$\mathcal{J}_1$ convergence to Lévy processes for dynamical systems modelled by exponential Young towers

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Abstract

This pre-preprint will eventually be part of a paper on "Superdiffusive homogenisation". It is being made available now for easy reference in other work.

1 Introduction

Tyran-Kamińska [14, 15] initiated the study of convergence to stable Lévy processes for deterministic dynamical systems. In particular, necessary and sufficient conditions for convergence in the Skorohod $\mathcal{J}_1$ topology were given in the setting of Gibbs-Markov maps (uniformly expanding maps with a countable alphabet). This was extended in Jung, Péne & Zhang [8] to the case of dynamical systems that are nonuniformly hyperbolic with exponential tails in the sense of Young [17]. Vector-valued observables of Gibbs-Markov maps were considered by Chevyrev *et al.* [7].

The results in [8] are restricted to scalar observables. Also, their results are formulated specifically for dispersing billiards with flat cusps. In this paper, we extend to the case of vector-valued observables. At the same time, we formulate the results in an abstract setting to facilitate future applications.

Notation We write $a_n \ll b_n$ if there are constants $C > 0$, $n_0 \geq 1$ such that $a_n \leq C b_n$ for all $n \geq n_0$. As usual, $a_n = o(b_n)$ means that $a_n/b_n \to 0$ and and $a_n \sim b_n$ means that $a_n/b_n \to 1$.

Define the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\} \subset \mathbb{R}^d$ where $|\;|$ denotes Euclidean distance. We denote by $B_a(c)$ the open ball of radius a centred at c.

2 Setup

We consider dynamical systems (f, Σ, μ) that are nonuniformly hyperbolic with exponential tails in the sense of Young [17]. In particular, Σ is a metric space with

Borel probability measure μ, and $f : \Sigma \to \Sigma$ is a mixing measure-preserving transformation. Moreover, there is an inducing set $Y \subset \Sigma$ with $\mu(Y) > 0$ and a return time $\tau : Y \to \mathbb{Z}^+$ (not necessarily the first return time) such that $F = f^\tau$ maps Y into Y and the tail probabilities $\mu(\tau > n)$ decays exponentially with n. In addition, there is an at most countable partition $\{Y_j : j \geq 1\}$ such that Y_j is a union of stable leaves and that the quotient map $\bar{F} : \bar{Y} \to \bar{Y}$ is a full-branch Gibbs-Markov map with respect to the corresponding partition $\{\bar{Y}_j\}$.

We are interested in proving a statistical limit law for vector-valued observables satisfying certain properties described below.

First, we suppose that there is a function $R : \Sigma \to \mathbb{Z}^+$ constant on sets of the form $f^k Y_j$ for all $j \geq 1$ and $0 \leq k \leq (\tau|_{Y_j}) - 1$. We require that R is regularly varying with exponent $\alpha \in (1, 2)$. That is

$$\mu(R > t) \sim \ell(t) t^{-\alpha} \quad \text{as } t \to \infty \tag{2.1}$$

where $\ell : (0, \infty) \to (0, \infty)$ is a slowly varying function.

Introduce $b_n > 0$ such that $b_n^\alpha \sim n\ell(b_n)$ as $n \to \infty$. Given $e > 0$, define $D_n^e = \{b_n^{-1} R > e\}$. We require that for every $e > 0$, there exist $\theta_1 > 1$ and $C > 0$ such that

$$\mu(D_n^e \cap f^{-j} D_n^e) \leq C n^{-\theta_1} \quad \text{for all } 1 \leq j \leq n. \tag{2.2}$$

This assumption means that large values of R are not too clustered.

Let $\{\Sigma_i, i \in \mathcal{I}\}$ be a finite collection of disjoint subsets of Σ such that the Σ_i are unions of sets $f^k Y_j$ as above. We suppose that

$$\mu(R 1_{\Sigma_i} > t) \sim c_i \ell(t) t^{-\alpha} \quad \text{as } t \to \infty \tag{2.3}$$

for each $i \in \mathcal{I}$, where $c_i > 0$, and $\sum_{i \in \mathcal{I}} c_i = 1$.

Next, fix vectors $\omega_i \in \mathbb{R}^d \setminus \{0\}$ for $i \in \mathcal{I}$. Define

$$Z : \Sigma \to \mathbb{R}^d, \qquad Z = \sum_{i \in \mathcal{I}} \omega_i R 1_{\Sigma_i}.$$

We assume without loss that the ω_i are distinct. (Otherwise, combine the Σ_i and add the c_i corresponding to a common value of ω_i.) Then Z is regularly varying with spectral measure ν on $\mathbb{S}^{d-1}$ given by

$$\nu = \Big(\sum_{i \in \mathcal{I}} c_i |\omega_i|^\alpha \Big)^{-1} \sum_{i \in \mathcal{I}} c_i |\omega_i|^\alpha \delta_{\hat{\omega}_i},$$

where $\hat{\omega}_i = \omega_i / \omega_i \in \mathbb{S}^{d-1}$. This means that

$$\lim_{t \to \infty} \frac{\mu(|Z| > rt,\ Z/|Z| \in E)}{\mu(|Z| > t)} = r^{-\alpha} \nu(E) \tag{2.4}$$

for all $r > 0$ and all Borel sets $E \subset \mathbb{S}^{d-1}$ with $\nu(\partial E) = 0$.

Let G_α be the corresponding d-dimensional α-stable law with characteristic function

$$\mathbb{E}\, e^{is\cdot G_\alpha} = \exp\Big\{ -\int_{\mathbb{S}^{d-1}} |s\cdot x|^\alpha \Big(1 - i\,\mathrm{sgn}(s\cdot x)\tan\frac{\pi\alpha}{2}\Big)\cos\frac{\pi\alpha}{2}\,\Gamma(1-\alpha)\,d\nu(x)\Big\}$$

for $s \in \mathbb{R}^d$. Then Z is in the domain of attraction of G_α. That is, if $Z_1, Z_2, \dots$ are i.i.d. copies of Z, then $b_n^{-1}\big(\sum_{j=1}^n Z_j - n\int_\Sigma Z\,d\mu\big) \to_d G_\alpha$.

