Asymptotic Evaluation of the sat-Function for r-Stars

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Abstract

Let S_m^r be the r-uniform set system on m vertices consisting of all r-tuples containing a given vertex. We determine the asymptotic behaviour of $\mathrm{sat}(n, S_m^r)$ for all r and m thus extending a result of Erdős, Füredi and Tuza.

Key words: generalised stars; saturated hypergraphs

An r-graph G is a pair $(V(G), E(G))$ where the *edge-set* $E(G)$ is a family of r-subsets of the *vertex set* $V(G)$. In the usual way we define the notion of subgraph, isomorphism, complementary graph $\overline{G}$, size $|G| = |E(G)|$, etc. A graph isomorphic to G is called a G-graph. Given an r-graph H we say that G is *monotonically H-saturated* (another common name is *strongly H-saturated*) if the addition of any new r-tuple to $E(G)$ creates at least one new H-subgraph. If, besides, G doesn't contain H as a subgraph then G is called *H-saturated*. Next, $\mathrm{sat}(n, H)$ ($\text{m-sat}(n, H)$) is defined to be the minimal size of an H-saturated (monotonically H-saturated) graph on n vertices.

The *star* S_m^r, $m > r \geq 2$, has $[m]$ as the vertex set and $\{E \in [m]^{(r)} : E \ni m\}$ as the edge set. In other words, S_m^r has m vertices and its $\binom{m-1}{r-1}$ edges are the r-tuples containing some fixed vertex, which is called the *centre*.

The exact values of $\mathrm{sat}(n, S_m^r)$ are known only for S_m^2, any m, (see (2)) and for S_4^3 (see (1)).

The asymptotic behaviour of $\mathrm{sat}(n, S_{r+1}^r)$ was found by Erdős, Füredi and Tuza (1, Theorem 2). Exploiting their ideas we extend this result to all stars.

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Theorem 1 *Let $m > r \geq 2$ and $S = S_m^r$. Then*

$$\frac{m-r}{2}\binom{n}{r-1} \geq \mathrm{sat}(n,S) \geq \text{m-sat}(n,S) \geq \frac{m-r}{2}\binom{n}{r-1} - O(n^{r-4/3}).(1)$$

PROOF. Let us provide a construction of an S-saturated graph $G = G_{m,n}^r$ proving the upper bound. Partition the vertex set $[n]$ into $n' = \lceil n/(m-r+1) \rceil$ *blocks* $B_1, \ldots, B_{n'}$ of size $m-r+1$ each except possibly the last one. The edge set is

$$E(G) = \left\{ F \in [n]^{(r)} : |F \cap B_j| \geq 2,\ j = \min\{i \in [n'] : F \cap B_i \neq \emptyset\} \right\}.$$

Thus every edge of G has at least two common points with some B_j and intersects no B_i with $i < j$.

Let us show that $S \not\subset G$. Suppose not and we have an S-subgraph $S' \subset G$ centered at x. Let

$$j = \min\{i \in [n'] : V(S') \cap B_i \neq \emptyset\}. \tag{2}$$

Choose an r-set $F \ni x$ so that it contains one vertex from B_j and some $r-1$ vertices in $V(S') \setminus B_j$ which is possible since $|V(S') \setminus B_j| \geq r-1$. We obtain a contradiction as on one hand F contains the centre x and must belong to S while on the other hand $F \notin E(G)$ by the definition.

Now, if we add any extra edge F to G then the set $Y = F \cup B_j$ spans a copy of S centered at x where B_j is the first block intersecting F and $\{x\} = F \cap B_j$. Indeed, every $F' \in Y^{(r)}$ containing x either equals F or intersects B_j in at least two points and so belongs to $E(G)$.

Therefore we conclude that G is S-saturated. To prove the desired upper bound $|G_{m,n}^r| \leq \frac{m-r}{2}\binom{n}{r-1}$ we observe, for $r = 2$, that each vertex of the 2-graph $G_{m,n}^2$ has degree at most $m-r$ while, for $r \geq 3$, we use induction and the equality $|G_{m,n+1}^r| = |G_{m,n}^r| + |G_{m-1,n}^{r-1}|$.

Trivially, $\mathrm{sat}(n,S) \geq \text{m-sat}(n,S)$.

Finally, let G be a minimum monotonically S-saturated graph on $V = [n]$. By the definition, the addition to G of any edge $F \in E(\overline{G})$ creates at least one S-subgraph $S' \subset G + F$. Let $\mathcal{S}(F)$ be the set of all such subgraphs S' created by F.

Let $\mathcal{F}(F)$ denote the set of edges in $\overline{G}$ which intersect $F \in E(G)$ in $r-1$

points and create a copy of S containing F as an edge. Formally,

$$\mathcal{F}(F) = \left\{F' \in E(\overline{G}) : |F \cap F'| = r-1,\ \exists S' \in \mathcal{S}(F')\ F \in E(S')\right\}, \quad F \in E(G).$$

Also we define

$$\begin{aligned} \mathcal{F}(G') &= \cup_{F \in E(G')} \mathcal{F}(F),\ G' \subset G, \\ \partial F &= F^{(r-1)}, \qquad\qquad F \in [n]^{(r)}, \\ \partial G' &= \cup_{F \in E(G')} \partial F, \quad \text{an } r\text{-graph } G'. \end{aligned}$$

