On the connectivity of extremal Ramsey graphs

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Received: 1 December 2006, Accepted: 31 October 2007

Abstract

An (r, b)-graph is a graph that contains no clique of size r and no independent set of size b. The set of extremal Ramsey graphs $ERG(r, b)$ consists of all (r, b)-graphs with $R(r, b) - 1$ vertices, where $R(r, b)$ is the classical Ramsey number. We show that any $G \in ERG(r, b)$ is $r - 1$ vertex connected and $2r - 4$ edge connected for $r, b \geq 3$.

1 Introduction

Let $R(r, b)$ be the classical Ramsey number. An (r, b)-*graph* is a graph that contains no clique of size r and no independent set of size b. Let $ERG(r, b)$ (to abbreviate *extremal Ramsey graphs*) consist of all (r, b)-graphs with $R(r, b) - 1$ vertices. In this paper we identify $G \in ERG(r, b)$ with a red-blue coloring of the complete graph and we denote the graphs induced by the color classes as G_{red} and G_{blue}.

In his talk at the 2005 British Combinatorial Conference, David Penman established various properties of extremal Ramsey graphs and conjectured that for $k \geq 3$ every graph $G \in ERG(k, k)$ is connected. Here we will observe that this conjecture is true. In fact, its validity follows fairly straightforwardly from a result of Xiaodong, Zheng, and Radziszowski [7] (see

*Web: `http://www.math.cmu.edu/~pikhurko`. Partially supported by the National Science Foundation, Grant DMS-0457512.

Section 2). For the sake of completeness we present a short self-contained proof, which follows by adopting the methods in [7]. (More recently, Shane Malik and David Penman [4] reported to the authors that they had independently proved this conjecture.)

Theorem 1 *Let $r \geq 3$, $b \geq 2$, and $G \in ERG(r,b)$. The red graph G_{red} is connected.*

Proof. Assume that $b \geq 3$, for otherwise $G_{\mathrm{red}} = K_{r-1}$ is clearly connected.

Suppose on the contrary that $V(G) = V_1 \cup V_2$ is a proper partition of the vertices with only blue edges across. Pick $x_i \in V_i, i = 1, 2$. Create a new graph G' by adding a new vertex x and coloring the incident edges as follows. Color xx_1 and xx_2 blue. Color xy with $y \in V_i - x_i$ by the color of $x_i y$. Finally, recolor x_1x_2 red.

Since we have strictly increased the number of vertices, G' must contain either a red K_r or a blue K_b. First suppose there is a red K_r on a set R. R must contain x. Indeed, the only difference between $G' - x$ and G is the color of the edge x_1x_2 and this solitary red edge joining V_1 and V_2 cannot be in a red K_r for $r \geq 3$. Since xx_1 and xx_2 are blue, $x_1, x_2 \notin R$ and $R - x$ lies inside one part, say V_1. But then $R - x + x_1$ is a red K_r in G, a contradiction. Next suppose we have a blue K_b on a set B. We must have $x \in B$. At least one of x_1, x_2 is not in B since x_1x_2 is red now. Next suppose $x_1 \notin B$. But then $R - x + x_1$ is a blue K_b in G (because x_1x_2 is blue in the original graph G.) This contradiction proves the theorem. ∎

In the remainder of the paper, we explore how connected the red graph of $G \in ERG(r,b)$ must be. We show that for $r, b \geq 3$, the (vertex) connectivity $\kappa(G_{\mathrm{red}}) \geq r - 1$ and the edge connectivity $\lambda(G_{\mathrm{red}}) \geq 2r - 4$. It is no doubt that these bounds are very far from best possible, which may be even exponential in $\min(r, b)$. However, it is not clear how to get any essential improvement.

2 Vertex Connectivity

We will use the following result of Xiaodong, Zheng, and Radziszowski [7, Theorem 3] which builds upon the ideas from Burr, Erdős, Faudree, and Schelp [1].