Let $\widetilde{Z} = Z - \int_\Sigma Z\,d\mu$. We define the sequence of processes

$$W_n^Z(t) = b_n^{-1}\sum_{j=0}^{[nt]-1} \widetilde{Z}\circ f^j, \quad t \in [0,1],$$

on (Σ,μ). Let L_α denote the α-stable Lévy process corresponding to the stable law G_α. We regard W_n^Z and L_α as random elements in the càdlàg space $D([0,1],\mathbb{R}^d)$. The strong $\mathcal{J}_1$ Skorohod topology [13, 16] on $D([0,1],\mathbb{R}^d)$ is metrized by

$$d(u_1,u_2) = \inf_\lambda \Big(\sup_{[0,1]}|u_1\circ\lambda - u_2| + \sup_{[0,1]}|\lambda - \mathrm{Id}|\Big).$$

Theorem 2.1. *Assume that $f : \Sigma \to \Sigma$ is mixing and nonuniformly hyperbolic with exponential tails in the sense of Young [17], and that conditions* (2.1) *to* (2.3) *are satisfied. Then $W_n^Z \to_w L_\alpha$ in the strong $\mathcal{J}_1$ topology.*

Remark 2.2. For a more general class of observables, we can consider integrable observables $V : \Sigma \to \mathbb{R}^d$ with $\int_\Sigma V\,d\mu = 0$ such that $H = V - \widetilde{Z}$ has the property that

$$b_n^{-1}\max_{k\le n}\Big|\sum_{j=0}^{k-1} H\circ f^j\Big| \to_p 0 \quad \text{on } (X,\mu).$$

Define

$$W_n^V(t) = b_n^{-1}\sum_{j=0}^{[nt]-1} V\circ f^j, \quad t\in[0,1].$$

Then it is immediate that $W_n^V \to_w L_\alpha$ in the strong $\mathcal{J}_1$ topology.

It is standard, see for example [3, 7, 10, 11], that in many situations of interest, Hölder observables lead to a first return observable V satisfying these assumptions.

3 Preliminaries about Gibbs-Markov maps and Young towers

Recall that $F = f^\tau : Y \to Y$ is a full-branch Gibbs-Markov map with ergodic invariant probability measure μ_Y. For standard facts about Gibbs-Markov maps, we refer to [1, 2].

Proposition 3.1. *Let $g \in L^2(Y)$ with $\int_Y g\,d\mu_Y = 0$, and suppose that g is constant on partition elements. Then*

$$\Big|\max_{1\le k\le n}\Big|\sum_{j=0}^{k-1} g\circ F^j\Big|\Big|_{L^2(Y)} \le Cn^{1/2}|g|_{L^2(Y)}$$

where $C>0$ is a constant independent of g and n.

Proof. Let $\{Y_j : j\ge 1\}$ be the partition for F. For $\theta\in(0,1)$, we define the symbolic metric $d_\theta(y,y') = \theta^{s(y,y')}$ where $s(y,y')$ is the least integer $n\ge 0$ such that $F^n y$ and $F^n y'$ lie in distinct partition elements. Given $v : Y\to\mathbb{R}$ continuous, we define $\|v\|_\theta = \|v\|_\infty + |v|_\theta$ where $|v|_\theta = \sup_{y\ne y'}|v(y)-v(y')|/d_\theta(y,y')$.

Define the transfer operator $P : L^1(Y)\to L^1(Y)$ (so $\int_Y Pv\,w\,d\mu_Y = \int_Y v\,w\circ F\,d\mu_Y$ for $v\in L^1(Y)$, $w\in L^\infty(Y)$). Then $(Pv)(y) = \sum_j p(y_j)v(y_j)$ where y_j is the unique preimage of y under $F|Y_j$. There exists $\theta\in(0,1)$, fixed from now on, and $C>0$, such that $0<p(y)\le C\mu(Y_j)$ and $|p(y)-p(y')|\le C\mu(Y_j)d_\theta(y,y')$ for all $y,y'\in Y_j$, $j\ge 1$. It follows easily that $\|Pg\|_\theta \ll |g|_{L^1(Y)} \le |g|_{L^2(Y)}$.

There exist constants $\gamma\in(0,1)$, $C>0$ such that $\|P^n v\|_\theta \ll \gamma^n\|v\|_\theta$ for all continuous $v : Y\to\mathbb{R}$ with $\int_Y v\,d\mu_Y = 0$ and all $n\ge 1$. Define $\chi = \sum_{n=1}^\infty P^n g$. Then $\|\chi\|_\infty \le \sum_{n=1}^\infty \|P^{n-1}Pg\|_\theta \ll \|Pg\|_\theta \ll |g|_{L^2(Y)}$.

Now define the martingale-coboundary decomposition

$$g = m + \chi\circ F - \chi$$

where $|m|_{L^2(Y)} \ll |g|_{L^2(Y)}$. Since $m\in\ker P$, it follows easily that $\big|\sum_{j<n} m\circ F^j\big|_{L^2(Y)} = n^{1/2}\|m|_{L^2(Y)}$. Moreover, $\{m\circ F^j,\ j\ge 0\}$ is a sequence of reverse martingale differences, so by Doob's inequality, $\big|\max_{k\le n}|\sum_{j<k} m\circ F^j|\big|_{L^2(Y)} \le 2n^{1/2}|m|_{L^2(Y)}$. Hence

$$\Big|\max_{k\le n}\Big|\sum_{j<k} g\circ F^j\Big|\Big|_{L^2(Y)} \le 2n^{1/2}|m|_{L^2(Y)} + 2|\chi|_{L^\infty(Y)} \ll n^{1/2}|g|_{L^2(Y)}$$

as required. ■

Define the Young tower $f_\Delta : \Delta\to\Delta$,

$$\Delta = \{(y,\ell)\in Y\times\mathbb{Z} : 0\le \ell<\tau(y)\}, \qquad f_\Delta(y,\ell) = \begin{cases}(y,\ell+1) & \ell\le\tau(y)-2\\ (Fy,0) & \ell=\tau(y)-1\end{cases}$$

with ergodic f_Δ-invariant probability measure $\mu_\Delta = (\mu_Y\times\text{counting})/\bar\tau$ where $\bar\tau = \int_Y \tau\,d\mu_Y$.

