# An Exact Turán Result for the Generalized Triangle

Oleg Pikhurko*
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213
Web: `http://www.math.cmu.edu/~pikhurko`

May 27, 2007

**Abstract**

Let $\Sigma_k$ consist of all $k$-graphs with three edges $D_1, D_2, D_3$ such that $|D_1 \cap D_2| = k-1$ and $D_1 \bigtriangleup D_2 \subseteq D_3$. The exact value of the Turán function $\mathrm{ex}(n, \Sigma_k)$ was computed for $k = 3$ by Bollobás [Discrete Math., **8** (1974) 21–24] and for $k = 4$ by Sidorenko [Math Notes, **41** (1987) 247–259].

Let the $k$-graph $T_k \in \Sigma_k$ have edges

$$\{1, \dots, k\},\ \{1, 2, \dots, k-1, k+1\},\ \text{and}\ \{k, k+1, \dots, 2k-1\}.$$

Frankl and Füredi [J. Combin. Theory (A), **52** (1989) 129–147] conjectured that there is $n_0 = n_0(k)$ such that $\mathrm{ex}(n, T_k) = \mathrm{ex}(n, \Sigma_k)$ for all $n \geq n_0$ and had previously proved this for $k = 3$ in [Combinatorica, **3** (1983) 341–349]. Here we settle the case $k = 4$ of the conjecture.

**Abbreviated title:** Generalized Triangle

**MSC:** Primary 05D05, Secondary 05C35

*Partially supported by the National Science Foundation, Grant DMS-0457512.

# 1 Introduction

Let $\mathcal{T}_k$ be the family of all *k-graphs* (i.e. $k$-uniform set systems) with three edges such that one edge contains the symmetric difference of the other two. The family $\Sigma_k$ consists of all $k$-graphs with three edges $D_1, D_2, D_3$ such that $|D_1 \cap D_2| = k-1$ and $D_1 \triangle D_2 \subseteq D_3$. Also, let the *generalized triangle* $T_k$ be the $k$-graph with edges

$$\{1,\ldots,k\},\ \{1,2,\ldots,k-1,k+1\},\ \text{and}\ \{k,k+1,\ldots,2k-1\}.$$

Please note that $T_k \in \Sigma_k$ and $\Sigma_k$ is a subfamily of $\mathcal{T}_k$. For $k=2$ we obtain in each case the single triangle $K_3^2$. Also, we have $\Sigma_3 = \mathcal{T}_3$.

Let $\mathcal{F}$ be a family of $k$-graphs. A $k$-graph $G$ is *$\mathcal{F}$-free* if it does not contain any member of $\mathcal{F}$ as a subgraph. The *Turán function* $\mathrm{ex}(n,\mathcal{F})$ is the maximum size of an $\mathcal{F}$-free $k$-graph of order $n$:

$$\mathrm{ex}(n,\mathcal{F}) = \max\left\{|G| : G \subseteq \binom{[n]}{k},\ G \text{ is } \mathcal{F}\text{-free}\right\},$$

where $[n] = \{1,\ldots,n\}$. Please note that we identify hypergraphs with their edge sets. For a $k$-graph $F$, we write $\mathrm{ex}(n,F)$ for $\mathrm{ex}(n,\{F\})$.

As a possible generalization of the problem of computing $\mathrm{ex}(n,K_3^2)$, which was solved by Mantel [14] and Turán [23], to 3-graphs, Katona [9] suggested to consider $\mathrm{ex}(n,\mathcal{T}_3)$. This function was determined by Bollobás [1] who showed that for any $n \geq 3$ the complete *balanced* 3-partite 3-graph (that is, the sizes of any two parts differ at most by one) is the unique extremal graph. Thus

$$\mathrm{ex}(n,\mathcal{T}_3) = \left\lfloor \frac{n}{3} \right\rfloor \times \left\lfloor \frac{n+1}{3} \right\rfloor \times \left\lfloor \frac{n+2}{3} \right\rfloor. \tag{1}$$

In turn, Bollobás conjectured that the same is true for $k \geq 4$, that is, the value of $\mathrm{ex}(n,\mathcal{T}_k)$ is given by the balanced $k$-partite $k$-graph. De Caen [2] raised the problem of computing $\mathrm{ex}(n,\Sigma_k)$.

Sidorenko [21] settled the case $k=4$ of Bollobás' conjecture. In fact, Sidorenko proved that forbidding $\Sigma_4$ alone suffices for the upper bound, that is,

$$\mathrm{ex}(n,\mathcal{T}_4) = \mathrm{ex}(n,\Sigma_4) = \left\lfloor \frac{n}{4} \right\rfloor \times \left\lfloor \frac{n+1}{4} \right\rfloor \times \left\lfloor \frac{n+2}{4} \right\rfloor \times \left\lfloor \frac{n+3}{4} \right\rfloor, \quad n \geq 4, \tag{2}$$

with the complete balanced 4-partite 4-graph being the unique extremal graph.

Shearer [20] showed that Bollobás' conjecture fails for $k \geq 10$.

Frankl and Füredi [6] proved various results on $\Sigma_k$-free $k$-graphs for $k = 5, 6$. In particular, they computed the exact value of $\mathrm{ex}(n, \Sigma_5)$ for all $n$ divisible by 11 and the exact value of $\mathrm{ex}(n, \Sigma_6)$ for all $n$ divisible by 12. For these $n$ the extremal graphs are blow-ups of the unique $(11, 5, 4)$ and $(12, 6, 5)$ Steiner systems.

Clearly, $\mathrm{ex}(n, T_k) \geq \mathrm{ex}(n, \Sigma_k)$. The super-saturation technique of Erdős and Simonovits [4] (or, alternatively, the proof of Lemma 9 here) shows that for any fixed $k$ we have

$$\mathrm{ex}(n, T_k) - \mathrm{ex}(n, \Sigma_k) = o(n^k). \tag{3}$$

Frankl and Füredi [6] conjectured that, for any fixed $k \geq 4$, if $n \geq n_0(k)$ is sufficiently large, then in fact

$$\mathrm{ex}(n, T_k) = \mathrm{ex}(n, \Sigma_k). \tag{4}$$

Previously, Frankl and Füredi [5] had proved the case $k = 3$ of (4). Very recently, Keevash and Mubayi [11] presented a different proof of $\mathrm{ex}(n, T_3) = \mathrm{ex}(n, \Sigma_3)$, which shows that we can take $n_0(3) = 33$.

In this paper, we settle the conjecture for $k = 4$.

**Theorem 1** *There is $n_0$ such that for all $n \geq n_0$ we have $\mathrm{ex}(n, T_4) = \mathrm{ex}(n, \Sigma_4)$ and, moreover, the complete balanced 4-partite 4-graph is the unique extremal 4-graph.*

Unfortunately, the bound on $n_0$ given by our proof of Theorem 1 is rather large.

Our argument can be modified to prove the case $k = 3$ of the conjecture. We do not describe the corresponding modifications since they are fairly obvious. (And there are already two shorter proofs of this case.) The conjecture is still open for $k \geq 5$. Our method seems promising in attacking the cases $k = 5$ or 6, given the above results of Frankl and Füredi [6] on $\Sigma_k$-free graphs. Unfortunately, we have not been able to settle these cases.

## 2 An Outline of the Proof

Here we sketch our argument establishing Theorem 1 as well as give some important definitions needed in the proof.

Two $k$-graphs $F$ and $G$ of the same order are *$m$-close* if we can add or remove at most $m$ edges to/from the first hypergraph and make it isomorphic to the second. In other words, for some bijection $\sigma : V(F) \to V(G)$ the symmetric difference between $\sigma(F) = \{\sigma(D) : D \in F\}$ and $G$ has at most $m$ edges.

Let $\mathcal{F}$ be a family of $k$-graphs. Its *Turán density* is

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{k}}.$$

(The limit is known to exist, see Katona, Nemetz, and Simonovits [10].)

Let us call the family $\mathcal{F}$ *$s$-stable* if for any $\varepsilon > 0$ there are $c > 0$ and $n_0$ such that for arbitrary $\mathcal{F}$-free $k$-graphs $G_1, \dots, G_{s+1}$ of the same order $n \geq n_0$, each of size at least $(\pi(\mathcal{F}) - c)\binom{n}{k}$, some two are $\varepsilon\binom{n}{k}$-close. Please note that if $\mathcal{F}$ is $s$-stable for some $s$ then it is also $t$-stable for any $t > s$. Let the bare term *stable* mean 1-stable.

