A new generalization of Mantel's theorem to k-graphs

Dhruv Mubayi*
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL 60607

Oleg Pikhurko†
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213-3890
Web: `http://www.math.cmu.edu/~pikhurko`

October 30, 2006

Abstract

Let the k-graph Fan^k consist of k edges that pairwise intersect exactly in one vertex x, plus one more edge intersecting each of these edges in a vertex different from x. We prove that, for n sufficiently large, the maximum number of edges in an n-vertex k-graph containing no copy of Fan^k is $\prod_{i=1}^{k} \lfloor \frac{n+i-1}{k} \rfloor$, which equals the number of edges in a complete k-partite k-graph with almost equal parts. This is the only extremal example. This result is a special case of our more general theorem that applies to a larger class of excluded configurations.

1 Introduction

The first theorem in extremal graph theory is Mantel's 1907 result, which determines the maximum number of edges in a triangle-free graph on n vertices (cf. Turán [22]). There are several possible generalizations of this problem to k-uniform hypergraphs (*k-graphs* for short). One was suggested by Katona [9] and Bollobás [1] (see Frankl-Füredi [4, 5], de Caen [2], Sidorenko [20], Shearer [19], Keevash-Mubayi [10], Pikhurko [16]). Another extension, the so-called *expanded*

*Partially supported by NSF Grant DMS-0400812, and an Alfred P. Sloan Research Fellowship.
†Partially supported by NSF Grant DMS-0457512.

triangle, was studied by Frankl [3] and Keevash-Sudakov [11]. In this paper we provide yet another generalization.

Let Fan^k be the k-graph comprising $k+1$ edges $E_1, \ldots, E_k, E$, with $E_i \cap E_j = \{x\}$ for all $i \neq j$, where $x \notin E$, and $|E_i \cap E| = 1$ for all i. In other words, k edges share a single common vertex x and the last edge intersects each of the other edges in a single vertex different from x. Please note that Fan^2 is simply a triangle, and in this sense Fan^k generalizes the definition of K_3. There is another, perhaps more subtle way that Fan^k is an extension of K_3.

Call a hypergraph *simple* if every two edges share at most one vertex. One of the formulations of the celebrated Erdős-Faber-Lovász conjecture states that the minimum number of edges in a simple k-graph that is not k-partite is $k+1$. Kahn [8] proved this with $k+1$ replaced by $(1+o(1))\,k$, but the question of the exact value remains open. If the conjecture is true, then Fan^k is a simple k-graph that is not k-partite with the minimum number of edges, and in this sense it generalizes a 2-graph triangle.

For $l \geq k$, let $T_l^k(n)$ be the complete l-partite k-graph with part sizes $\lfloor n/l \rfloor$ or $\lceil n/l \rceil$: every edge of $T_l^k(n)$ has at most one vertex in each of the l parts, and all edges subject to this restriction are present. Let

$$t_l^k(n) = |T_l^k(n)|.$$

(We identify a k-graph with its edge set.) It is convenient to agree that $T_l^k(n) = \emptyset$ and $t_l^k(n) = 0$ if $l < k$. Given a k-graph F, we write $\mathrm{ex}(n, F)$ for the maximum number of edges in an n-vertex k-graph containing no copy of F. Mantel proved that $\mathrm{ex}(n, \mathrm{Fan}^2) = t_2^2(n)$ for all positive n. Here we generalize this to $k > 2$, for large n.

Theorem 1 *Let $k \geq 3$. Then, for all sufficiently large n, the maximum number of edges in an n-vertex k-graph containing no copy of* Fan^k *is $t_k^k(n) = \prod_{i=1}^{k} \lfloor \frac{n+i-1}{k} \rfloor$. The only k-graph for which equality holds is $T_k^k(n)$.*

Our approach to proving Theorem 1 comes from two recent papers by the current authors [13, 15]. Although the paper [15] has been accepted by the Journal of Combinatorial Theory, Series B, its publication is suspended for an indefinite period of time because of a disagreement over the copyright between the author and the publisher. We feel that the approach is quite versatile and may be applicable to other hypergraph Turán problems. Therefore, we give a complete description of the method and provide self-contained proofs for any claims from [15].