For $n\ge 1$, define the lap number $N_n : \Delta\to\mathbb{Z}^+$ to be the integer satisfying

$$\sum_{j=0}^{N_n(x)-1}\tau(F^j y) \le n+\ell < \sum_{j=0}^{N_n(x)}\tau(F^j y) \tag{3.1}$$

for $x=(y,\ell)\in\Delta$.

Proposition 3.2. *There exists $C > 0$ such that $\big| \max_{1\le k\le n} |N_k - k\bar\tau^{-1}|\big|_{L^2(\Delta)} \le Cn^{1/2}$ for all $n \ge 1$.*

Proof. Write $\tau_k = \sum_{j=0}^{k-1} \tau\circ F^j$. Then, for $x = (y,\ell) \in \Delta$,

$$\tau_{N_n(x)}(y) - \tau(y) \le \sum_{j=0}^{N_n(x)-1} \tau(F^j y) - \ell \le n \le \sum_{j=0}^{N_n(x)} \tau(F^j y) - \ell \le \tau_{N_n(x)}(y) + \tau(F^{N_n(x)}y).$$

Hence,

$$\begin{aligned} |n\bar\tau^{-1} - N_n(x)| &\le \bar\tau^{-1}|\tau_{N_n(x)}(y) - \bar\tau N_n(x)| + \bar\tau^{-1}\max\{\tau(y), \tau(F^{N_n(x)}y)\} \\ &\le \bar\tau^{-1}\max_{0\le k\le n}|\tau_k(y) - k\bar\tau| + \bar\tau^{-1}\max_{0\le k\le n}\tau(F^k y) \end{aligned}$$

where we used that $N_n \le n$. Now, $|\max_{0\le k\le n}\tau(F^k y)|_{L^2(\Delta)} \le n^{1/2}|\tau|_{L^2(\Delta)}$. Applying Proposition 3.1 with $g = \tau - \bar\tau$,

$$\big|\max_{k\le n} |\tau_{N_k(x)}(y) - \bar\tau N_k(x)|\big|_{L^2(\Delta)} \ll n^{1/2}.$$

The result follows. ■

Let $\mathcal{A}$ consist of subsets of Σ that are unions of partition elements. Let $\mathcal{B}$ consist of subsets of Σ that are unions of local stable leaves (in other words 1_B is constant along local stable leaves for such subsets B). Clearly, $\mathcal{A} \subset \mathcal{B}$.

Theorem 3.3. *Suppose that f is mixing. Then there exist constants $C > 0$, $\gamma \in (0,1)$ such that*

$$\big|\mu\big(A\cap f^{-n}B) - \mu(A)\mu(B)| \le C\gamma^n$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $n \ge 1$.

Proof. In general, the Young towers associated to f are mixing only up to a finite cycle. However, since f is mixing, we can reduce by [5, Section 10] (see also [4, Section 4.1]) to the case when $f_\Delta : \Delta \to \Delta$ is mixing.

The observables 1_A and 1_B on Σ lift to observables on the two-sided tower. Since $A, B \in \mathcal{B}$, the observables project to observables on the one-sided tower obtained by quotienting stable leaves. Moreover, 1_A is dynamically Hölder. By Young [17],

$$\big|\mu\big(A\cap f^{-n}B) - \mu(A)\mu(B)| \ll \|1_A\|\, |1_B|_\infty\, \gamma^n.$$

The result follows since $\|1_A\| = |1_A|_\infty \le 1$ and $|1_B|_\infty \le 1$. ■

4 Proof of Theorem 2.1

In this section, we prove Theorem 2.1. We largely follow the approach in [8, Section 4] which was written specifically for billiards with flat cusps in the case $d = 1$. Our verification of Condition II below is more dynamical than the probabilistic argument in [8].

Let $\mathcal{N}_n = \sum_{j=1}^n \delta_{(\frac{j}{n}, b_n^{-1}\widetilde{Z}\circ f^{j-1})}$, where $\widetilde{Z} = Z - \int_\Sigma Z\,d\mu$. Then $\mathcal{N}_n$ is a random point process on $(0,\infty)\times(\mathbb{R}^d\setminus\{0\})$.

The Lévy measure Π corresponding to the Lévy process L_α is given by

$$\Pi(B) = \alpha\int_{\mathbb{S}^{d-1}}\int_0^\infty 1_B(rx)r^{-\alpha-1}\,dr\,d\nu(x).$$

Let $\mathcal{N}$ be the Poisson point process on $(0,\infty)\times(\mathbb{R}^d\setminus\{0\})$ with mean measure Leb $\times\Pi$.

By [14, Theorem 4.1] (see also [15, Theorem 1.2] and [8, Proposition 4.4]), to prove Theorem 2.1 it is enough to verify two conditions:

Condition I (Point process convergence). $\mathcal{N}_n \to_w \mathcal{N}$ as $n\to\infty$ in the space of point measures defined on $(0,\infty)\times(\mathbb{R}^d\setminus\{0\})$.

Condition II (Vanishing small values). For every $\gamma > 0$,

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\mu\biggl(\max_{1\le k\le n}\Bigl|\sum_{j=0}^{k-1}\bigl(\widetilde{Z}1_{\{|\widetilde{Z}|<b_n\epsilon\}}\bigr)\circ f^j - k\int_\Sigma \widetilde{Z}1_{\{|\widetilde{Z}|<b_n\epsilon\}}\,d\mu\Bigr| > b_n\gamma\biggr) = 0.$$

These conditions are verified in the next two subsections.

4.1 Point process convergence

In this subsection, we verify Condition I. We follow [8, Section 4.5] using [12, Theorem 2.1]. It is enough to prove convergence of $\mathcal{N}_n$ to $\mathcal{N}$ on

$$(0,\infty)\times U, \qquad U = \mathbb{R}^d\setminus\overline{B_{a_0}(0)}$$

for each fixed $a_0 > 0$.