Let us return to Theorem 1. By (2) and (3) we know $\mathrm{ex}(n, T_4)$ asymptotically but we have to compute the exact value for all large $n$. We proceed by showing that $T_4$ is stable.

This is not an easy task so first we prove that $\Sigma_4$ is stable. This is done by modifying Sidorenko's proof of (2), so let us very briefly recall his argument. It will be presented here somewhat differently than usually, in order to be more conformal with our strengthening.

Let $G$ be a maximum $\Sigma_4$-free 4-graph of order $n$. If some two vertices of $G$ are not covered by a common edge, we can delete one and clone (that is, duplicate) the other without introducing any forbidden subgraph nor decreasing the size of $G$. This *symmetrization* trick can be applied to whole groups of vertices, showing that it suffices to prove the upper bound (2) for 4-graphs $G$ constructed in the following way. Take $m \leq n$ and a $\Sigma_4$-free 4-graph $H$ on $[m]$ such that every two vertices are covered by an edge. Choose positive integers $v_1 + \dots + v_m = n$ and *blow-up* $H$ correspondingly, that is, replace each vertex $i \in [m]$ by a set $V_i$ of size $v_i$ and replace each edge of $H$ by the corresponding complete 4-partite 4-graph. Clearly, the obtained 4-graph $G$ has size $n^4 \, \lambda_H(\frac{v_1}{n}, \dots, \frac{v_m}{n})$, where

$$\lambda_H(y_1, \dots, y_m) = \sum_{D\in H} \prod_{i\in D} y_i$$

is the *Lagrange polynomial* of $H$. Hence, $\mathrm{ex}(n, \Sigma_4) \leq n^4 \, \Lambda_H$, where

$$\Lambda_H = \max\{\lambda_H(y_1, \dots, y_m) : y_i \in \mathbb{R},\ y_i \geq 0,\ y_1 + \dots + y_m = 1\} \tag{5}$$

is the *Lagrangian* of $H$. Take any real vector $\boldsymbol{y} = (y_1, \dots, y_m)$ attaining the maximum in (5). By deleting vertices of $H$ if necessary, we can assume without loss of generality that each $y_i$ is strictly positive. It is not hard to see that all partial derivatives $\frac{\partial}{\partial_i}\lambda_H(\boldsymbol{y})$ must be the same and, in view of the trivial identity $\sum_{i=1}^m y_i \frac{\partial}{\partial_i}\lambda_H(\boldsymbol{y}) = 4\Lambda_H$, are all equal to $4\Lambda_H$. Moreover, our assumptions on $H$ imply that every triple of vertices is contained in at most one edge. Hence,

$$4m\,\Lambda_H = \sum_{i=1}^m \frac{\partial}{\partial_i}\lambda_H(\boldsymbol{y}) \le \lambda_{\binom{[m]}{3}}(\boldsymbol{y}) \le \Lambda_{\binom{[m]}{3}} \le \frac{1}{m^3}\binom{m}{3}.$$

The case $m = 5$ leads to a contradiction, see Section 3.5 here. So, $\Lambda_H \le \sup\{\frac{(m-1)(m-2)}{24m^3} : m \in \mathbb{N} \setminus \{5\}\} = \frac{1}{4^4}$. This implies (2) for all $n$ divisible by 4. Some extra work is needed to prove (2) for all $n$. We refer the Reader to [21] for further details.

The main difficulty in proving the stability of $\Sigma_4$ via Sidorenko's argument is that the Symmetrization phase (or even just one step of it) may in principle change the hypergraph essentially. In particular, it is difficult to see why it should preserve the property of being close to a 4-partite 4-graph. We overcome this by iteratively and constantly removing vertices whose degree becomes too small at any step of Symmetrization. This ensures that every time we are about to apply a deletion-cloning step we have the large minimum degree. Then, when we reverse one step, each undeleted vertex must have many incident edges, which forces all of them to fit perfectly into the 4-partition we already have. All details of the proof that $\Sigma_4$ is stable can be found in Sections 3–4.

For any $k$, the stability of $T_k$ easily follows from the stability of $\Sigma_k$, see Section 5. Unfortunately, we do not know how prove the stability of $T_4$ directly, without doing this for $\Sigma_4$ first.

The stability of $T_4$ implies by (2) and (3) that any maximum $T_4$-free graph $G$ is close to a complete 4-partite 4-graph $P$ on the same vertex set. We choose $P$ (not necessarily balanced) to maximize $|G \cap P|$. If $G \subseteq P$, we are done. Otherwise, there is at least one *bad* edge (i.e. an edge of $G \setminus P \neq \emptyset$). Roughly speaking, we argue in Section 6 that each bad edge forces many *missing* edges (i.e. the edges in $P \setminus G$). The inequality $|G \setminus P| \ge |P \setminus G|$ allows us to find a vertex $x$ belonging to $\Omega(n^{k-1})$ bad edges. These bad edges block almost all properly placed edges containing $x$, so $|G \cap P|$ can be strictly increased by moving $x$ to another part of $P$, a contradiction.

Let us remark here that the fact that stability may help in exact computation of the hypergraph Turán function was observed and used by Füredi and Simonovits [7], Keevash and Mubayi [11], Keevash and Sudakov [13, 12], Mubayi and Pikhurko [15], Pikhurko [17], and others.

As we have already mentioned, we were not able to settle other open cases of Frankl and Füredi's conjecture. However, we hope that our approach (or some parts of it) may be useful in attacking the cases $k \geq 5$ of the conjecture.

# 3 Symmetrization

Here we describe the Symmetrization process.

Let $\delta = \pi(\Sigma_4)$, which is 3/32 by (2). Suppose that reals $c_2, c_1, c_0 > 0$, and an integer $n_0$ satisfy $1 \gg c_2 \gg c_1 \gg c_0 \gg \frac{1}{n_0}$, where $b \gg a$ means that $a > 0$ is a sufficiently small real, depending on $b$. Let $n \geq n_0$ and $G$ be an arbitrary $\Sigma_4$-free 4-graph on $[n]$ with at least $(\delta - c_0)\binom{n}{4}$ edges.

Informally speaking, we iterate the following two-part procedure. During *Cleaning Up*, we consecutively remove vertices whose relative degree is too small. When we reach a sufficiently large minimum degree, we apply *Merging*: if there are two groups of vertices such that no edge of $G$ intersects both, we 'merge' them together and return to Cleaning Up (even if there are other groups that can be merged). If there is nothing to merge, then the whole procedure terminates.

Here is the formal description (and the analysis) of this procedure, which we call *Symmetrization*. It consists of several parts. For the ease of reference, we put each part into a separate section.

## 3.1 The Initial Configuration and the Maintained Properties

Initially, let $G_0 = G$, $V_0 = U_0 = [n]$, and $P_{0,u} = \{u\}$ for $u \in [n]$.

Suppose that, after $i$ iterations of Cleaning Up and Merging, we have a 4-graph $G_i$ on $V_i$, a subset $U_i \subseteq V_i$, and a partition $\mathcal{P}_i = \{P_{i,u} : u \in V_i\}$ of $V_i$, where $P_{i,u}$ denotes the part containing $u$. Thus the same set $P_{i,u}$ is listed $|P_{i,u}|$ times in the sequence $(P_{i,v})_{v \in V_i}$.

Let $l$ denote the total number of iterations until we stop. It will be the case that the following *Properties 1–4* hold for every $i \leq l$:

1. For any $0 \leq j \leq i$, $U_j \cap V_i$ is a *transversal* for the partition $\{P_{j,u} \cap V_i : u \in V_i\}$, that is, it intersects every part in precisely one vertex. (In particular, $U_i$ is a transversal for the partition $\{P_{i,u} : u \in V_i\}$.)

2. $G[U_i] = G_i[U_i]$, where e.g. $G[U_i]$ denotes the subgraph of $G$ induced by the set $U_i$.

3. $G_i$ is the union of the complete 4-partite 4-graphs with the parts from $\mathcal{P}_i$ corresponding to the edges of $G[U_i]$. (In other words, no edge of $G_i$ can intersect a part in more than one vertex while permuting vertices inside any part we get an automorphism of $G_i$.)