So suppose that we wish to prove that $\mathrm{ex}(n, F) = t_l^k(n)$ for a given F. The method has four steps:

Step 1. Define an appropriately chosen family $\mathcal{F}$ of k-graphs such that $F \in \mathcal{F}$. There is no general recipe for $\mathcal{F}$. A particular property that $\mathcal{F}$ should possess is that any F-free k-graph of order n can be made $\mathcal{F}$-free by removing $o(n^k)$ edges. Then $\mathrm{ex}(n, F) = \mathrm{ex}(n, \mathcal{F}) + o(n^k)$ but, hopefully, $\mathrm{ex}(n, \mathcal{F})$ is easier to analyze.

Step 2. Prove that $\mathcal{F}$ is *stable* with respect to $T_l^k(n)$. Loosely speaking, this means that every $\mathcal{F}$-free k-graph G on n vertices with close to $\mathrm{ex}(n, \mathcal{F})$ edges can be transformed to $T_l^k(n)$ without changing too many edges.

Step 3. From the stability of $\mathcal{F}$, deduce the stability of F. (We use the property of $\mathcal{F}$ from Step 1, whose proof is combined with Step 3 in this article.)

Step 4. Using the stability of F, deduce the exact result $\mathrm{ex}(n, F) = t_l^k(n)$. This technique was first employed by Simonovits [21] to determine $\mathrm{ex}(n, F)$ exactly for color-critical 2-graphs F. Recently, stability has been used to determine exact results for several hypergraph Turán problems [6, 10, 11, 12, 15, 16].

The next three sections give the details of Steps 2–4, culminating in a proof of Theorem 1. Actually, our main result, Theorem 3 proved in Section 4, determines the exact extremal function for a more general configuration which includes Fan^k as a special case. We next define the family used in Step 1.

Fix $l \geq k \geq 2$. Let $\mathcal{F}_l^k$ be the set of all minimal k-graphs F such that there is an l-set C, called the *core*, such that at least one edge $D \in F$ lies entirely in C and every pair of vertices of C is covered by an edge of F. (Of course, it suffices to consider only pairs not inside D.) Let F_l^k be the k-graph with edges: $[k]$ and $E_{ij} \cup \{i, j\}$ over all pairs $\{i, j\} \in \binom{[l]}{2} \setminus \binom{[k]}{2}$, where E_{ij} are pairwise disjoint $(k-2)$-sets disjoint from $[l]$. Clearly, $F_l^k \in \mathcal{F}_l^k$. Note that

- $\mathcal{F}_k^k = \{F_k^k\}$ and F_k^k is the k-graph of one edge,
- $\mathcal{F}_l^2 = \{K_l^2\}$,
- $F_{k+1}^k = \mathrm{Fan}^k$.

For $l \geq k \geq 2$ a particular $\mathcal{F}_{l+1}^k$-free k-graph is $T_l^k(n)$. It is easy to see this, since if $T_l^k(n)$ contains a copy of $F \in \mathcal{F}_{l+1}^k$, then the vertex set in $T_l^k(n)$ playing the role of C must have at most one point in each part of $T_l^k(n)$ but there are not enough parts to accommodate these $l+1$ vertices. Consequently, the maximum size of an n-vertex $\mathcal{F}_{l+1}^k$-free k-graph is at least $t_l^k(n)$. In fact, we have an equality:

Theorem 2 *Let $n \geq l \geq k \geq 2$, and let G be an n-vertex $\mathcal{F}_{l+1}^k$-free k-graph. Then $|G| \leq t_l^k(n)$, and if equality holds then $G = T_l^k(n)$.*

This result can be proved by a straightforward modification of the proof of Theorem 1 in

[13]. Also, one can obtain it as a by-product of our proof of Theorem 4 below (see the remark following the inequality (4)).

The main theorem of the current paper is the following extension of Theorem 1.

Theorem 3 *Let $l \geq k \geq 2$. Then, for all sufficiently large n, we have $\mathrm{ex}(n, F_{l+1}^k) = t_l^k(n)$ and $T_l^k(n)$ is the unique maximum F_{l+1}^k-free k-graph of order n.*

Let us specify here the notation we are going to use. We write $V(G)$ for the vertex set of a k-graph G. Given a vertex $x \in V(G)$, the *link* of x is the $(k-1)$-graph

$$L_G(x) = \{S \setminus \{x\} : S \in G,\ S \ni x\},$$

and the *degree* is $\deg_G(x) = |L_G(x)|$. The *codegree* of x and y, written $\mathrm{codeg}_G(x, y)$, is the number of edges in G containing both x and y, and the *neighborhood* of x is

$$N_G(x) = \{y : \mathrm{codeg}(x, y) > 0,\ y \neq x\}.$$

Given $X \subset V(G)$, let $e_G(X)$ be the number of edges in G that contain at least two vertices from X. In all cases above, we omit the subscript G if the k-graph G is obvious from context. For $S \subset V(G)$, we write $G[S]$ for the hypergraph induced by G on S. Two k-graphs F and G of the same order are *m-close* if we can add or remove at most m edges from the first graph and make it isomorphic to the second; in other words, for some bijection $\sigma : V(F) \to V(G)$ the symmetric difference between $\sigma(F) = \{\sigma(D) : D \in F\}$ and G has at most m edges.