Fix a_0 and let

$$A_n = \{|\widetilde{Z}| > a_0 b_n\}.$$

Let $\mathcal{W}$ be the ring of subsets of U generated by sets of the type $\{x\in\mathbb{R}^d : a < |x| < a',\ x/|x|\in E\}$, where $a_0 < a < a'$ and $E\subset\mathbb{S}^{d-1}$ is open with $\nu(\partial E) = 0$. Note that $\mathcal{W}$ generates the Borel sigma-algebra on U and that $\Pi(\partial W) = 0$ for all $W\in\mathcal{W}$.

For a collection $\mathcal{F}$ of measurable subsets of Σ, define

$$\mathcal{Q}_p(\mathcal{F}) = \sup_{\substack{A\in\mathcal{F}\\ B\in\sigma(\bigcup_{j\ge p} f^{-j}\mathcal{F})}}\bigl|\mu(A\cap B) - \mu(A)\mu(B)\bigr|.$$

Let $J_n = b_n^{-1}\widetilde{Z}$. By [12, Theorem 2.1], to prove Condition I it suffices to prove the following:

Lemma 4.1. *(a)* $\lim_{n\to\infty} \mu\big(J_n^{-1}W|A_n\big) = \Pi(W|U)$ *for all* $W \in \mathcal{W}$.

(b) $\mathcal{Q}_1(J_n^{-1}\mathcal{W}_0) = o(\mu(A_n))$ *for every finite subset* $\mathcal{W}_0$ *of* $\mathcal{W}$.

Proof of Lemma 4.1(a). Without loss, we may suppose that $W = \{x \in \mathbb{R}^d : a < |x| < a',\ x/|x| \in E\}$ where $a_0 < a < a'$ and $E \subset \mathbb{S}^{d-1}$ is open with $\nu(\partial E) = 0$. Since $Z - \widetilde{Z}$ is constant, it follows from (2.4) and the definition of Π that

$$\lim_{n\to\infty} \mu\big(J_n^{-1}W|A_n\big) = \lim_{n\to\infty} \frac{\mu\big(|\widetilde{Z}| \in (ab_n, a'b_n),\ \widetilde{Z}/|\widetilde{Z}| \in E\big)}{\mu(|\widetilde{Z}| > a_0 b_n)}$$

$$= \lim_{n\to\infty} \frac{\mu\big(|Z| \in (ab_n, a'b_n),\ Z/|Z| \in E\big)}{\mu(|Z| > a_0 b_n)} = a_0^{\alpha}(a^{-\alpha} - (a')^{-\alpha})\nu(E).$$

On the other hand,

$$\Pi(W|U) = \frac{\Pi(W)}{\Pi(U)} = \frac{(a^{-\alpha} - (a')^{-\alpha})\nu(E)}{a_0^{-\alpha}},$$

so the result is proved. ■

Next, we prove Lemma 4.1(b).

Proposition 4.2. *There is a constant* $C > 0$ *such that* $\mu(A_n) \sim Cn^{-1}$ *as* $n \to \infty$.

Proof. Since $Z - \widetilde{Z}$ is constant, it suffices to redefine $A_n = \{|Z| > a_0 b_n\}$.

For $i \in \mathcal{I}$, let $a_i = a_0/|\omega_i|$ and $c_i' = c_i a_i^{-\alpha}$. By (2.3) and using that ℓ is slowly varying,

$$\mu(R1_{\Sigma_i} > a_i b_n) = c_i'\ell(a_i b_n) b_n^{-\alpha} \sim c_i'\ell(a_i b_n)\ell(b_n)^{-1} n^{-1} \sim c_i n^{-1}.$$

Hence $\mu(A_n) = \sum_{i\in\mathcal{I}} \mu(R1_{\Sigma_i} > a_i b_n) \sim Cn^{-1}$ with $C = \sum_{i\in\mathcal{I}} c_i'$. ■

Let $\theta_1 > 1$ be as in (2.2). Without loss, we can suppose that $\theta_1 \in (1,2)$.

Proposition 4.3. *There exists* $C > 0$ *such that* $\mu(A_n \cap f^{-k}A_n) \le Cn^{-\theta_1}$ *for all* $k, n \ge 1$.

Proof. By Theorem 3.3, $\big|\mu(A_n \cap f^{-k}A_n) - \mu(A_n)^2\big| \ll \gamma^k$ for all $k \ge 1$. By Proposition 4.2, $\mu(A_n)^2 \ll n^{-2} \le n^{-\theta_1}$. Hence $\mu(A_n \cap f^{-k}A_n) \ll n^{-\theta_1}$ uniformly in $k \ge n$.

Set $e = a_0/(2\sum_{i\in\mathcal{I}} |\omega_i|)$. Then

$$A_n \subset \{2|Z| > a_0 b_n\} \subset \{R > e b_n\} = D_n^e$$

with D_n^e as in (2.2). Hence $A_n \cap f^{-k}A_n \subset D_n^e \cap f^{-k}D_n^e$ and, by (2.2), $\mu(A_n \cap f^{-k}A_n) \ll n^{-\theta_1}$ uniformly in $1 \le k \le n$.

Combining the estimates for $k \le n$ and $k \ge n$ yields the desired result. ■

Now fix $\mathcal{W}_0 \subset \mathcal{W}$ finite, and let $\mathcal{Q}_{n,p} = \mathcal{Q}_p(J_n^{-1}\mathcal{W}_0)$.

Proposition 4.4. *$\mathcal{Q}_{n,p} \le C\gamma^p$ for all $n, p \ge 1$.*

Proof. Let $A \in J_n^{-1}\mathcal{W}_0$, $B \in \sigma\big(\bigcup_{j\ge p} f^{-j}J_n^{-1}\mathcal{W}_0\big)$. Write $B = f^{-p}B'$ where $B' \in \sigma\big(\bigcup_{j\ge 0} f^{-j}J_n^{-1}\mathcal{W}_0\big)$. Then $|\mu(A\cap B) - \mu(A)\mu(B)| \ll \gamma^p$ by Theorem 3.3. ■

Let $\tau_{A_n}(x) = \min\{n \ge 1 : f^n(x) \in A_n\}$ for $x \in \Sigma$.