As G is monotonically S-saturated we conclude that

$$\mathcal{F}(G) = V^{(r)} \setminus E(G). \tag{3}$$

Choose an integer $k = k(n)$, to be specified later, such that $k \to \infty$ and $k/n \to 0$. On the vertex set V we define two subgraphs $G_0, G_1 \subset G$; G_0 is a maximal subgraph of G with $|\mathcal{F}(G_0)| \le k|G_0|$ and G_1 consists of the edges of G not in G_0: $E(G_1) = E(G) \setminus E(G_0)$. By the maximality of G_0 for every $F \in E(G_1)$ we have

$$|\mathcal{F}(F) \setminus \mathcal{F}(G_0)| > k. \tag{4}$$

From (3) and the proved upper bound in (1) we conclude that $|\mathcal{F}(G)| = \binom{n}{r} - |G| = \binom{n}{r} - O(n^{r-1})$. Taking into the account that $\mathcal{F}(G) = \mathcal{F}(G_0) \cup \mathcal{F}(G_1)$ and $|\mathcal{F}(G_0)| \le k|G_0| = O(kn^{r-1})$ we obtain

$$|X| = \binom{n}{r} - O(kn^{r-1}), \tag{5}$$

where $X = \mathcal{F}(G_1) \setminus \mathcal{F}(G_0)$.

Let $Z = V^{(r-1)} \setminus \partial G_1$. We claim that

$$|Z| = O(k^{1/2} n^{r-3/2}). \tag{6}$$

Suppose not. Then the average value of $z(D) = |\{E \in Z : E \supset D\}|$ over all $D \in V^{(r-2)}$ is greater than $O(k^{1/2} n^{1/2})$. For any $E, E' \in Z$ with $|E \cap E'| = r-2$ we have $F = E \cup E' \notin X$, because otherwise at least one of $E, E' \in \partial F$ is covered by an edge of $S' \in \mathcal{S}(F)$ which then is necessarily an edge of G_1 (as it intersects $F \in \mathcal{F}(G_1) \setminus \mathcal{F}(G_0)$ in $r-1$ vertices). Therefore, we have at least $\binom{r}{2}^{-1} \sum_{D \in V^{(r-2)}} \binom{z(D)}{2}$ r-sets not in X, which exceeds $\binom{n}{r-2} O(kn)$ by the convexity of $\binom{x}{2}$. This contradicts (5) and proves the claim.

Let

$$g_1(E) = |\{F \in E(G_1) : F \supset E\}|, \quad E \in \partial G_1.$$

Take any $F \in E(G_1)$. Let $\partial F = \{E_1, \ldots, E_r\}$. We claim that all but at most two of $g_1(E_i)$'s are larger than $k/6$. Suppose not, say $g_1(E_i) \le k/6$, $i = 1, 2, 3$. Take $F' \in \mathcal{F}(F) \setminus \mathcal{F}(G_0)$ and any $S' \in \mathcal{S}(F')$ containing F as an edge. Let $F' = E_i \cup \{x\}$, some $i \in [r]$, $x \in V \setminus F$. The star S' contains $r-2$ edges of the form $E_j \cup \{x\}$, $j \ne i$. These edges cannot be in G_0 and so contribute at least 1 to $g_1(E_1) + g_1(E_2) + g_1(E_3)$. In total, each $\{x\} \cup E_j \in E(G_1)$ is counted at most twice. (Once it occurs then at most 2 edges of the form $\{x\} \cup E_i$ can belong to $E(\overline{G})$.) But this contradicts (4). The claim is proved.

Define

$$\begin{aligned} W &= \{E \in \partial G_1 : g_1(E) \le m - r - 1\}, \\ T &= \{F \in E(G_1) : W \cap \partial F \ne \emptyset\}. \end{aligned}$$

We claim that $|W| = O(k^{1/2} n^{r-3/2})$. Suppose not. Note that for $E, E' \in W$ with $|E \cap E'| = r-2$ we necessarily have $F = E \cup E' \notin X$ for otherwise in an $S' \in \mathcal{S}(F)$ centered at x, say $x \in E$, there are $m - r$ edges (necessarily in $E(G_1)$) different from F and covering E. Thus there are at least $\binom{r}{2}^{-1} \sum_{D \in V^{(r-2)}} \binom{w(D)}{2}$ edges not in X, where $w(D) = |\{E \in W : E \supset D\}|$, $D \in V^{(r-2)}$. Using the convexity of the $\binom{x}{2}$-function as before we can argue that there more than $O(kn^{r-1})$ edges not in X contradicting (5). The claim is established.

Every $E \in W$ is contained in at most $m - r - 1$ edges $F \in E(G_1)$ so $|T| = O(k^{1/2} n^{r-3/2})$. For every $F \in E(G_1) \setminus T$ we have $\sum_{E \in \partial F} \frac{1}{g_1(E)} \le \frac{2}{m-r} + (r-2)\frac{6}{k}$. Note the following easy identity

$$\begin{aligned} |\partial G_1| &= \sum_{F \in E(G_1) \setminus T} \left(\sum_{E \in \partial F} \frac{1}{g_1(E)} \right) + \sum_{F \in T} \left(\sum_{E \in \partial F} \frac{1}{g_1(E)} \right) \\ &\le \left(\frac{2}{m-r} + O(1/k) \right) |G_1| + r|T|. \end{aligned}$$

We know, see (6), that $|\partial G_1| = \binom{n}{r-1} - O(k^{1/2} n^{r-3/2})$. Hence

$$\frac{m-r}{2} \binom{n}{r-1} - |G| = O(k^{1/2} n^{r-3/2} + |G|/k) = O(k^{1/2} n^{r-3/2} + n^{r-1}/k).$$

Taking $k = \lfloor n^{1/3} \rfloor$ we obtain the required. $\square$

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