The notation $a \pm b$ means a number between $a - b$ and $a + b$.

2 Step 2: $\mathcal{F}_{l+1}^k$ is stable

Our goal in this section is to prove the following stability result.

Theorem 4 *For any $l \geq k \geq 2$ and $\delta > 0$ there exist $\varepsilon > 0$ and M such that the following holds for all $n > M$: If G is an n-vertex $\mathcal{F}_{l+1}^k$-free k-graph with at least $t_l^k(n) - \varepsilon n^k$ edges, then G is δn^k-close to $T_l^k(n)$.*

The proof of Theorem 4 has many similarities to that in [13, Theorem 3]. Thus we will refer to [13] for proofs of some claims, when the arguments are identical. In particular, we use the following facts shown in [13].

Equation (1) in [13]: For any $l \geq k \geq 2$ and $0 \leq s \leq n$ we have

$$t_{l-1}^k(n-s) + s \cdot t_{l-1}^{k-1}(n-s) \leq t_l^k(n). \tag{1}$$

Hint. The left-hand side of (1) is the number of edges in the complete l-partite k-graph with one part of size s and other part sizes being $\lfloor \frac{n-s}{l-1} \rfloor$ and $\lceil \frac{n-s}{l-1} \rceil$. ∎

Claim 1 in [13]: For any $l \geq k \geq 2$ and $\delta > 0$ there are $\varepsilon > 0$ and M such that, for any l-partite k-graph of order $n \geq M$ and size at least $t_l^k(n) - \varepsilon n^k$, the number of vertices in each part is

$$\left(\frac{1}{l} \pm \delta\right) n. \quad ∎ \tag{2}$$

Proof of Theorem 4. Our proof uses induction on $k + l$. It is convenient to start with the trivial base case $l = k - 1$ which formally satisfies the conclusion of the theorem: F_k^k is the k-graph of one edge, and $T_{k-1}^k(n)$ has no edges. The other base case $k = 2$ is the content of the Simonovits stability theorem [21], so we further assume that $l \geq k > 2$.

Let $\delta = \delta_l > 0$ be given. Our goal is to obtain $\varepsilon = \varepsilon_l$ and $M = M_l$ satisfying the theorem. We choose the constants in this order:

$$\delta_l \gg \delta_{l-1} \gg \varepsilon_{l-1} \gg \varepsilon_l \gg \frac{1}{M_{l-1}} \gg \frac{1}{M_l},$$

where $a \gg b$ means that $b > 0$ is sufficiently small depending on a (and k, l). In particular, we assume that $\varepsilon_{l-1}, M_{l-1}$ demonstrate the validity of the theorem for $l-1$, $k-1$, and δ_{l-1}. Suppose that $n > M_l$. Let G be an $\mathcal{F}_{l+1}^k$-free k-graph on n vertices with

$$|G| \geq t_l^k(n) - \varepsilon_l n^k. \tag{3}$$

Pick a vertex $x \in V(G)$ of maximum degree Δ. Let $N = N(x)$ be the neighborhood of x, that is, the set of vertices $y \neq x$ for which $\mathrm{codeg}_G(x, y) > 0$. Consider the k-graph $G[N]$ induced by N, and suppose that it contains a copy H of a member of $\mathcal{F}_l^k$. Let $C \subset V(H)$ be the core of H, and $D \subset C$ for some $D \in G$. Form H' from H by adding the vertex x and edges containing each pair $\{x, v\}$ with $v \in C$. These edges exist by the definition of N. Therefore H' contains a member of $\mathcal{F}_{l+1}^k$ with core $C \cup \{x\}$, which is a contradiction. Consequently, $G[N]$ is $\mathcal{F}_l^k$-free.

Next consider the $(k-1)$-graph L, where $L = L(x)$ is the link of x. Suppose that L contains a copy H of a member of $\mathcal{F}_l^{k-1}$. Enlarge every edge of H to contain x. The resulting k-graph contains a copy of some $H' \in \mathcal{F}_{l+1}^k$ with core $C \cup \{x\}$, a contradiction. Therefore L is $\mathcal{F}_l^{k-1}$-free.