4. If $i \geq 1$, then $U_i \subseteq U_{i-1}$ and $V_i \subseteq V_{i-1}$.

Trivially, Properties 1–4 are valid for $i = 0$.

## 3.2 Cleaning Up

Let us describe the *Cleaning Up* part. Suppose we have already performed $i \geq 0$ rounds of Cleaning Up and Merging. Let $G_i$, $V_i$, $U_i$, and $\mathcal{P}_i$ describe the current configuration.

Initially, we set $G_i' = G_i$, $V_i' = V_i$, and $n_i' = |V_i'|$. These three variables (with primes) will be iteratively updated while $G_i$, $V_i$, $U_i$, and $\mathcal{P}_i$, remain unchanged (throughout the whole proof).

If every $u \in V_i'$ is incident to at least $(\delta - c_1)\binom{n_i'-1}{3}$ edges of $G_i'$, then we have finished Cleaning Up. The final values of $G_i'$, $V_i'$, and $n_i'$ serve as the input for the second part, Merging.

Suppose that some $u \in V_i'$ has $G_i'$-degree less than $(\delta - c_1)\binom{n_i'-1}{3}$.

We are going to remove some vertex $v$ of $P_{i,u}$ from $G_i'$. It does not really matter which vertex we delete, since all vertices of $P_{i,u}$ look the same with respect to $G_i'$ by Property 3. However, it is convenient to note which vertices come from which parts. Also, when we analyze the whole Symmetrization procedure in Section 4, we restrict our attention the final set $V_l$ and our proof needs that if $P_{i,w}$ is merged into $P_{i,u}$ and $P_{i,w} \cap V_l \neq \emptyset$, then $P_{i,u} \cap V_l \neq \emptyset$. The last property is formally stated and proved in Lemma 6. For these purposes we require that the following always holds during Cleaning Up:

$$P_{j,v} \cap U_j \cap V_i' \neq \emptyset, \qquad \forall\, v \in V_i', \quad \forall\, j \in [0, i], \tag{6}$$

where $[s, t]$ denotes the interval $\{s, s+1, \ldots, t\}$ for integers $s \leq t$. Informally, (6) states that if all vertices of some part $P_{j,v}$ are later removed, then the special vertex $u \in P_{j,v} \cap U_j$ is removed last.

If we have not removed any vertices in the current (i.e. $i$-th) Cleaning Up step, then (6) holds by Property 1. In order to maintain the validity of (6), we select a vertex $v$ of $P_{i,u}$ to be deleted as follows (assuming that (6) holds beforehand). By (6), $P_{i,u} \cap U_i \cap V_i' \neq \emptyset$, so assume that $u \in U_i \cap V_i'$. Initially let $j = i$ and $v = u$. If $P_{j,v} \cap V_i' = \{v\}$, we are done: select this $v$. Otherwise, let $h \in [j-1]$ be the largest index such that $P_{j,v} \cap V_i'$ intersects two parts of $\mathcal{P}_h$. (Such $h$ exists because $|P_{j,v} \cap V_i'| \geq 2$, $\mathcal{P}_0$ consists of singletons while, as we will see, each partition is obtained from the previous one by gluing some two parts together.) One of these two parts of $\mathcal{P}_h$ is $P_{h,v}$. Let the other be $P_{h,w}$. By the inductive assumption (6) we can assume that $w \in U_h \cap V_i'$. Redefine $j = h$ and $v = w$ and repeat. Clearly, $j$ strictly decreases each time, so this selection procedure always terminates.

Having selected $v$, we remove it from $G_i'$ (with all incident edges) and from $V_i'$. We redefine $n_i' = |V_i'|$; thus it decreases by 1. One can check that (6) remains valid after the removal.

We keep repeating the above removal step until the minimal degree of $G_i'$ becomes at least $(\delta - c_1)\binom{n_i'-1}{3}$.

This finishes the description of Cleaning Up.

### 3.3 Merging

Now, let us describe the *Merging* part. Suppose first that we have some distinct $u, w \in U_i \cap V_i'$ such that no edge of $G_i'$ intersects both $P_{i,u}$ and $P_{i,w}$. (Note that by Property 3 this is equivalent to no edge of $G$ containing both $u$ and $w$.) Without loss of generality assume that

$$d_{G_i'}(u) \geq d_{G_i'}(w), \tag{7}$$

where, for example, $d_{G_i'}(u)$ is the $G_i'$-*degree* of $u$, that is, the number of edges of $G_i'$ that contain $u$. We let $U_{i+1} = (U_i \cap V_i') \setminus \{w\}$ and $V_{i+1} = V_i'$. For $v \in V_{i+1}$, let

$$P_{i+1,v} = \begin{cases} P_{i,u} \cup P_{i,w} \cap V_{i+1}, & \text{if } v \in P_{i,u} \cup P_{i,w}, \\ P_{i,v} \cap V_{i+1}, & \text{otherwise.} \end{cases}$$

The 4-graph $G_{i+1}$ is defined by blowing-up $G[U_{i+1}]$ in the obvious way. Informally speaking, we merge $P_{i,w}$ into $P_{i,u}$ with $w$ being erased from the new transversal set $U_{i+1}$. After this (even if there are other pairs that can be merged), we start the new round of Cleaning Up and Merging.

If no $u$ and $w$ as above exist, then we are done with Merging as well as with the whole Symmetrization: let $G_{i+1} = G'_i$, $U_{i+1} = U_i \cap V'_i$, $V_{i+1} = V'_i$, and $P_{i+1,v} = P_{i,v} \cap V'_i$ for $v \in V'_i$, and stop.

This finishes the description of Merging.

## 3.4 Verifying Properties 1–4

Let us note some important properties of the above steps. First of all, let us point out that during Cleaning Up we repeat removals until we have attained the large minimum degree while in the Merging step we glue parts *only once*. This gives us the crucial property that whenever we merge two parts together, we have the minimum degree condition beforehand, having just done Cleaning Up. Once a vertex is removed from $V'_i$, it never returns and all incident to it edges are discarded in the remaining stages of Symmetrization.

Suppose that Properties 1–4 are valid for some $i$. We claim that they remain true for $i+1$, that is, after another round of Cleaning Up and Merging is applied. Let $G'_i$ and $V'_i$ be the final values after Cleaning Up is finished.

Property 4 is valid for $i+1$ since $U_{i+1} \subseteq U_i \cap V'_i \subseteq U_i$ and $V_{i+1} = V'_i \subseteq V_i$. To verify Property 1 for $j \in [0, i]$, recall that $U_j$ is a transversal for the partition $\{P_{j,v} \cap V'_i : v \in V'_i\}$ by (6) while if two parts are merged together then we take care to remove the corresponding vertex from the transversal set. Property 2 remains valid in view of

$$G_{i+1}[U_{i+1}] = G'_i[U_{i+1}] = G_i[U_{i+1}] = G[U_{i+1}],$$

where we used the established facts $U_{i+1} \subseteq U_i$ and $G_i[U_i] = G[U_i]$. Finally, Property 3 follows from the definition of Merging.

## 3.5 Analyzing the Final Configuration

Suppose that we have done $l$ steps in total. Of course, we have some freedom while performing these steps so the final configuration may vary. However, the following claims are always true, irrespective of the choices we made.

**Lemma 2** *For each $i \in [0, l]$, the 4-graph $G_i$, as a blown-up $G[U_i]$, is $\Sigma_4$-free.* ∎

**Lemma 3** *Merging does not decrease size:* $|G_{i+1}| \geq |G'_i|$ *for any* $0 \leq i \leq l-1$.

*Proof.* We can view the above Merging as removing all edges incident to $P_{i,w}$ and then cloning $u$ the appropriate number of times. The claim now follows from (7) and Property 3 of $G_i$. (Note that no edge can intersect both $P_{i,u}$ and $P_{i,w}$ by the choice of $u$ and $w$.) ∎

**Lemma 4** $|V_l| \geq (1 - \frac{2c_0}{c_1})n$.

*Proof.* Let $r = n - |V_l|$ be the number of the removed vertices. The number of the corresponding removed edges is at most

$$(\delta - c_1)\binom{n-1}{3} + \cdots + (\delta - c_1)\binom{n-r}{3} = (\delta - c_1)\left(\binom{n}{4} - \binom{n-r}{4}\right). \qquad (8)$$

Between any two consecutive removals, the 4-graph may change as the result of Merging but its number of edges does not decrease by Lemma 3.

