Comment on “Reliability of the 0-1 test for chaos”

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Recently Hu, Tung, Gao and Cao investigated the 0-1 test for chaos. By looking at random data and at data stemming from the one-dimensional logistic map they concluded that the test is unreliable. We explain why their criticism is unfounded.

PACS numbers: 05.45.-a
Keywords: Chaos, dynamical systems, nonlinear time series analysis

Recently a paper appeared [1] in which the 0-1 test for chaos [2, 3] was investigated with respect to its reliability. The authors accepted the reliability for distinguishing between regular and fully developed chaos. However, their conclusions were largely negative. In this Comment we refute the criticism on the reliability.

We first recall the 0-1 test, following [3] where we gave a modified and simplified version of the original test [2]; the modified test deals better with noisy data. Consider a scalar observable $\Phi(n)$. In an experiment, $\Phi(n)$ is a discrete set of measurement data. Choose $c > 0$ and define

$$p(n) = \sum_{j=1}^{n} \Phi(j) \cos jc, \tag{1}$$

for $n = 1, 2, \ldots$ Define the mean square displacement $M(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} [p(j+n) - p(j)]^2$ and the asymptotic growth rate $K = \lim_{n\to\infty} \log M(n)/\log n$. If the underlying dynamics is regular (ie periodic or quasi-periodic) then $K = 0$; if the underlying dynamics is chaotic then $K = 1$. (We refer to [2, 3] for the justification of these statements.)

In practice, let N denote the amount of data. Throughout this Comment, we compute $M(n)$ for $N/100 \le n \le N/10$ to ensure that $0 \ll n \ll N$. Then we compute K by performing a least square fit for $\log M(n)$ versus $\log n$. This is done for 100 randomly chosen values of c and the median value of K is the final output of the test (cf. [3]).

Section II of [1] is devoted to “understanding” the test for chaos, namely that the test examines how the variance of the auxiliary process $p(n)$ scales with time. This interpretation is correct but was already present in the original paper. Indeed, the “$\mathbb{R}$-extension” discussed in [2, Section 5] corresponds exactly to the “fluctuation analysis” in [1]. As mentioned explicitly in [2, Section 5], the necessity to subtract off the mean is a nontrivial obstruction, whereas a crucial property of the 0-1 test (which is an “$\mathbf{SE}(2)$-extension”) is that the mean automatically vanishes (due to rotation symmetry in the plane). A deeper understanding of the 0-1 test would recognise the issue with the mean which underlies the development of the test in [2].

The mathematics behind the test establishes a dichotomy for deterministic systems: bounded or diffusive behaviour of the mean zero random walk. This is one of the strengths of the test, a binary diagnostic 0 or 1, with no grey areas in principle. (There are always grey areas in practice for any test.) Hence Section III of [1] which highlights the “misclassification” of $1/f^\alpha$ noise (it is shown that the 0-1 test yields $K = 1$ independent of the choice of α) has missed the point. Moreover, it is a misapplication of the test; as clearly stated in both our manuscripts [2, 3] the test is designed for *deterministic* (but possible noisy) systems. (Indeed, the paper [2] is entitled *“A new test for chaos in deterministic systems”*.) The problem of distinguishing between chaotic and stochastic dynamics —although of obvious practical importance — is different from the problem we are working on and hence is not grounds for criticism.

Section IV of [1] discusses the “edge of chaos” and “weak chaos” in the context of the logistic map $x_{n+1} = ax_n(1 - x_n) + \sigma\eta_n$ where a is the bifurcation parameter and $\eta_n \sim \mathcal{N}(0, 1)$. It is claimed that the 0-1 test cannot distinguish properly the edge of chaos at $a = a_\infty = 3.569945672...$ and weak chaos at $a = a_\infty + 0.001$. We refute this claim below on three different levels: (i) For the parameters considered in [1] a visual version of our test is effective; (ii) For most practical purposes the automated test is effective with a moderate amount of data; (iii) In theory the automated test works with probability one. For brevity, we restrict to clean data $\sigma = 0$ (see [3] for noisy data).