Lemma 4.5. *$\mathcal{Q}_{n,1} \le \mathcal{Q}_{n,p+1} + \mu(A_n \cap \{\tau_{A_n} \le p\}) + \mu(A_n)\mu(\tau_{A_n} \le p)$ for all $n, p \ge 1$.*

Proof. This is identical to [8, Lemma 4.9]. We give the short argument for completeness.

Let $A \in J_n^{-1}\mathcal{W}_0$ and $B \in \sigma\big(\bigcup_{j\ge 1} f^{-j}J_n^{-1}\mathcal{W}_0\big)$. Note that $A \subset A_n$ by definition of $\mathcal{W}$.

Suppose that $\mathcal{W}_0 = \{W_1, \dots, W_K\}$. Observe that there exists a function $g : (\{0,1\}^K)^{\mathbb{N}} \to \{0,1\}$ such that $1_B = g(Y_1, Y_2, \dots)$, where $Y_i = \big(1_{J_n^{-1}W_1}, \dots, 1_{J_n^{-1}W_K}\big) \circ f^i$. Define $B' \in \sigma\big(\bigcup_{j\ge p+1} f^{-j}(J_n^{-1}\mathcal{W}_0)\big)$ by

$$1_{B'} = g(0, \dots, 0, Y_{p+1}, Y_{p+2}, \dots).$$

Then $|\operatorname{Cov}(1_A, 1_{B'})| \le \mathcal{Q}_{n,p+1}$. Moreover, $|1_B - 1_{B'}| \le 1_{\{\tau_{A_n}\le p\}}$, so

$$\begin{aligned}\big|\operatorname{Cov}(1_A, 1_B) - \operatorname{Cov}(1_A, 1_{B'})\big| &\le \mu\big(A \cap \{\tau_{A_n} \le p\}\big) + \mu(A)\mu(\tau_{A_n} \le p) \\ &\le \mu(A_n \cap \{\tau_{A_n} \le p\}) + \mu(A_n)\mu(\tau_{A_n} \le p).\end{aligned}$$

The result follows. ■

Proof of Lemma 4.1(b). Let $p_n = [n^{\theta_2}]$ where $0 < \theta_2 < \theta_1 - 1 < 1$. Applying Proposition 4.3,

$$\begin{aligned}\mu(A_n \cap \{\tau_{A_n} \le p_n\}) &= \mu\Big(A_n \cap \bigcup_{k=1}^{p_n} f^{-k}A_n\Big) \\ &\le \sum_{k=1}^{p_n} \mu(A_n \cap f^{-k}A_n) \ll p_n n^{-\theta_1} = o(n^{-1}).\end{aligned}$$

Using Proposition 4.2 and invariance of μ under f,

$$\mu(\tau_{A_n} \le p_n) = \mu\Big(\bigcup_{k=1}^{p_n} f^{-k}A_n\Big) \le \sum_{k=1}^{p_n} \mu(f^{-k}A_n) = p_n\mu(A_n) \ll p_n n^{-1} = o(1).$$

Hence it follows from Proposition 4.2 and Lemma 4.5 that $\mathcal{Q}_{n,1} \le \mathcal{Q}_{n,p_n+1} + o(\mu(A_n))$. By Propositions 4.2 and 4.4, $\mathcal{Q}_{n,p_n+1} = o(n^{-1}) = o(\mu(A_n))$, so $\mathcal{Q}_{n,1} = o(\mu(A_n))$. ■

4.2 Vanishing small values

In this subsection, we verify Condition II. It is convenient to work at the level of the tower $f_\Delta : \Delta \to \Delta$. Define the measure-preserving semiconjugacy $\pi : \Delta \to \Sigma$, $\pi(y,\ell) = f^\ell y$. Let $\Delta_i = \pi^{-1}\Sigma_i$ $i \in \mathcal{I}$. Abusing notation, we denote lifted observables $R \circ \pi : \Delta \to \mathbb{Z}^+$, $\widetilde{Z} \circ \pi : \Delta \to \mathbb{R}^d$ and so on simply by R, $\widetilde{Z}$ and so on.

Fix $i \in \mathcal{I}$ and set $R^{(i)} = R1_{\Delta_i}$. Let

$$\widetilde{R}^{(i)}_\xi = R^{(i)} 1_{\{\omega_i R \in B_\xi(\bar{Z})\}} - \int_\Delta R^{(i)} 1_{\{\omega_i R \in B_\xi(\bar{Z})\}}\, d\mu_\Delta, \quad \xi > 0.$$

The main step in the proof is:

Lemma 4.6. *There is a constant $C > 0$ such that*

$$\limsup_{n\to\infty} b_n^{-1} \Big| \max_{1\le k\le n} \Big| \sum_{j=0}^{k-1} \widetilde{R}^{(i)}_{\epsilon b_n} \circ f^j_\Delta \Big| \Big|_{L^1(\Delta)} \le C\epsilon^{1-\alpha/2} \quad \textit{for all } \epsilon > 0.$$

Most of the remainder of this subsection is concerned with proving Lemma 4.6. At the end of the subsection, we show that Condition II follows from the lemma.

We begin with the following consequence of condition (2.2).

Proposition 4.7. *Let $\theta_1 > 1$ be as in condition* (2.2) *and let $\beta \in (0,\alpha)$. Then $\mu_\Delta(R > n$ and $R \circ f^j > n) \ll n^{-\beta\theta_1}$ for all $1 \le j \le n^\beta$.*

Proof. Recall that $b_n^\alpha \sim n\ell(b_n)$ so there exists $K > 0$ such that $b_n \le Kn^{1/\beta}$ for all n. Taking $e = 1/K$ in (2.2),

$$\begin{aligned} &\mu_\Delta(R > n^{1/\beta} \text{ and } R \circ f^j > n^{1/\beta}) \\ &\qquad \le \mu_\Delta(R > K^{-1}b_n \text{ and } R \circ f^j > K^{-1}b_n) \ll n^{-\theta_1}, \end{aligned}$$

for all $1 \le j \le n$. The result follows. ∎

Let $1 \le \eta \le \xi$. Define $H = H_\xi : Y \to [0,\infty)$, $s = s_\eta : Y \to \mathbb{Z}$,

$$H(y) = \sum_{0\le j\le\tau(y)-1} (R^{(i)} 1_{\{\omega_i R \in B_\xi(\bar{Z})\}})(y,j), \quad s(y) = \#\{0 \le j \le \tau(y)-1 : R(y,j) > \eta\}.$$