We can assume that for any $m \geq n_0/2$, we have $\mathrm{ex}(m, \Sigma_4) \leq (\delta + c_0)\binom{m}{4}$. (Recall that $n_0$ is sufficiently large depending on $c_0$.) Suppose first that $|V_l| \geq n_0/2$. Then we have $|G_l| \leq (\delta + c_0)\binom{n-r}{4}$ since $G_l$ is $\Sigma_4$-free by Lemma 2. The initial 4-graph $G_0 = G$ has at least $(\delta - c_0)\binom{n}{4}$ edges by our assumption on $G$. Hence, by (8)

$$(\delta - c_0)\binom{n}{4} \leq (\delta - c_1)\left(\binom{n}{4} - \binom{n-r}{4}\right) + (\delta + c_0)\binom{n-r}{4}.$$

Simplifying, we obtain that $c_1\left(\binom{n}{4} - \binom{n-r}{4}\right) \leq c_0\left(\binom{n}{4} + \binom{n-r}{4}\right) \leq 2c_0\binom{n}{4}$. Thus

$$\frac{2c_0}{c_1} \geq 1 - \frac{\binom{n-r}{4}}{\binom{n}{4}} \geq 1 - \left(\frac{n-r}{n}\right)^4.$$

Hence, $r \leq (1 - (1 - \frac{2c_0}{c_1})^{1/4})\, n < \frac{2c_0}{c_1}\, n$ as required.

Note that the case $|V_l| < n_0/2$ can never occur because then $r > n/2$, which is at least $\frac{2c_0}{c_1}\, n$ in view of $c_0 \ll c_1$. Therefore, the moment when we removed the $\lceil \frac{2c_0}{c_1}\, n \rceil$-th vertex would contradict the above calculations. This completes the proof of the lemma. ∎

**Lemma 5** *Under the assumptions and notation of Section 3, the partition* $\mathcal{P}_l$ *has precisely* 4 *parts, each of size between* $\frac{n}{4} - c_2 n$ *and* $\frac{n}{4} + c_2 n$*, while* $G_l$ *is the complete* 4*-partite* 4*-graph.*

*Proof.* To prove the lemma, we have to apply another iterative procedure. In order not to mess up the previous notation, we will be updating only certain variables $y_i$, $i \in [n]$.

Initially, let $y_u = \frac{|P_{l,u}|}{|V_l|}$ for $u \in U_l$ (while all other $y_v$ are set to zero). For brevity, let $H = G[U_l] = G_l[U_l]$. Clearly, $\sum_{i=1}^n y_i = 1$. We view $\boldsymbol{y} = (y_1, \ldots, y_n)$ as vertex weights. For a set $D \subseteq [n]$, let $y_D = \prod_{i \in D} y_i$. For $u \in U_l$, let

$$H_u = \{D : D \not\ni u,\ D \cup \{u\} \in H\}$$

denote the *link graph* of $u$. The *weighted degree* $\lambda_i(\boldsymbol{y})$ of $i \in U_l$ is $\sum_{D \in H_i} y_D$. Please note that $\lambda_i(\boldsymbol{y})$ does not depend on the weight $y_i$ of the vertex itself. In fact, $\lambda_i(\boldsymbol{y})$ is equal to the $i$-th partial derivative of the Lagrange polynomial $\lambda_H(\boldsymbol{y}) = \sum_{D \in H} y_D$.

Initially, we have by Lemma 4 that

$$\lambda_H(\boldsymbol{y}) = \frac{|G_l|}{|V_l|^4} \geq \frac{(\delta - c_0)\binom{n}{4} - \frac{2c_0}{c_1} n \binom{n-1}{3}}{n^4} \geq \frac{\delta}{24} - \frac{c_0}{2c_1}. \tag{9}$$

We repeat as long as possible the following *Regularization* procedure, which, roughly speaking, makes the weighted degrees almost equal except for those vertices $i$ with $y_i = 0$.

Choose $i, j \in U_l$ such that $y_i > 0$, $y_j > 0$, and $\lambda_i(\boldsymbol{y}) - \lambda_j(\boldsymbol{y}) \geq c_1$. (If no such $i, j$ exist we stop.) Let

$$d = \min\left(\tfrac{c_1}{n},\ y_j\right).$$

Replace $y_i$ by $y_i + d$ and $y_j$ by $y_j - d$. Observe that $\lambda_H(\boldsymbol{y})$ changes by

$$\Delta = d(\lambda_i(\boldsymbol{y}) - \lambda_j(\boldsymbol{y})) - d^2 \lambda_{ij}(\boldsymbol{y}),$$

where $\lambda_{ij}(\boldsymbol{y}) = \sum_{\{a,b,i,j\} \in H} y_a y_b$. As, very roughly, $\lambda_{ij}(\boldsymbol{y}) \leq (\sum_{i=1}^n y_i)^2 = 1$ and $d \leq c_1/2$, we have $\Delta \geq dc_1/2$. This and the definition of $d$ implies that during each such transformation either $\lambda_H(\boldsymbol{y})$ increases by at least $\frac{c_1^2}{2n}$ or at least one new $y_j$ becomes 0 (while $\lambda_H(\boldsymbol{y})$ does not decrease). The latter can happen at most $n$ times. As $\lambda_H(\boldsymbol{y})$ never decreases, the former case takes place at most

$$\frac{c_0/(2c_1)}{c_1^2/(2n)} = c_0 n / c_1^3$$

times. Indeed, initially we had (9) while at the end we have $\lambda_H(\boldsymbol{y}) \leq 4^{-4}$ since $H$ is $\Sigma_4$-free.

Thus the total amount the shifted weight in the above procedure is at most $\frac{c_1}{n} \times (c_0 n / c_1^3 + n)$, which is at most $c_2/2$ since $c_2 \gg c_1 \gg c_0$.

When we are done with the Regularization procedure, define

$$A = \{u \in U_l : y_i > 0\}.$$

For any $i, j \in A$ we have

$$|\lambda_i(\boldsymbol{y}) - \lambda_u(\boldsymbol{y})| \le c_1. \tag{10}$$

The identity

$$\sum_{i \in A} y_i \lambda_i(\boldsymbol{y}) = 4\,\lambda_H(\boldsymbol{y})$$

(and $\sum_{i=1}^n y_i = 1$) implies that for every $i \in A$ we have

$$|\lambda_i(\boldsymbol{y}) - 4\,\lambda_H(\boldsymbol{y})| \le c_1.$$

Let $a = |A|$. Note that no two edges of $H[A] = G[A]$ can intersect in 3 vertices (since every two vertices of $U_l$ are covered by an edge of $G$ by the definition of Merging). We thus have

$$4\,\lambda_H(\boldsymbol{y})\,a - c_1 a \le \sum_{i \in A} \lambda_i(\boldsymbol{y}) \le \lambda_{\binom{A}{3}}(\boldsymbol{y}) \le \Lambda_{\binom{A}{3}} \le \frac{1}{a^3}\binom{a}{3}. \tag{11}$$

To show the last inequality, pick any non-negative $x_1, \ldots, x_a$ summing up to 1 and maximizing $\lambda_{\binom{A}{3}}(\boldsymbol{x})$. Let $B = \{i \in [a] : x_i > 0\}$ and $b = |B|$. Assume that $b \ge 3$ for otherwise there is nothing to do. We have $x_i = x_j$ for all $i, j \in B$ (otherwise we get a contradiction by replacing $x_i$ and $x_j$ by $(x_i + x_j)/2$). The required inequality now follows from $b^{-3}\binom{b}{3} \le a^{-3}\binom{a}{3}$.

By (9) and (11),

$$\frac{1}{4^4} - \frac{c_0}{2c_1} \le \lambda_H(\boldsymbol{y}) \le \frac{(a-1)(a-2)}{24a^3} + \frac{c_1}{4}.$$

As $c_1 \gg c_0$ are sufficiently small, this implies that $a = 4$ or 5.