Set $s = n - |N|$ and let $X = V(G) \setminus N$. Note that $x \in X$. Since $G[N]$ is $\mathcal{F}_l^k$-free and L is $\mathcal{F}_l^{k-1}$-free, Theorem 2 implies that $|G[N]| \leq t_{l-1}^k(n-s)$ and $\Delta = |L| \leq t_{l-1}^{k-1}(n-s)$. This gives

$$\begin{aligned} |G| &\leq |G[N]| + s \cdot \Delta - e_G(X) \\ &\leq t_{l-1}^k(n-s) + s \cdot t_{l-1}^{k-1}(n-s) - e_G(X) \\ &\leq t_l^k(n) - e_G(X), \end{aligned} \tag{4}$$

where the last inequality follows from (1). (Recall that $e_G(X)$ is the number of edges of G that intersect X in at least two vertices.) At this stage, one can deduce the upper bound in Theorem 2 by induction on $k+l$ since, obviously, $e_G(X) \geq 0$. (A further routine analysis will also show that $T_l^k(n)$ is the unique extremal configuration for $\mathrm{ex}(n, \mathcal{F}_{l+1}^k)$.)

The inequalites (3) and (4) imply that

$$t_l^k(n) - \varepsilon_l n^k \leq t_{l-1}^k(n-s) + s \cdot t_{l-1}^{k-1}(n-s).$$

Note that the right-hand side is the size of the l-partite k-graph with n vertices such that one part has size s and the other $l-1$ parts are almost equal. From (2), we conclude that

$$s = \left(\frac{1}{l} \pm \delta_{l-1}\right) n. \tag{5}$$

Moreover, routine calculations show (alternatively, see Claim 2 in [13, Theorem 3]) that (3) and (4) imply that

$$\Delta = |L| > t_{l-1}^{k-1}(n-s) - \varepsilon_{l-1}(n-s)^{k-1}. \tag{6}$$

Now consider L. This $(k-1)$-graph has $n-s$ vertices. Since $n \geq M_l \gg M_{l-1}$, we have $n - s \geq M_{l-1}$ by (5). Because of (6) we may apply the induction hypothesis to the $\mathcal{F}_l^{k-1}$-free $(k-1)$-graph L. We conclude that there exists a Turán hypergraph $T_{l-1} \cong T_{l-1}^{k-1}(n-s)$ with vertex partition $N = W_1 \cup \ldots \cup W_{l-1}$ such that

$$|T_{l-1} \bigtriangleup L| \leq \delta_{l-1}(n-s)^{k-1}. \tag{7}$$

By (5) we conclude that for each $i \in [l-1]$ we have

$$|W_i| = \frac{n-s}{l-1} \pm 1 = \left(\frac{1}{l} \pm \delta_{l-1}\right) n. \tag{8}$$

Let $W_l = X$ and let T_l be the l-partite k-graph with the vertex partition $W_1 \cup \ldots \cup W_l$. By (5) and (8) T_l is $\frac{\delta_l}{2}\, n^k$-close to a $T_l^k(n)$ because we can transform one to the other by moving at most $\delta_{l-1} n \times l$ vertices between parts, thus changing at most $\delta_{l-1} l n \times \binom{n-1}{k-1} < (\delta_l/2) n^k$ edges.

We will show that

$$|G \setminus T_l| \leq \frac{\delta_l}{5}\, n^k. \tag{9}$$

This implies, in view of (3) and the inequality $|T_l| \leq t_l^k(n)$, that

$$|G \bigtriangleup T_l| = |T_l| - |G| + 2|G \setminus T_l| \leq \varepsilon_l\, n^k + \frac{2\delta_l}{5}\, n^k < \frac{\delta_l}{2}\, n^k,$$

and the desired bound $|G \bigtriangleup T_l^k(n)| \leq \delta_l n^k$ follows from the triangle inequality.

From (4) we conclude that $e_G(X) \leq \varepsilon_l n^k$. Suppose on the contrary to (9) that we have more than $\frac{\delta_l}{5}\, n^k - \varepsilon_l n^k > \frac{\delta_l}{6}\, n^k$ edges of G intersecting some part of N in at least two vertices.

By averaging there is an $i \in [l-1]$ such that $|B| \geq \frac{\delta_l}{6l} n^2$, where B consists of all 2-subsets of W_i covered by at least one edge of G. Assume that $i = l-1$ without loss of generality.