Note that $H = H' + H'' + H'''$, where

$$\begin{aligned} H'(y) &= 1_{\{s(y)=1\}} \sum_{j<\tau(y)} (R^{(i)} 1_{\{R>\eta\}} 1_{\{\omega_i R \in B_\xi(\bar{Z})\}})(y,j), \\ H''(y) &= \sum_{j<\tau(y)} (R^{(i)} 1_{\{R\le\eta\}} 1_{\{\omega_i R \in B_\xi(\bar{Z})\}})(y,j), \\ H'''(y) &= 1_{\{s(y)\ge2\}} \sum_{j<\tau(y)} (R^{(i)} 1_{\{R>\eta\}} 1_{\{\omega_i R \in B_\xi(\bar{Z})\}})(y,j). \end{aligned}$$

Proposition 4.8. *Let $\delta > 0$. There exists $M > 0$ and $C > 0$ such that*

(a) $|H'|_{L^2(Y)} \le C\ell(\xi)^{1/2}\xi^{1-\alpha/2}$,

(b) $|H''|_{L^2(Y)} \le C\eta^{1-\alpha/2+\delta}$,

(c) $|H'''|_{L^2(Y)} \le C\xi^{1+\delta}\eta^{\delta-\alpha\theta_1/2}$,

for all $1 \le \eta \le \xi$ satisfying $M\log\xi \le \eta^{\alpha/2}$ and $\xi > |\bar{Z}|$.

Proof. We use throughout that $R^{(i)} \le R$ for all i.
(a) Observe that $H' \le \xi_i$ where $\xi_i = |\omega_i|^{-1}(\xi + |\bar{Z}|)$. Also for $0 \le n \le \xi_i$,

$$\begin{aligned}\mu_Y(H' = n) &\le \mu_Y(y \in Y : R(y,j) = n \text{ for some } 0 \le j < \tau(y)\}) \\ &\le \int_Y \sum_{j=0}^{\tau(y)-1} 1_{\{y\in Y: R(y,j)=n\}}\, d\mu_Y = \bar{\tau}\mu_\Delta(R = n).\end{aligned}$$

Hence

$$|H'|^2_{L^2(Y)} = \sum_{n\le\xi_i} n^2\mu_Y(H' = n) \le \bar{\tau}\sum_{n\le\xi_i} n^2\mu_\Delta(R = n) \ll \sum_{n\le\xi_i} n\mu_\Delta(R \ge n).$$

Using (2.3) and applying Karamata's inequality, we conclude that $|H'|^2_{L^2(Y)} \ll \ell(\xi_i)\xi_i^{2-\alpha} \ll \ell(\xi)\xi^{2-\alpha}$ as required.
(b) Let $q > 2$, $\epsilon > 0$ and observe that $|R1_{\{R\le\eta\}}|^q_{L^q(\Delta)} \ll \sum_{j\le\eta} j^{q-1}\mu_\Delta(R \ge j) \ll \eta^{q+\epsilon-\alpha}$. Let $g_j(y) = (R1_{\{R\le\eta\}}) \circ f^j_\Delta(y, 0)$. Since f_Δ is measure-preserving,

$$\begin{aligned}\Big|\sum_{j<k} g_j\Big|^q_{L^q(Y)} &= \int_Y \Big|\sum_{j<k} g_j\Big|^q d\mu_Y \le \int_Y \sum_{\ell=0}^{\tau(y)-1}\Big|\sum_{j<k}(R1_{\{R\le\eta\}}) \circ f^j_\Delta(y,\ell)\Big|^q d\mu_Y(y) \\ &= \bar{\tau}\int_\Delta \Big|\sum_{j<k}(R1_{\{R\le\eta\}}) \circ f^j_\Delta\Big|^q d\mu_\Delta \le \bar{\tau}k^q|R1_{\{R\le\eta\}}|^q_{L^q(\Delta)} \ll k^q\eta^{q+\epsilon-\alpha}.\end{aligned}$$

Note also that $H'' \le \sum_{j<\tau} g_j$. Since τ has exponential tails, there exists $c_0 > 0$ such that

$$\begin{aligned}|H''|^2_{L^2(Y)} &\le \sum_{k=1}^\infty \Big|1_{\{\tau=k\}}\sum_{j<k} g_j\Big|^2_{L^2(Y)} \ll \sum_{k=1}^\infty e^{-c_0k}\Big|\sum_{j<k} g_j\Big|^2_{L^q(Y)} \\ &\ll \sum_{k=1}^\infty e^{-c_0k}(k\eta^{(q+\epsilon-\alpha)/q})^2 \ll \eta^{2(q+\epsilon-\alpha)/q}.\end{aligned}$$

The desired estimate follows for q and ϵ sufficiently close to 2 and 0.

(c) Since τ has exponential tails, there exists $c_1 > 0$ such that $k^2\mu_Y(\tau = k) \ll e^{-c_1 k}$. Let

$$M = \alpha\theta_1/c_1, \qquad \epsilon \in (0, \alpha/2).$$

Suppose that $1 \le \eta \le \xi$ satisfies $M \log \xi \le \eta^{\alpha/2}$. In particular, $M \log \xi \le \eta^{\alpha-\epsilon}$. For any $y \in \{\tau = k,\, s \ge 2\}$, there exist $0 \le j_1 < j_2 < k$ such that $R(y, j_1) > \eta$ and $R(y, j_2) > \eta$. By Proposition 4.7,

$$\begin{aligned}\mu_Y(\tau = k,\ s \ge 2) &\le \bar\tau \sum_{0\le j_1<j_2<k} \mu_\Delta(R \circ f_\Delta^{j_1} > \eta \text{ and } R \circ f_\Delta^{j_2} > \eta) \\ &\le k\bar\tau \sum_{1\le j<k} \mu_\Delta(R > \eta \text{ and } R \circ f_\Delta^{j} > \eta\} \ll k^2 \eta^{-(\alpha-\epsilon)\theta_1},\end{aligned}$$

for all $k \le \eta^{\alpha-\epsilon}$ and hence for $k \le M \log \xi$.