If $a = 5$, then $A$ spans at most one edge $D$ because no two edges of $H$ can intersect in three vertices. All vertices of $H$ outside $A$ have now weight 0, so $D$ is the only edge that contributes to $\lambda_H(\boldsymbol{y})$ (at the final stage). It follows that $y_D \ge 4^{-4} - \frac{c_0}{2c_1}$ and each vertex of $D$ has weight at least, for example, $\frac{1}{4} - c_2$. But then any pair $i, j$ where $i \in D$ and $j$ is the unique element of $A \setminus D$ contradicts (10): $\lambda_i(\boldsymbol{y}) \ge (\frac{1}{4} - c_2)^3$ while $\lambda_j(\boldsymbol{y}) = 0$.

Hence $a = 4$ and $A \in H$. We have $y_A \ge 4^{-4} - \frac{c_0}{2c_1}$. If follows that for each $i \in A$ we have $|y_i - \frac{1}{4}| \le \frac{c_2}{2}$.

Let us analyze the initial position. Since the Regularization procedure changed each $y_i$ by at most $\frac{c_2}{2}$, we have $\left|y_i - \frac{1}{4}\right| \le c_2$ for all $i \in A$.

Thus, in order to finish the proof of the lemma, we have to derive a contradiction from assuming that there is $j \in U_l \setminus A$ with $y_j > 0$ (initially). Recall that the Symmetrization stops when there is nothing to merge. Since we apply Cleaning Up before trying to merge anything, the final 4-graph $G_l$ has large minimum degree. In particular,

$$(n-r)^3 \lambda_j(\boldsymbol{y}) = d_{G_l}(j) \ge (\delta - c_1)\binom{n-r}{3}. \tag{12}$$

Since no edge of $H \setminus \{A\}$ can intersect $A$ in 3 vertices while $\sum_{i \in U_l \setminus A} y_i \le 1 - 4(\frac{1}{4} - c_2) = 4c_2$, we have

$$\lambda_j(\boldsymbol{y}) \le \binom{4}{2}\left(\frac{1}{4} + c_2\right)^2 4c_2 + 4\left(\frac{1}{4} + c_2\right)(4c_2)^2 + (4c_2)^3. \tag{13}$$

The inequalities (12) and (13) contradict our choice $1 \gg c_2 \gg c_1 \gg c_0$ and the fact that $r \le \frac{2c_0 n}{c_1}$ by Lemma 4. This finishes the proof of the lemma. ∎

Also, we will need the following easy corollary of Property 1.

**Lemma 6** *Suppose that $P_{i,w}$ was merged into $P_{i,u}$ during the i-th Merging step, that is, we have $u, w \in U_i$, $u \in U_{i+1}$, and $w \in V_{i+1} \setminus U_{i+1}$. Then, if $P_{i,u} \cap V_l = \emptyset$ then $P_{i,w} \cap V_l = \emptyset$.*

*Proof.* Suppose that $P_{i,w} \cap V_l \ne \emptyset$. Since $P_{i+1,u} \cap V_l \supseteq P_{i,w} \cap V_l$ is non-empty, we conclude by Property 1 that $u$, the unique vertex of $U_{i+1} \cap P_{i+1,u}$, belongs to $V_l$. Thus, $u \in P_{i,u} \cap V_l$ and this set is non-empty, implying the lemma. ∎

# 4 Reversing Symmetrization

Let $\delta = \pi(\Sigma_4) = \frac{3}{32}$. Here we will prove that $\Sigma_4$ is stable. In fact, we will show the following stronger claim.

**Theorem 7** *For any $c_1 > 0$ there are $c_0 > 0$ and $n_0$ such that any $\Sigma_4$-free 4-graph with $n \ge n_0$ vertices and at least $(\delta - c_0)\binom{n}{4}$ edges can be made 4-partite by removing at most $c_1 n$ vertices.*

*Proof.* Fix some sufficiently small constant $c_2 > 0$. (For example, $c_2 = 0.001$ will do with room to spare.) By decreasing $c_1$ if necessary, assume that $1 \gg c_2 \gg c_1 \gg c_0 \gg \frac{1}{n_0}$. Let $n \geq n_0$ and $G$ be an arbitrary $\Sigma_4$-free 4-graph on $[n]$ with at least $(\delta - c_0)\binom{n}{4}$ edges.

Apply Symmetrization to $G$ with respect to the constants $c_2, c_1, c_0, n_0$ as specified in Section 3. Let $G_i, V_i, U_i, \mathcal{P}_i$, for $i = 0, 1, \ldots, l$, describe the process.

By Lemma 5 we have $|U_l| = 4$; let us assume that $U_l = [4]$. Thus $G_l$ is the complete 4-partite graph with the parts $P_{l,1}, \ldots, P_{l,4}$. Let $V = V_l$ and $m = |V_l|$.

For $i \in [0, l]$, let $G_i'$ and $V_i'$ denote the configuration after the $i$-th Cleaning Up. We have $V_i' \supseteq V$ and $G_i'[V] = G_i[V]$. Thus the 4-graph $G_i[V]$ can be obtained from $G_i'$ by removing all vertices of $V_i' \setminus V$; this reduces the minimum degree by at most $|V_i' \setminus V| \binom{n-2}{2}$. It follows from Lemma 4 that the minimal degree of each $G_i[V]$, $i \in [0, l]$, is at least

$$(\delta - c_1)\binom{m-1}{3} - \frac{2c_0}{c_1} n \binom{n-2}{2} \geq (\delta - 2c_1)\binom{m-1}{3}. \tag{14}$$

Now, we reverse Merging. Let us call this process *Splitting*. Initially, define $\mathcal{Q}_l = \mathcal{P}_l$ and denote its parts by $Q_{l,u} = P_{l,u}$ for $u \in [4]$. Inductively, for $i = l-1, l-2, \ldots, 0$ we define a partition $\mathcal{Q}_i$ of $V$, namely $V = \cup_{j=1}^{4} Q_{i,j}$, such that the following *Properties I–III* hold.

I. For every $j \in [4]$ we have $j \in Q_{i,j}$.

II. For any $v \in V$ there is $u \in [4]$ such that $P_{i,v} \cap V \subseteq Q_{i,u}$, that is, $\mathcal{Q}_i$ is a coarser partition than $\mathcal{P}_i$ when restricted to $V$.

III. $G_i[V]$ is a 4-partite 4-graph (not necessarily complete or balanced) with parts $Q_{i,1}, \ldots, Q_{i,4}$.

Let us remark here that, unlike Symmetrization, the Splitting process does not change the vertex set: it remains $V = V_l$ all the time. As we have already said, once a vertex is deleted during Cleaning Up, it and all incident to it edges never reappear again. Also, please note that we maintain the same number of the parts (i.e. 4 parts) during Splitting.

If we can maintain Properties I–III, then, in view of $|[n] \setminus V| \leq \frac{2c_0}{c_1} n < c_1 n$, the partition $V = Q_{0,1} \cup \cdots \cup Q_{0,4}$ shows that $G_0[V] = G[V]$ is the required 4-partite subgraph of $G$, proving the theorem.

Clearly, Properties I–III hold for $i = l$. Suppose that for some $i \in [l]$ we have already defined a partition $\mathcal{Q}_i = \{Q_{i,j} : j \in [4]\}$ such that Properties I–III hold. We have to show how to find the desired partition $\mathcal{Q}_{i-1}$.

Note that the inequality (14) implies that $|G_i[V]| \ge (\delta - 2c_1)\binom{m}{4}$. Also, any 4-partite 4-graph of order $m$ has at most $(m/4)^4 = \delta m^4/4!$ edges, just slightly more than $(\delta - 2c_1)\binom{m}{4}$. Routine calculations show that, for example,

$$\left| |Q_{i,j}| - \frac{m}{4} \right| \le \frac{c_2}{30}\, m, \quad \forall j \in [4], \tag{15}$$

and

$$\left| d_{S_i}(x) - \delta \binom{m-1}{3} \right| \le \frac{c_2}{10} \binom{m-1}{3}, \quad \forall x \in V, \tag{16}$$

where $S_i$ is the complete 4-partite 4-graph with parts $Q_{i,1}, \ldots, Q_{i,4}$. By our assumption, $G_i[V] \subseteq S_i$. This, (14), and (16) imply that

$$d_{S_i \setminus H}(x) \le \left(2c_1 + \frac{c_2}{10}\right) \binom{m-1}{3}, \quad \forall\, x \in V. \tag{17}$$

Assume that at the $i$-th Merging step we have merged $P_{i-1,w}$ into $P_{i-1,u}$ with $u, w \in U_{i-1}$ so that $u \in U_i$ but $w \notin U_i$.