Let $w = (\frac{1}{l} - \delta_{l-1})n$. Recall that w is a lower bound on each $|W_i|$ by (5) and (8). For every choice of $x_1 \in W_1, \ldots, x_{l-2} \in W_{l-2}$ and $\{x_{l-1}, x_l\} \in B$, at least $w^{l-2} \times \frac{\delta_l}{6l} n^2$ choices in total, we consider a potential copy of $\mathcal{F}^k_{l+1}$ with core $C = \{x, x_1, \ldots, x_l\}$. (Recall that x is the chosen vertex of maximum degree.) As G is $\mathcal{F}^k_{l+1}$-free, at least one of the following must hold:

1. $K \notin G$, where $K = \{x, x_1, \ldots, x_{k-1}\}$.
2. A pair $\{x, x_i\}$ with $i \in [l]$ is not covered by an edge of G.
3. A pair $\{x_i, x_j\}$ with $\{i, j\} \neq \{l-1, l\}$ is not covered by an edge of G.

One of these three alternatives holds for at least one third of the choices of x_i's. If it is Alternative 1, then for each such K we have $K \setminus \{x\} \in T_{l-1} \setminus L$. Any fixed set K is counted at most n^{l-k+1} times. Now, since $\delta_{l-1} \ll \delta_l$, we obtain a contradiction to (7):

$$|T_{l-1} \setminus L| \geq \frac{1}{3} \times w^{l-2} \times \frac{\delta_l}{6l} n^2 \times n^{-l+k-1} > \delta_{l-1}(n-s)^{k-1}.$$

If it is Alternative 2, then we obtain a contradiction as follows. For every uncovered pair $\{x, x_i\}$, the vertex x_i belongs to at least w^{k-2} edges of the $(k-1)$-graph T_{l-1}. None of these edges belongs to L, for otherwise the pair $\{x, x_i\}$ would be covered by an edge of G. On the other hand, every edge $D \in T_{l-1} \setminus L$ appears this way for at most $(k-1)\, n^{l-1}$ choices of the sequence $(x_1, \ldots, x_l)$: we have to choose $x_i \in D$ and then the other $l-1$ vertices x_j. Thus we have

$$|T_{l-1} \setminus L| \geq \frac{1}{3} \times w^{l-2} \times \frac{\delta_l}{6l} n^2 \times w^{k-2} \times \frac{1}{(k-1)\, n^{l-1}} > \delta_{l-1}(n-s)^{k-1},$$

again a contradiction to (7). Finally, suppose that Alternative 3 appears frequently. Each pair $\{x_i, x_j\}$ belongs to at least w^{k-3} edges of $T_{l-1} \setminus L$. However, each such edge is counted at most $\binom{k-1}{2} n^{l-2}$ times. Hence,

$$|T_{l-1} \setminus L| \geq \frac{1}{3} \times w^{l-2} \times \frac{\delta_l}{6l} n^2 \times w^{k-3} \times \binom{k-1}{2}^{-1} n^{-l+2} > \delta_{l-1}(n-s)^{k-1}.$$

Again we obtain a contradiction to (7). This completes the proof of Theorem 4. ∎

3 Step 3: F^k_{l+1} is stable

Please note that Theorem 5 below is formally stronger than Theorem 4. However, it follows from Theorem 4 by an application of Lemma 4 from [15]. The last result indirectly relies on the recent Hypergraph Regularity Lemma of Gowers [7] or Nagle-Rödl-Schacht-Skokan [14, 18, 17].

For our particular hypergraph F^k_{l+1}, the recourse to such a complicated technique is not really necessary and we present a short and self-contained proof, similar to the proof of Lemma 3 in [15].

Theorem 5 *For any $l \ge k \ge 2$ and $\delta > 0$ there exist $\varepsilon > 0$ and M such that the following holds for all $n > M$: Any n-vertex F^k_{l+1}-free k-graph G with at least $t^k_l(n) - \varepsilon n^k$ edges is δn^k-close to $T^k_l(n)$.*

Proof. Given $\delta > 0$, let $\delta \gg \varepsilon \gg 1/M$.