Starting with (i), for the specific parameters a_∞ and $a_\infty + 0.001$ we computed K versus N up to $N = 50,000$. The results are shown in Fig. 1 and show a clear functional difference of $K(N)$. Also defining $p(n)$ as in (1) and $q(n) = \sum_{j=1}^{n} \Phi(j) \sin jc$, we plotted $p(n)$ versus $q(n)$ in Fig. 2. Here $N = 5000$ suffices for an effective visual

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test. Again this visual test was explicit in the original paper [2, Section 5]. We should emphasise that a major advantage of our test over methods such as [1] is that it is easily automated and in most cases it is not necessary to make a visual interpretation of the results.

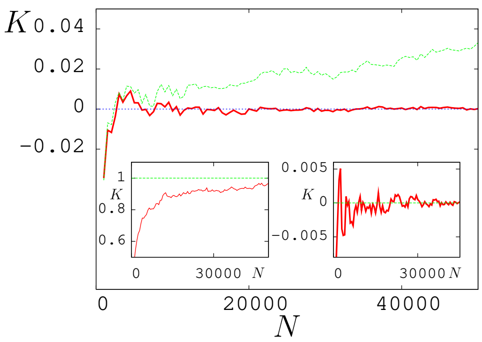

FIG. 1: (Color online) Plot of K as a function of N for the logistic map at $a = a_\infty$ (continuous line, online red) and $a = a_\infty + 0.001$ (dashed line, online green). Although the value of K is small in both cases, the behaviour of K as a function of N distinguishes the two cases. The corresponding graphs for fully developed chaos at $a = 3.99$ and regular dynamics at $a = 3.561$ are shown in the left and right insets respectively.

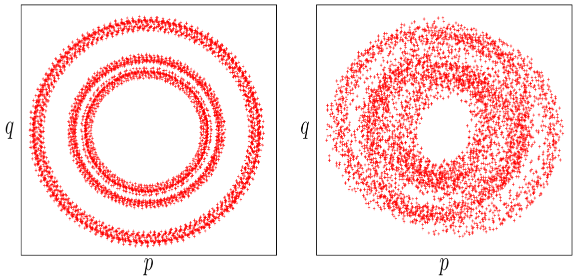

FIG. 2: (Color online) Plot of p versus q for the logistic map. (a) $a = a_\infty$; (b) $a = a_\infty + 0.001$. We used 5000 data points.

For (ii), see Fig. 3 where the results of our test are shown for $N = 5000$. The parameter a is incremented in steps of 0.01 for $3.5 \le a \le 4$ and it is clear that the automated test (ie the computed value of K) is effective for most values of a.

Turning to (iii), the dynamics of the logistic map is particularly well understood [4]: almost every value of a yields either a periodic attractor or a chaotic attractor satisfying the *Collet-Eckmann condition*. It then follows from results in [5] and [6] respectively that with probability one K converges to 0 and 1 as $N \to \infty$.

In conclusion, the claim that the 0-1 test is not reliable is unsubstantiated. The authors chose to focus on (a) situations where the test was not intended to apply, and (b) very specialised situations in tiny regions of parameter space. In this Comment we have reiterated the effectiveness of the 0-1 test outside these regions and established the efficacy of our test in situation (b). The claims in [1]

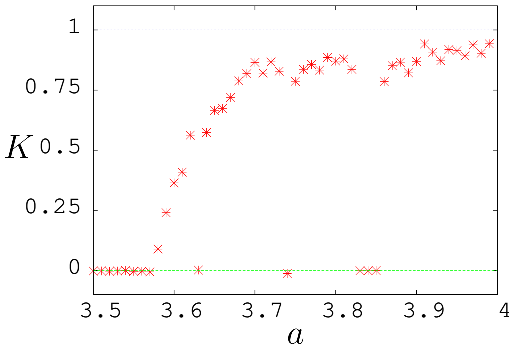

FIG. 3: (Color online) Plot of K versus a for the logistic map with $3.5 \le a \le 4$ increased in increments of 0.01. We used 5000 data points. The horizontal lines indicate $K = 0$ and $K = 1$.

are thus shown to be based on misunderstandings regarding the intended scope and implementation of the test.

We agree with the final comment of the authors that scientists should use all available methods to characterise and analyse complex data; for essentially deterministic data the 0-1 test is such a method. We do not claim that our test will replace traditional methods such as those relying on phase-space reconstruction [7, 8] but that it avoids certain well-documented drawbacks of such tests [9] and hence is likely to be advantageous in certain situations.

Acknowledgments

GAG is partly supported by ARC grant DP0452147. GAG and IM are partly supported by EPSRC grant GR/S22714/01.

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