Noting that $H'''(y) \ll 1_{\{s(y)\ge 2\}}\, \xi\tau(y)$,

$$\begin{aligned}|H'''|^2_{L^2(Y)} &\ll \xi^2 \int_Y 1_{\{s\ge 2\}}\, \tau^2 \, d\mu_Y \\ &\ll \xi^2 \sum_{k\le M\log\xi} k^2\mu_Y(\tau = k,\ s \ge 2) + \xi^2 \sum_{k>M\log\xi} k^2\mu_Y(\tau = k).\end{aligned}$$

Now,

$$\begin{aligned}\sum_{k\le M\log\xi} k^2\mu_Y(\tau = k,\ s \ge 2) &\ll \sum_{k\le M\log\xi} k^4\eta^{-(\alpha-\epsilon)\theta_1} \ll (\log\xi)^5\eta^{-(\alpha-\epsilon)\theta_1}, \\ \sum_{k>M\log\xi} k^2\mu_Y(\tau = k) &\ll \sum_{k>M\log\xi} e^{-c_1 k} \ll \xi^{-c_1 M} \le \eta^{-c_1 M} = \eta^{-\alpha\theta_1}.\end{aligned}$$

Hence, $|H'''|^2_{L^2(Y)} \ll \xi^2(\log\xi + 1)^5\eta^{-(\alpha-\epsilon)\theta_1}$ and the result follows for ϵ sufficiently small. ■

Corollary 4.9. *There is a constant $C > 0$ such that*

$$\limsup_{n\to\infty} b_n^{-1} n^{1/2} \|H_{\epsilon b_n}\|_{L^2(Y)} \le C\epsilon^{1-\alpha/2} \quad \textit{for all } \epsilon > 0.$$

Proof. Recall that $\theta_1 > 1$. Choose $\omega \in (0, 1/\alpha)$ sufficiently close to $1/\alpha$ that $\omega\alpha\theta_1 > 1$. We apply Proposition 4.8 with $\xi = \epsilon b_n$, $\eta = n^\omega$. (Note that for all $\epsilon > 0$, $|\bar Z| \in \mathbb{R}^d$, the constraints $1 \le \eta \le \xi$, $M \log \xi \le \eta^{\alpha/2}$, $\xi > |\bar Z|$ are satisfied for n sufficiently large.) Then

$$|H'|_{L^2(Y)} \ll \epsilon^{1-\alpha/2} b_n \Big(\frac{\ell(\epsilon b_n)}{\ell(b_n)}\Big)^{1/2} \big(\ell(b_n) b_n^{-\alpha}\big)^{1/2} \sim \epsilon^{1-\alpha/2} b_n n^{-1/2}.$$

Also, $|H''|_{L^2(Y)} \ll n^{\omega(1-\alpha/2+\delta)} = o(b_n n^{-1/2})$ for $\delta > 0$ sufficiently small. Finally, $|H'''|_{L^2(Y)} \ll b_n^{1+\delta} n^{\delta\omega - \omega\alpha\theta_1/2}$. Since $\omega\alpha\theta_1 > 1$, we can shrink δ if necessary so that $b_n^\delta n^{\delta\omega} n^{-\omega\alpha\theta_1/2} = o(n^{-1/2})$. Hence, $|H'''|_{L^2(Y)} = o(b_n n^{-1/2})$. ■

Define $\psi_\xi:\Delta\to[0,\infty)$ by $\psi_\xi(y,\ell)=\sum_{0\le j<\ell}(R^{(i)}1_{\{\omega_i R\in B_\xi(\bar Z)\}})(y,j)$.

Corollary 4.10. *There exists $C>0$ such that*

$$\limsup_{n\to\infty} b_n^{-1}\big|\max_{k\le n}\psi_{\epsilon b_n}\circ f_\Delta^k\big|_{L^1(\Delta)}\le C\epsilon^{1-\alpha/2}\quad \textit{for all } \epsilon>0.$$

Proof. Define $h_\xi:\Delta\to[0,\infty)$ by $h_\xi(y,\ell)=H(y)$. Then $\max_{k\le n}h_\xi\circ f_\Delta^k(y,\ell)\le \max_{k\le n}H_\xi(F^k y)$. Since τ has exponential tails,

$$\begin{aligned}\big|\max_{k\le n}h_\xi\circ f_\Delta^k\big|_{L^1(\Delta)} &\le \bar\tau^{-1}\int_Y \tau\max_{k\le n}H_\xi\circ F^k\,d\mu_Y\\ &\ll \big|\max_{k\le n}H_\xi\circ F^k\big|_{L^2(Y)}\ll n^{1/2}|H_\xi|_{L^2(Y)}.\end{aligned}$$

Hence it follows from Corollary 4.9 that $\limsup_{n\to\infty}b_n^{-1}|\max_{k\le n}h_{\epsilon b_n}\circ f_\Delta^k|_{L^1(\Delta)}\ll \epsilon^{1-\alpha/2}$. Since $0\le\psi_{\epsilon b_n}\le h_{\epsilon b_n}$, we obtain the desired estimate for $\max_{k\le n}\psi_{\epsilon b_n}\circ f_\Delta^k$. ■

Define $Q_{n,\xi}:\Delta\to\mathbb{R}$ by $Q_{n,\xi}=N_n\int_Y H_\xi\,d\mu_Y-n\int_\Delta R^{(i)}1_{\{\omega_i R\in B_\xi(\bar Z)\}}\,d\mu_\Delta$ where N_n is the lap number (3.1).