Suppose that $n > M$ and G is an n-vertex F^k_{l+1}-free k-graph with at least $t^k_l(n) - \varepsilon n^k$ edges. Let G' be obtained from G by deleting all edges that contain a pair of vertices whose codegree is at most $l^3\binom{n}{k-3}$. Since the number of pairs of vertices is $\binom{n}{2}$, we have

$$|G \setminus G'| \le l^3 \binom{n}{k-3} \times \binom{n}{2} < \epsilon\, n^k < \frac{\delta}{2}\, n^k. \tag{10}$$

Now we argue that G' is $\mathcal{F}^k_{l+1}$-free. Suppose on the contrary that G' contains a copy of some $F \in \mathcal{F}^k_{l+1}$ with core C and edge $D \subset C$. Since every pair of vertices $x, y \in C$ is contained in an edge of G', we have, by $l \ge k \ge 2$,

$$\mathrm{codeg}_G(x,y) \ge l^3 \binom{n}{k-3} > \left(\binom{l+1}{2}(k-2) + l + 1\right)\binom{n}{k-3}.$$

Hence we can greedily choose edges of G containing all pairs in $\binom{C}{2} \setminus \binom{D}{2}$, so that these edges intersect C in precisely two vertices and are pairwise disjoint outside C. The resulting set of $\binom{l+1}{2} - \binom{k}{2}$ edges, together with D, forms a copy of F^k_{l+1} in G, a contradiction.

We have

$$|G'| > |G| - \epsilon\, n^k \ge (t^k_l(n) - \varepsilon n^k) - \epsilon\, n^k = t^k_l(n) - 2\varepsilon n^k.$$

We apply Theorem 4 to G' and conclude that G' is $\frac{\delta}{2}\, n^k$-close to $T^k_l(n)$. By (10), G and $T^k_l(n)$ are δn^k-close. The proof is complete. ∎

4 Step 4: Proof of Theorem 3

Proof. If $k = 2$, then Theorem 3 is precisely the Turán theorem [22]. Thus let us assume that $l \ge k \ge 3$. Choose small $c \gg c' \gg \delta > 0$. Let n be large.

Let G be an F^k_{l+1}-free k-graph on $[n]$ with $|G| = t^k_l(n)$. We will show that G is l-partite. This implies the theorem because $T^k_l(n)$ is the unique l-partite k-graph on n vertices with $t^k_l(n)$ edges, and the addition of any edge to $T^k_l(n)$ yields a copy of F^k_{l+1}.

Let $W_1 \cup \cdots \cup W_l$ be a partition of $[n]$ such that

$$f = \sum_{D \in G} \Big| \{ i \in [l] : D \cap W_i \neq \emptyset \} \Big|$$

is maximum possible. Let T be the complete l-partite k-graph on $W_1 \cup \cdots \cup W_l$. Let us call the edges in $T \setminus G$ *missing* and the edges in $G \setminus T$ *bad*. As $|T| \leq t_l^k(n) = |G|$, the number of bad edges is at least the number of missing edges.

By Theorem 5, there is an l-partite k-graph which is δn^k-close to G. Consequently, $f \geq k(|G| - \delta n^k)$. On the other hand,

$$f \leq k|G \cap T| + (k-1)|G \setminus T| = k|G| - |G \setminus T|.$$

This implies that $|G \setminus T| \leq k\delta n^k$ and, in view of $|T| \leq |G|$,

$$|T \setminus G| \leq k\delta n^k. \tag{11}$$

Thus we have $|T| \geq |G \cap T| \geq t_l^k(n) - k\delta n^k$. From (2) we conclude that for each $i \in [l]$ we have, for example, $\big|\, |W_i| - \frac{n}{l} \,\big| \leq \frac{n}{2l}$.

If $G \subset T$, then we are done. Thus, let us assume that B is non-empty, where the 2-graph B consists of all *bad* pairs, that is, pairs of vertices which come from the same part W_i and are covered by an edge of G.

For distinct vertices x, y call the pair $\{x, y\}$ *sparse* if G has at most $\left(\binom{l+1}{2}(k-2) + l + 1\right)\binom{n}{k-3}$ edges containing both x and y; otherwise $\{x, y\}$ is called *dense*. It is easy to see that if we have a fixed $(l+1)$-set $C \subset V(G)$ containing at least one edge $D \in G$, then at least one pair of vertices $\{x, y\}$ from $\binom{C}{2} \setminus \binom{D}{2}$ is sparse. (For otherwise we can greedily build a copy of F_{l+1}^k in G with the core C.)

Let A consist of those $z \in V(G)$ which are incident to at least cn^{k-1} missing edges.

Claim 1 Any bad pair $\{x_0, x_1\}$ intersects A.

Proof of Claim. Assume without loss of generality that $x_0, x_1 \in W_1$ are covered by $D \in G$.