Let us recycle some of the previous variables by denoting $H = G_i[V]$, $H' = G_{i-1}[V]$, $A = P_{i-1,u} \cap V$, and $B = P_{i-1,w} \cap V$. Thus, $H$ is obtained from $H'$ by merging $B$ into $A$. By Property II, assume without loss of generality that $A \cup B \subseteq Q_{i,1}$. Define $W_1 = Q_{i,1} \setminus B$ and $W_j = Q_{i,j}$ for $j \in [2, 4]$. We know that $H$ is 4-partite and we want to prove the same claim about $H'$. Recall that $U_l = [4]$ is a transversal for $\mathcal{Q}_i$.

If $B = \emptyset$, then we are trivially done: $H' = H$ and we can let $Q_{i-1,j} = W_j$ for $j \in [4]$ (that is, $\mathcal{Q}_{i-1} = \mathcal{Q}_i$). So, suppose that $B \ne \emptyset$. By Lemma 6, we have $A \ne \emptyset$. Also, Property 1 of Section 3.1 shows that $u, w \in V$.

By (15) we know that each of $W_2, W_3, W_4$ has about $\frac{m}{4}$ elements. However, although $W_1 \supseteq A$ and $B$ are non-empty, one of them may be very small. Thus we have to be extra careful when doing any estimates involving $|W_1|$ and $|B|$.

The hypergraph $H'[V \setminus B] = H[V \setminus B]$ is 4-partite with parts $W_1, \ldots, W_4$. To finish the proof of the theorem, it is enough to show that we can add $B$ to some part $W_j$, thus letting $Q_{i-1,j} = W_j \cup B$ and $Q_{i-1,h} = W_h$ for $h \ne j$. Properties I–II will hold automatically (note that $B \cap [4] = \emptyset$ by Property 1 of Section 3.1), so we have to care about Property III only.

Let us show that

$$F_1 = \emptyset \text{ and } |F_j| \le c_2\binom{m-1}{3}|B|, \quad \forall\, j \in [2,4], \tag{18}$$

where

$$F_j = \{D \in H' : D \cap B \ne \emptyset,\ |D \cap W_j| \ge 2\}.$$

Suppose first on the contrary to our claim that (18) does not hold for some $j \in [2,4]$. Assume without loss of generality that $j = 4$. Fix any $v_1 \in A$. We know that $A$ is non-empty. (This is why we needed Lemma 6.) The following holds for any choices of an edge $D \in F_4$, of vertices $v_2 \in W_2$, $v_3 \in W_3$, and of two distinct vertices $v_4, v_4' \in D \cap W_4$: at least one of the two quadruples $\{v_1, v_2, v_3, v_4\}$ and $\{v_1, v_2, v_3, v_4'\}$ is missing from $H$. (Indeed, their symmetric difference is $\{v_4, v_4'\}$ so together with $D$ they form a forbidden subgraph.) On the other hand, any such missing edge is counted at most $\binom{m}{2}|B|$ times, which is a crude upper bound on the number of ways to pick an edge $D$ that contains a given vertex of $W_4$ and intersects $B$. Since $W_2$ and $W_3$ have at least $(\frac{1}{4} - \frac{c_2}{30})m$ vertices each by (15), we have identified at least

$$\frac{c_2\binom{m-1}{3}|B| \times ((\frac{1}{4} - \frac{c_2}{30})m)^2}{\binom{m}{2}|B|} \ge \frac{c_2}{9}\binom{m-1}{3}$$

edges of $S_i \setminus H$ which contain $v_1$. This contradicts (17).

Suppose next that $F_1 \ne \emptyset$. Fix one edge $D \in F_1$ and choose distinct $v_1, v_1' \in W_1 \cap D$. For every choice of $v_h \in W_h$ for $2 \le h \le 4$, at least one of the edges $\{v_1, v_2, v_3, v_4\}$ or $\{v_1', v_2, v_3, v_4\}$ is missing from $H$. At least half of the missing edges contain, say, $v_1$. Thus, we have identified at least $\frac{1}{2}\,((\frac{1}{4} - \frac{c_2}{30})m)^3$ edges of $S_i \setminus H$ containing $v_1$, contradicting (17). This finishes the proof of (18).

Note that every edge of $H'$ intersects $B$ in at most one vertex. By (14), when applied to $H' = G_{i-1}[V]$, we have at least $|B| \times (\delta - 2c_1)\binom{m-1}{3}$ $H'$-edges intersecting $B$. By (18), we have $|F_1 \cup F_2 \cup F_3 \cup F_4| \le 3c_2\binom{m-1}{3}|B|$. The remaining edges through $B$ are each intersecting 3 different parts $W_j$. Therefore there is a 3-set $J \subseteq [4]$ such that

$$|F_J| \ge \frac{\delta - 2c_1 - 3c_2}{4}\binom{m-1}{3}|B| \ge \frac{\delta}{5}\binom{m-1}{3}|B|,$$

where

$$F_J = \{D \in H' : D \cap B \ne \emptyset \text{ and } \forall\, h \in J\ |D \cap W_h| = 1\}\,.$$

Let $\{j\} = [4] \setminus J$. Also, recall that any permutation that moves only vertices of $B$ is an automorphism of $H'$.

First, suppose that there is a vertex $x \in W_j$ such that for some (equivalently, arbitrary) $y \in B$ the pair $\{x, y\}$ is covered by an edge of $H'$. For every $D \in F_J$ with $y \in D$, $S_i$ contains $D' = (D \cup \{x\}) \setminus \{y\}$. The symmetric difference of $D$ and $D'$ is $\{x, y\}$. Hence $D'$ cannot belong to $H'$. Since $D' \cap B = \emptyset$, we have $D' \notin H$. Thus we have identified at least $|F_J|/|B| \geq \frac{\delta}{5}\binom{m-1}{3}$ edges of $S_i \setminus H$ that pass through some fixed vertex $x$. Again, this contradicts (17). Thus, let us assume that

$$\text{no edge of } H' \text{ intersects both } B \text{ and } W_j. \tag{19}$$

In particular, we conclude that $F_j = \emptyset$.

Let us define $Q_{i-1,j} = W_j \cup B$ and $Q_{i-1,h} = W_h$ for $h \in J$. Let $S_{i-1}$ be the complete 4-partite 4-graph with the parts $Q_{i-1,1}, \ldots, Q_{i-1,4}$.

We claim that this choice does the job. As we have already mentioned, only Property III needs justification.

By (18) and (19) the number of edges in $H' \setminus S_{i-1}$ containing any $x \in B$ is at most $3c_2\binom{m-1}{3}$. Thus, by (14),

$$d_{H' \cap S_{i-1}}(x) \geq (\delta - 2c_1 - 3c_2)\binom{m-1}{3}, \quad \forall\, x \in B. \tag{20}$$

Since $W_j \cup B \supseteq Q_{i,j}$, we have by (15)

$$|B \cup W_j| \geq \left(\frac{1}{4} - \frac{c_2}{30}\right) m. \tag{21}$$

This implies that each $W_h$ with $h \in J$ has at least $(\frac{1}{4} - 2\sqrt{c_2})m$ vertices for otherwise we obtain from (20) and (21) that

$$\left(\frac{1}{4} - 2\sqrt{c_2}\right) m \times \frac{1}{4}\left(m - \left(\frac{1}{4} - \frac{c_2}{30}\right) m - \left(\frac{1}{4} - 2\sqrt{c_2}\right) m\right)^2 \geq (\delta - 2c_1 - 3c_2)\binom{m-1}{3},$$

which is impossible for $1 \gg c_2 \gg c_1$ as routine calculations show.

Suppose on the contrary to Property III that some $F_h$ with $h \in J$ is non-empty, say $F_h \ni D$. Fix distinct $v_h, v_h' \in D \cap W_h$.

