i\theta(N + [b_n] - b_n)/a_n} E(e^{i\theta Y}) d\theta.$$
These satisfy

\[ |J_1| \leq \int_{-\alpha_n \epsilon}^{\alpha_n \epsilon} \lambda(\theta/a_n)^n - \mathbb{E}(e^{i\theta Y}) \, d\theta, \quad |J_2| \leq \int_{|\theta| \geq \alpha_n \epsilon} |\mathbb{E}(e^{i\theta Y})| \, d\theta. \]

Let \( \alpha' \in (0, \alpha) \). By Lemma 3.6(i) and Potter’s bounds, there exists \( c' > 0 \) such that \( |\lambda(\theta/a_n)|^n \leq e^{-c'|\theta|^\alpha} \) which is integrable. Hence \( J_1 \to 0 \) uniformly in \( N \) by (A.1) and the dominated convergence theorem. Also \( J_2 \to 0 \) uniformly in \( N \). This completes the proof.

References