It is easy to see that for any choice of $x_i \in W_i \setminus D$ for $i \in [2, l]$ (at least $(\frac{n}{2l} - k)^{l-1} > (\frac{n}{3l})^{l-1}$ choices), at least one pair $\{x_i, x_j\}$ with $\{i, j\} \neq \{0, 1\}$ is sparse or the k-tuple $\{x_1, x_2, \ldots, x_k\}$ is missing for otherwise we obtain a copy of F_{l+1}^k. (In fact, we can make stronger claims but this one suffices.)

If the second alternative occurs at least a half of the time, then $x_1 \in A$. Indeed, any k-tuple $D \ni x_1$ is counted at most n^{l-k} times (the number of ways to choose $x_{k+1}, \ldots, x_l$), so x_1 belongs to at least $\frac{1}{2}\,(\frac{n}{3l})^{l-1}/n^{l-k} \geq cn^{k-1}$ missing edges, as required.

So, suppose that for at least half of the choices, the first alternative holds, i.e., there is a sparse pair. Each such pair $\{x_i, x_j\}$ appears, very roughly, at most n^{l-3} times unless $\{x_i, x_j\} \cap \{x_0, x_1\} \neq \emptyset$ when the pair is counted at most n^{l-2} times. There are two further alternatives to consider.

If at least a quarter of the time, the found sparse pair is disjoint from $\{x_0, x_1\}$, then we obtain at least $\frac{1}{4}(\frac{n}{3l})^{l-1}/n^{l-3} \geq cn^2$ sparse pairs, each intersecting two parts W_i. But this leads to a contradiction to (11): each such sparse pair in contained in at least, say, $(\frac{n}{3l})^{k-2}$ missing edges while each missing edge contains at most $\binom{k}{2}$ sparse pairs. Hence, at least a quarter of the time, the sparse pair intersects $\{x_0, x_1\}$, so one of these vertices, say x_0, is in at least $\frac{1}{8}\left(\frac{n}{3l}\right)^{l-1}/n^{l-2}$ sparse pairs, which implies that $x_0 \in A$. The claim has been proved. ▌

Considering vertices from A, we obtain at least $|A| \times cn^{k-1}/k$ missing edges and, consequently, at least $|A| \times cn^{k-1}/k$ bad edges. Let $\mathcal{B}$ consist of the pairs $(D, \{x, y\})$, where $\{x, y\} \in B$, $D \in G$ and $x, y \in D$. (Thus D is a bad edge.) As each bad edge contains at least one bad pair, we conclude that $|\mathcal{B}| \geq |A| \times cn^{k-1}/k$. For any $(D, \{x, y\}) \in \mathcal{B}$, we have $\{x, y\} \cap A \neq \emptyset$ by Claim 1. If we fix x and D, then, obviously, there are at most $k-1$ ways to choose a bad pair $\{x, y\} \subset D$. By Claim 1, some vertex $x \in A$, say $x \in W_1$, belongs to at least

$$\frac{|\mathcal{B}|}{(k-1)\,|A|} \geq \frac{c}{k(k-1)}\, n^{k-1}$$

bad edges, each intersecting W_1 in another vertex y.

Let $Y_1 \subset W_1$ be the neighborhood of x in the 2-graph B. Let $Z_1 \subset Y_1$ be the set of those vertices z for which $\{x, z\}$ is dense. The number of edges containing x and some vertex of $Y_1 \setminus Z_1$ is at most $l^3 n^{k-2} < \frac{c}{2k(k-1)}\, n^{k-1}$. Consequently, the number of bad edges containing x and some vertex of Z_1 is at least $\frac{c}{2k(k-1)}\, n^{k-1}$. Therefore $|Z_1| \geq \frac{c}{2k(k-1)} n \geq c'n$.

Let Z_j consist of those $z \in W_j$ for which $\{x, z\}$ is dense, $j \in [2, l]$. If $|Z_j| \geq c'n$ for each $j \in [2, l]$, then every l-tuple $(x_1, x_2, \ldots, x_l)$ with $x_j \in Z_j$ (at least $(c'n)^l$ choices) generates a sparse pair not containing x or the edge $\{x_1, \ldots, x_k\}$ is missing. The latter alternative cannot happen, say, at least half of the time because otherwise we obtain more than $\frac{1}{2}\,(c'n)^l/n^{l-k} > k\delta n^k$ missing edges, a contradiction to (11). Thus at least half of the time, we obtain a sparse pair disjoint from x. This gives at least $\frac{1}{2}\,(c'n)^l/n^{l-2}$ sparse pairs, each intersecting some two parts, which leads to a contradiction to (11).