Suppose first that, for example, $|W_j| \geq \frac{m}{100}$. Then for each choice of $v_f \in W_f$, $f \in [4] \setminus \{h\}$, either $E = \{v_1, v_2, v_3, v_4\}$ or $(E \cup \{v_h'\}) \setminus \{v_h\}$ is missing from $H$. This way we identify at least $\frac{1}{2}((\frac{1}{4} - 2\sqrt{c_2})m)^2 \frac{m}{100}$ edges of $S_i \setminus H$ incident either to $v_h$ or to $v_h'$, which contradicts (17). So, assume $|W_j| < \frac{m}{100}$. This implies that $j = 1$ and, by (21), that

$|B| \geq (0.24 - \frac{c_2}{30})m$. Since $H'[V \setminus B]$ is a 4-partite graph, we conclude, by (14) applied to $H' = G_{i-1}[V]$, that each of $v_h$ and $v'_h$ is in at least

$$(\delta - 2c_1)\binom{m-1}{3} - |W_1|\binom{m}{2}$$

$H'$-edges intersecting $B$, of which at most $3c_2\binom{m-1}{3}m$ edges can intersect some part $W_f$ in more than one vertex by (18) and (19). The number of the remaining edges is strictly more than

$$\frac{1}{2} \times \frac{(m - |W_1| - |W_h| - |B|)^2}{4} \times |B|,$$

that is, more than the half of the size of the complete 3-partite 3-graph with parts $B$, $W_f$, and $W_g$, where $\{1, f, g, h\} = [4]$. So some two of the edges via $v_h$ and $v'_h$ coincide on the complement of $W_h$. This means that their symmetric difference is exactly $\{v_h, v'_h\}$, giving us a copy of $\Sigma_4$. This final contradiction proves Property III and completes the proof of the theorem. ∎

A direct corollary of (2) and Theorem 7 is the following.

**Theorem 8** $\Sigma_4$ *is stable.* ∎

# 5 The Stability of $\Sigma_k$ Implies the Stability of $T_k$

The below Lemma 9 holds for any $k$. It directly follows from Pikhurko [17, Lemma 4]. Since the latter result relies on the so-called *Removal Lemma*, whose proof uses the Hypergraph Regularity Lemma (see Gowers [8], Nagle, Rödl, Schacht, and Skokan [16, 18, 19], Tao [22], Elek and Szegedy [3]), we give an alternative self-contained proof of Lemma 9. Informally, Lemma 9 follows from the fact that every $\Sigma_k$-free graph can be made $T_k$-free by removing a small proportion of edges.

**Lemma 9** *Let $k \geq 3$. If $\Sigma_k$ is $s$-stable, then $T_k$ is $s$-stable.*

*Proof.* Let $\varepsilon > 0$ be given. By the definition of $s$-stability, there are $c > 0$ and $n_0$ such that of any $s+1$ $\Sigma_k$-free $k$-graphs, each having $n \geq n_0$ vertices and at least $(\pi(\Sigma_k) - c)\binom{n}{k}$ edges, some two are $\frac{\varepsilon}{2}\binom{n}{k}$-close to each other. We can assume that $c < \frac{\varepsilon}{2}$.

Let $G_1, \ldots, G_{s+1}$ be arbitrary $T_k$-free $k$-graphs on the same vertex set $[n]$, $n \geq n_0$, each having at least $(\pi(T_k) - \frac{c}{2})\binom{n}{k}$ edges.

Take some $i \in [s+1]$. Call a pair $\{x,y\} \in \binom{[n]}{2}$ *$i$-sparse* if there are at most $(k-1)\binom{n}{k-3}$ $G_i$-edges containing both $x$ and $y$. Let $G_i'$ be obtained from $G_i$ by removing all edges containing sparse pairs, at most $\binom{n}{2} \times (k-1)\binom{n}{k-3} < \frac{c}{2}\binom{n}{k}$ edges. Thus,

$$|G_i'| \geq (\pi(T_k) - c)\binom{n}{2}, \quad \forall\, i \in [s+1]. \tag{22}$$

We claim that $G_i'$ is $\Sigma_k$-free. Suppose on the contrary that $E_1 \triangle E_2 = \{x,y\} \subseteq E_3$ with $E_1, E_2, E_3 \in G_i'$. As the edge $E_3$ has not been deleted, the pair $\{x,y\}$ was not sparse (with respect to $G_i$). Thus $G_i$ has more than $(k-1)\binom{n}{k-3}$ edges containing both $x, y$, of which at least one, say $E_4$, must be disjoint from the $(k-1)$-set $E_1 \cap E_2$. But then $E_1, E_2, E_4 \in G_i$ form a $T_k$-subgraph, a contradiction.

By (22) and the $s$-stability of $\Sigma_k$, we conclude that some two $k$-graphs, say $G_i'$ and $G_j'$, are $\frac{\varepsilon}{2}\binom{n}{k}$-close. But then $G_i$ and $G_j$ are $(\frac{\varepsilon}{2} + \frac{c}{2} + \frac{c}{2})\binom{n}{k}$-close. Since $c < \frac{\varepsilon}{2}$ and $\varepsilon$ is arbitrary, we conclude that $T_k$ is $s$-stable, as required. ∎

Theorem 8 and Lemma 9 imply the following result.

**Corollary 10** *$T_4$ is stable.* ∎

The stability of $T_3$ was established by Keevash and Mubayi [11].

## 6 The Stability of $T_4$ Gives the Exact Result

Now we are ready to prove Theorem 1.

*Proof of Theorem 1.* Let $1 \gg c_1 \gg c_0 \gg \frac{1}{n_0}$. Let $n \geq n_0$ and let $G$ be a maximum $T_4$-free 4-graph on $[n]$. It is enough to prove the theorem under the additional assumption that the minimal degree of $G$ is at least $t(n) - t(n-1)$, where $t(n)$ is the function in the right-hand side of (2). (Otherwise an "improving induction" argument, see Keevash and Sudakov [13, Page 680], leads to a contradiction.)

Let $[n] = W_1 \cup \dots \cup W_4$ be a partition such that $|P \cap G|$ is maximum, where $P$ is the complete 4-partite 4-graph with parts $W_1, \dots, W_4$. Assume that $G \setminus P \neq \emptyset$ for otherwise we are easily done. We are going to exhibit a contradiction.

By the stability of $T_4$ (Corollary 10) we have

$$|G \setminus P| \leq c_0 n^4. \tag{23}$$

Let $w_i = |W_i|$. It easily follows from (23) and $|G| \geq t(n)$ that, for example, each $w_i \geq \frac{n}{5}$.

The edges of $P \setminus G$ are called *missing*. The edges of $G \setminus P$ are called *bad*. A pair $\{x, y\}$ is *bad* if $x, y$ lie in the same part $W_i$ and $\{x, y\}$ is covered by an edge of $G$.

Clearly, $P$ does not contain $T_4$ as a subgraph. By the maximality of $G$,

$$|G \setminus P| \geq |P \setminus G|, \tag{24}$$

that is, the number of bad edges is at least the number of missing edges.

Let $A \subseteq V$ consist of all vertices which belong to at least $c_1 n^3$ missing edges.

**Claim 1** Each bad pair intersects $|A|$.

*Proof of Claim.* Let $\{x_1, x_1'\}$ be a bad pair, say $x_1, x_1' \in W_1$ with $\{x_1, x_1', y, z\} \in G$. For all choices of $x_i \in W_i \setminus \{y, z\}$, $i = 2, 3, 4$, at least one of the edges $\{x_1, x_2, x_3, x_4\}$ or $\{x_1', x_2, x_3, x_4\}$ is missing. This means that we have encountered at least

$$(w_2 - 2)(w_3 - 2)(w_4 - 2) > 2c_1 n^3$$

missing edges that intersect $\{x_1, x_1'\}$. One of $x_1$ and $x_1'$ belongs to at least a half of these edges, giving the required. ∎

Let $B$ be the 2-graph consisting of all bad pairs. Thus, by Claim 1, $A$ is an edge-dominating set for $B$. Since $B$ is non-empty, $A$ is non-empty. We have, for example,

$$|A| \leq c_1^2 n, \tag{25}$$

for otherwise we encounter at least $c_1^2 n \times c_1 n^3 / \binom{4}{2} > c_0 n^4$ missing edges, a contradiction to (23). It follows that the number of bad pairs $|B| \leq c_1^2 n^2$.