Proposition 4.11. *There exists $C>0$ such that*

$$\limsup_{n\to\infty} b_n^{-1}\big|\max_{k\le n}|Q_{k,\epsilon b_n}|\big|_{L^2(\Delta)}\le C\epsilon^{1-\alpha/2}\quad \textit{for all } \epsilon>0.$$

Proof. Observe that $Q_{n,\xi}=(N_n-n\bar\tau^{-1})\int_Y H_\xi\,d\mu_Y$. Hence

$$|Q_{n,\xi}|\le |N_n-n\bar\tau^{-1}|\int_Y H_\xi\,d\mu_Y\le |N_n-n\bar\tau^{-1}|\,|H_\xi|_{L^2(Y)}.$$

The result follows from Proposition 3.2 and Corollary 4.9. ■

Proof of Lemma 4.6. Let $\widetilde H_\xi=H_\xi-\int_Y H_\xi\,d\mu_Y$. Define $G_{n,\xi}:Y\to\mathbb{R}$ and $g_{n,\xi}:\Delta\to\mathbb{R}$,

$$G_{n,\xi}=\max_{k\le n}\sum_{j<k}\widetilde H_\xi\circ F^j,\quad g_{n,\xi}(y,\ell)=G_{n,\xi}(y).$$

By Proposition 3.1 and Corollary 4.9,

$$\limsup_{n\to\infty}b_n^{-1}|G_{n,\epsilon b_n}|_{L^2(Y)}\ll \limsup_{n\to\infty}b_n^{-1}n^{1/2}|H_{\epsilon b_n}|_{L^2(Y)}\ll\epsilon^{1-\alpha/2}.$$

Since τ has exponential tails,

$$\int_\Delta |g_{n,\epsilon b_n}|\,d\mu_\Delta=\bar\tau^{-1}\int_Y\tau|G_{n,\epsilon b_n}|\,d\mu_Y\ll |G_{n,\epsilon b_n}|_{L^2(Y)},$$

so

$$\limsup_{n\to\infty}b_n^{-1}\int_\Delta|g_{n,\epsilon b_n}|\,d\mu_\Delta\ll\epsilon^{1-\alpha/2}. \tag{4.1}$$

Next, by the definition of the lap number N_n,

$$\sum_{j=0}^{n-1}(R^{(i)}1_{\{\omega_i R\in B_\xi(\bar Z)\}})(f_\Delta^j(y,\ell)) = \sum_{j=0}^{N_n(y,\ell)-1} H_\xi(F^jy)+\psi_\xi(f_\Delta^n(y,\ell))-\psi_\xi(y,\ell),$$

and so

$$\sum_{j=0}^{n-1}\widetilde R_\xi^{(i)}(f_\Delta^j(y,\ell)) = \sum_{j=0}^{N_n(y,\ell)-1} \widetilde H_\xi(F^jy)+\psi_\xi(f_\Delta^n(y,\ell))-\psi_\xi(y,\ell)+Q_{n,\xi}(y,\ell).$$

Since $N_n\le n$ for all n,

$$\max_{k\le n}\Big|\sum_{j=0}^{k-1}\widetilde R_\xi^{(i)}\circ f_\Delta^j(y,\ell)\Big| \ll \max_{k\le n}\Big|\sum_{j<k}\widetilde H_\xi(F^jy)\Big| + \max_{k\le n}\psi_\xi(f_\Delta^k(y,\ell))+\max_{k\le n}|Q_{k,\xi}(y,\ell)|.$$

In other words,

$$\max_{k\le n}\Big|\sum_{j=0}^{k-1}\widetilde R_\xi^{(i)}\circ f_\Delta^j\Big| \ll g_{n,\xi}+\max_{k\le n}\psi_\xi\circ f_\Delta^k+\max_{k\le n}|Q_{k,\xi}|.$$

Setting $\xi=\epsilon b_n$, the result follows from (4.1), Corollary 4.10 and Proposition 4.11. ■

We can now complete the verification of Condition II. First notice that

$$R^{(i)}1_{\{|\widetilde Z|<\xi\}} = R^{(i)}1_{\{|\omega_i R^{(i)}-\int_\Delta Z\,d\mu|<\xi\}} = R^{(i)}1_{\{\omega_i R\in B_\xi(\bar Z)\}}.$$

Hence

$$Z1_{\{|\widetilde Z|<\xi\}} = \sum_i \omega_i R^{(i)}1_{\{\omega_i R\in B_\xi(\bar Z)\}}$$

and so

$$Z1_{\{|\widetilde Z|<\xi\}} - \int_\Delta Z1_{\{|\widetilde Z|<\xi\}}d\mu_\Delta = \sum_i \omega_i\widetilde R_\xi^{(i)}.$$

By Lemma 4.6,

$$\limsup_{n\to\infty} b_n^{-1}\Big|\max_{1\le k\le n}\Big|\sum_{j=0}^{k-1}\big(Z1_{\{|\widetilde Z|<\epsilon b_n\}}\big)\circ f_\Delta^j - k\int_\Delta Z1_{\{|\widetilde Z|<\epsilon b_n\}}d\mu_\Delta\Big|\Big|_{L^1(\Delta)} \ll \epsilon^{1-\alpha/2}.$$

Also, by Proposition 3.1 with $g=1_{\{|\widetilde Z|<\epsilon b_n\}}-\int_\Delta 1_{\{|\widetilde Z|<\epsilon b_n\}}d\mu_\Delta$,

$$\Big|\max_{1\le k\le n}\Big|\sum_{j=0}^{k-1}1_{\{|\widetilde Z|<\epsilon b_n\}}\circ f_\Delta^j - k\int_\Delta 1_{\{|\widetilde Z|<\epsilon b_n\}}d\mu_\Delta\Big|\Big|_{L^2(\Delta)} \ll n^{1/2}.$$

Combining these two estimates yields

$$\limsup_{n\to\infty} b_n^{-1}\Big|\max_{1\le k\le n}\Big|\sum_{j=0}^{k-1}\big(\widetilde Z1_{\{|\widetilde Z|<\epsilon b_n\}}\big)\circ f_\Delta^j - k\int_\Delta \widetilde Z1_{\{|\widetilde Z|<\epsilon b_n\}}d\mu_\Delta\Big|\Big|_{L^1(\Delta)} \ll \epsilon^{1-\alpha/2}.$$

Hence Condition II follows from Markov's inequality and the fact that $\pi:\Delta\to\Sigma$ is a measure-preserving semiconjugacy.

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