Hence, assume that, for example, $|Z_2| < c'n$. This means that all but at most $c'n$ pairs $\{x, z\}$ with $z \in W_2$ are sparse, that is, there are at most $l^3 n^{k-2} + c'n^{k-1} < 2c'n^{k-1}$ G-edges containing x and intersecting W_2. Let us contemplate moving x from W_1 to W_2. Some edges of G may decrease their contribution to f. But each such edge must contain x and intersect W_2 so the corresponding decrease is at most $2c'n^{k-1}$. On the other hand, the number of edges of G

containing x, intersecting $W_1 \setminus \{x\}$, and disjoint from W_2 is at least $\left(\frac{c}{k(k-1)} - 2c'\right) n^{k-1} > 2c' n^{k-1}$. Hence, by moving x from W_1 to W_2 we strictly increase f, a contradiction to the choice of the parts W_i. The theorem is proved. ∎

Acknowledgment

We thank the referees for their careful reading of the manuscript.

References

[1] B. Bollobás, *Three-graphs without two triples whose symmetric difference is contained in a third*, Discrete Math. **8** (1974), 21–24.

[2] D. de Caen, *Uniform hypergraphs with no block containing the symmetric difference of any two other blocks*, Congres. Numer. **47** (1985), 249–253.

[3] P. Frankl, *Asymptotic solution of a Turán-type problem*, Graphs Combin. **6** (1990), 223–227.

[4] P. Frankl and Z. Füredi, *A new generalization of the Erdős-Ko-Rado theorem*, Combinatorica **3** (1983), 341–349.

[5] P. Frankl and Z. Füredi, *Extremal problems whose solutions are the blowups of the small Witt-designs*, J. Combin. Theory (A) **52** (1989), 129–147.

[6] Z. Füredi and M. Simonovits, *Triple systems not containing a Fano configuration*, Combin. Prob. Comput. **14** (2005) 467–488.

[7] W. T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi's theorem*, Manuscript, 2005.

[8] J. Kahn, *Coloring nearly-disjoint hypergraphs with $n+o(n)$ colors*, J. Combin. Theory (A) **59** (1992), 31–39.

[9] G. O. H. Katona, *Extremal problems for hypergraphs*, Combinatorics, vol. 56, Math. Cent. Tracts, 1974, pp. 13–42.

[10] P. Keevash and D. Mubayi, *Stability results for cancellative hypergraphs*, J. Combin. Theory (B), **92** (2004), 163–175.

[11] P. Keevash and B. Sudakov, *On a hypergraph Turán problem of Frankl*, Combinatorica, **25** (2005) 673–706.

[12] P. Keevash and B. Sudakov, *The exact Turán number of the Fano plane*, Combinatorica, **25** (2005) 561–574.

[13] D. Mubayi, *A hypergraph extension of Turán's theorem*, J. Combin. Theory (B), **96** (2006) 122–134.

[14] B. Nagle, V. Rödl, and M. Schacht, *The counting lemma for regular k-uniform hypergraphs*, Random Struct. Algorithms, **28** (2006) 113–179.

[15] O. Pikhurko, *Exact computation of the hypergraph Turán function for expanded complete 2-graphs*, `arXiv:math.CO/0510227`. This paper was accepted by *J. Comb. Th. (B)* but its publication is suspended, because of disagreement over copyright, see `http://www.math.cmu.edu/~pikhurko/Copyright.html`, 2005.

[16] O. Pikhurko, *An exact Turán result for the generalized triangle*, Submitted, 2005.

[17] V. Rödl and J. Skokan, *Regularity lemma for k-uniform hypergraphs*, Random Struct. Algorithms **25** (2004), 1–42.

[18] V. Rödl and J. Skokan, *Applications of the regularity lemma for uniform hypergraphs*, Random Struct. Algorithms, **28** (2006) 180–194.

[19] J. B. Shearer, *A new construction for cancellative families of sets*, Electronic J. Combin. **3** (1996), 3pp.

[20] A. F. Sidorenko, *The maximal number of edges in a homogeneous hypergraph containing no prohibited subgraphs*, Math Notes **41** (1987), 247–259, Translated from *Mat. Zametki*.

[21] M. Simonovits, *A method for solving extremal problems in graph theory, stability problems*, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, 1968, pp. 279–319.

[22] P. Turán, *On an extremal problem in graph theory (in Hungarian)*, Mat. Fiz. Lapok **48** (1941), 436–452.