By the definition of $A$, there are at least $|A| c_1 n^3/4$ missing edges and consequently at least $|A| c_1 n^3/4$ bad edges. Call a bad pair $\{x, y\}$ *dense* if there are at least $3n$ edges of $G$ containing it. (All such edges are bad by definition.) Let $\mathcal{M}$ consist of all pairs $(E, D)$, where $E$ is a bad edge and $D \subseteq E$ is a dense bad pair. Each bad edge contains at least one bad pair while *sparse* pairs (that is, bad pairs which are not dense) belong to at most $3n \times |B| \leq 3n \times c_1^2 n^2$ bad edges. Hence

$$|\mathcal{M}| \geq \frac{|A| c_1 n^3}{4} - 3c_1^2 n^3 \times \binom{4}{2} > \frac{|A| c_1 n^3}{5}.$$

On the other hand, for each pair $(E, D) \in \mathcal{M}$ we have $D \cap A \neq \emptyset$ by Claim 1. It follows that some vertex $x_1 \in A$ belongs to $D$ for at least $c_1 n^3/5$ pairs $(E, D) \in \mathcal{M}$.

Assume that, for example, $x_1 \in W_1$. Let $Y \subseteq W_1 \setminus \{x_1\}$ consist of those $y \in W_1$ such that $\{x_1, y\}$ is a dense bad pair. We have $|Y| \geq \frac{1}{5}c_1 n^3/\binom{n}{2} > c_1 n/3$.

Let $x_i \in W_i$, $i = 2, 3, 4$ be arbitrary. If $\{x_1, x_2, x_3, x_4\} \in G$, then for each $y \in Y$ the 4-tuple $\{y, x_2, x_3, x_4\}$ is a missing edge: this follows from the fact that the pair $\{x_1, y\}$ is dense, so there must be a $G$-edge $D \ni x_1, y$ disjoint from $\{x_2, x_3, x_4\}$. Hence, for at most $c_1 n^3$ choices of $x_2, x_3, x_4$ we have $\{x_1, x_2, x_3, x_4\} \in G$, for otherwise we get at least $c_1 n^3 \times c_1 n/3 > c_0 n^4$ missing edges, contradicting (23) and (24). Thus we have shown that

$$d_{G\cap P}(x_1) \leq c_1 n^3. \tag{26}$$

Now we argue that we can move $x_1$ to some other part and increase $|G \cap P|$, which would be a contradiction to the choice of the partition $W_1, \ldots, W_4$. We know that the minimal degree of $G$ is at least $t(n) - t(n-1) \geq (\delta - c_0)\binom{n}{3}$, where $\delta = \pi(T_4) = \frac{3}{32}$.

If there are at least $c_1 n^3$ $G$-edges $D$ containing $x_1$ such that $D \setminus \{x_1\}$ intersects some part $W_i$, $i \in [4]$, in at least two vertices, then this creates at least $c_1 n^2$ bad pairs. This in turn forces the dominating set $A$ to have size at least $c_1 n > c_1^2 n$, which contradicts (25).

So, let us assume otherwise. This means by (26) that there are at least $(\delta - c_0)\binom{n}{3} - 2c_1 n^3 > 3c_1 n^3$ edges of $G$ which contain $x_1$, intersect $W_1 \setminus \{x_1\}$, and then intersect some two of the parts $W_2$, $W_3$, and $W_4$. Without loss of generality assume that at least a third of these edges intersect both $W_2$ and $W_3$. Thus, if we move $x_1$ to $W_4$ and update $P$ correspondingly, then all these edges will belong to $P \cap G$. On the other hand, the number of edges that $P \cap G$ loses during this move is at most $c_1 n^3$ by (26). Thus $|P \cap G|$ strictly increases, which contradicts the choice of $P$, completing the proof. ∎

# 7 Concluding Remarks

One of the difficulties in extending our approach to $k = 5$ or $k = 6$ is that the corresponding analog of Theorem 7 is false. Let us outline an example for $k = 5$. Take the (unique) maximum $\Sigma_5$-free 5-graph $G$ of order $n = 11m$, which is a blow-up of the $(11, 5, 4)$ Steiner system, see [6]. Choose some 5 parts spanning no edge and add into them $m$ vertex-disjoint 5-edges, each transversing these 5 parts. For each added edge $D$ remove all $G$-edges $D'$ with $|D' \cap D| = 4$. The obtained 5-graph $H$ is still $\Sigma_5$-free and has size $(\pi(\Sigma_5) + o(1))\binom{n}{5}$. However, in order to make $H$ into a subgraph of a blown-up

Steiner system we have to remove either an almost whole part or at least one vertex from each added edge, in either case at least $(\frac{1}{11}+o(1))n$ vertices.

Still, we believe that the following is true.

**Conjecture 11** *Both $\Sigma_5$ and $\Sigma_6$ are stable.*

# Acknowledgment

The author thanks the referees for the careful reading which greatly improved the presentation of this paper.

# References


[1] B. Bollobás, *Three-graphs without two triples whose symmetric difference is contained in a third*, Discrete Math. **8** (1974), 21–24.

[2] D. de Caen, *Uniform hypergraphs with no block containing the symmetric difference of any two other blocks*, Congres. Numer. **47** (1985), 249–253.

[3] G. Elek and B. Szegedy, *Limits of hypergraphs, removal and regularity lemmas. A non-standard approach*, Manuscript, 2007.

[4] P. Erdős and M. Simonovits, *Supersaturated graphs and hypergraphs*, Combinatorica **3** (1983), 181–192.

[5] P. Frankl and Z. Füredi, *A new generalization of the Erdős-Ko-Rado theorem*, Combinatorica **3** (1983), 341–349.

[6] P. Frankl and Z. Füredi, *Extremal problems whose solutions are the blowups of the small Witt-designs*, J. Combin. Theory (A) **52** (1989), 129–147.

[7] Z. Füredi and M. Simonovits, *Triple systems not containing a Fano configuration*, Combin. Prob. Computing **14** (2005), 467–488.

[8] W. T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi's theorem*, Manuscript, 2005.

[9] G. O. H. Katona, *Extremal problems for hypergraphs*, Combinatorics, vol. 56, Math. Cent. Tracts, 1974, pp. 13–42.

[10] G. O. H. Katona, T. Nemetz, and M. Simonovits, *On a graph problem of Turán (In Hungarian)*, Mat. Fiz. Lapok **15** (1964), 228–238.

[11] P. Keevash and D. Mubayi, *Stability results for cancellative hypergraphs*, J. Combin. Theory (B) **92** (2004), 163–175.

[12] P. Keevash and B. Sudakov, *The Turán number of the Fano plane*, Combinatorica **25** (2005), 561–574.

[13] P. Keevash and B. Sudakov, *On a hypergraph Turán problem of Frankl*, Combinatorica **25** (2006), 673–706.

[14] W. Mantel, *Problem 28*, Winkundige Opgaven **10** (1907), 60–61.

[15] D. Mubayi and O. Pikhurko, *A new generalization of Mantel's theorem to $k$-graphs*, J. Combin. Theory (B) **97** (2007), 669–678.

[16] B. Nagle, V. Rödl, and M. Schacht, *The counting lemma for regular $k$-uniform hypergraphs*, Random Struct. Algorithms **28** (2005), 113–179.

[17] O. Pikhurko, *Exact computation of the hypergraph Turán function for expanded complete 2-graphs*, E-print `arXiv:math.CO/0510227`, accepted by *J. Comb. Th. (B)*. The publication is suspended because of a disagreement over copyright, see `http://www.math.cmu.edu/~pikhurko/Copyright.html`, 2005.

[18] V. Rödl and J. Skokan, *Regularity lemma for $k$-uniform hypergraphs*, Random Struct. Algorithms **25** (2004), 1–42.

[19] V. Rödl and J. Skokan, *Applications of the regularity lemma for uniform hypergraphs*, Random Struct. Algorithms **28** (2006), 180–194.

[20] J. B. Shearer, *A new construction for cancellative families of sets*, Electronic J. Combin. **3** (1996), 3pp.

[21] A. F. Sidorenko, *The maximal number of edges in a homogeneous hypergraph containing no prohibited subgraphs*, Math Notes **41** (1987), 247–259, Translated from *Mat. Zametki.*

[22] T. Tao, *A variant of the hypergraph removal lemma*, J. Combin. Theory (A) **113** (2006), 1257–1280.

[23] P. Turán, *On an extremal problem in graph theory (in Hungarian)*, Mat. Fiz. Lapok **48** (1941), 436–452.