Theorem 2 (Xiaodong et al [7]) *If $2 \leq p \leq q$ and $3 \leq r$, then*

$$R(r, p + q - 1) \geq R(r, p) + R(r, q) + \begin{cases} r - 3, & \text{if } p = 2, \\ r - 2, & \text{if } p \geq 3. \end{cases}$$

In particular, the case $p = 2$ and $q = b - 1$ gives the original result of Burr et al [1, Theorem 1] (see also [7, Theorem 1] for a small correction, namely that (1) is not true for $b = 2$): for any $r \geq 2$ and $b \geq 3$,

$$R(r, b) \geq R(r, b - 1) + 2r - 3. \tag{1}$$

It follows that for any $G \in ERG(r,b)$, with $b \geq 3$, the minimum red degree

$$\delta(G_{\rm red}) \geq 2r - 4. \tag{2}$$

Indeed, take any vertex x and let $d_{G_{\rm red}}(x)$ denote its red degree. The blue neighborhood of x has $R(r,b) - 2 - d_{G_{\rm red}}(x)$ vertices and induces an $(r, b-1)$-graph. Hence,

$$R(r,b) - 2 - d_{G_{\rm red}}(x) \leq R(r, b-1) - 1,$$

and $d_{G_{\rm red}}(x) \geq 2r - 4$ by (1), as required.

Theorem 3 *Let $r, b \geq 3$ and $G \in ERG(r,b)$. The (vertex) connectivity of the red graph $G_{\rm red}$ satisfies $\kappa(G_{\rm red}) \geq r - 1$.*

Proof. Suppose on the contrary that we can find a partition $V(G) = V_1 \cup V_2 \cup X$ such that all edges between V_1 and V_2 are blue and $0 < |X| \leq r - 2$ (X is nonempty by Theorem 1). Let $p - 1$ and $q - 1$ be the sizes of a maximum blue clique inside V_1 and V_2 respectively. Assume without loss of generality that $p \leq q$.

If $p = 2$, then V_1 spans a red clique, so $|V_1| \leq r - 1$. The red degree of each vertex of V_1 is at most $|V_1| - 1 + |X| \leq 2r - 4$. By (2), all these inequalities are equalities; in particular, $|V_1| = r - 1$, $|X| = r - 2$, and all edges between V_1 and X are red. But then V_1 plus any vertex from $X \neq \emptyset$ spans a red K_r, a contradiction.

So assume $p \geq 3$. We have $(p-1) + (q-1) \leq b - 1$, and

$$R(r,p) - 1 + R(r,q) - 1 \geq |V_1| + |V_2| = V(G) - |X| = R(r,b) - 1 - |X|.$$

It follows from Theorem 2 that $|X| \geq r - 1$, a contradiction. ∎

Corollary 4 *Let $r \geq b \geq 3$ and $G \in ERG(r,b)$. The red graph $G_{\rm red}$ is Hamiltonian.*

Proof. The Chvátal-Erdős condition [2] states that a graph has a Hamiltonian cycle if it is k-connected and does not contain a set of $k+1$ independent points. By Theorem 3, this condition is satisfied for $G_{\rm red}$ when $r \geq b \geq 3$. ∎

3 Edge Connectivity

Theorem 5 *Let $r, b \geq 3$ and $G \in ERG(r,b)$. The edge connectivity of the red graph $G_{\rm red}$ satisfies*

$$\lambda(G_{\rm red}) \geq \min\{\delta(G_{\rm red}), \kappa(G_{\rm red}) + r - 3\}. \tag{3}$$

Note that we have the trivial upper bound $\lambda(G_{\text{red}}) \leq \delta(G_{\text{red}})$. Our lower bounds for $\delta(G_{\text{red}})$ and $\kappa(G_{\text{red}})$ imply $\lambda(G_{\text{red}}) \geq 2r - 4$.

Proof. For $r = 3$, the statement $\lambda(G_{\text{red}}) \geq \min\{\delta(G_{\text{red}}), \kappa(G_{\text{red}})\} = \kappa(G_{\text{red}})$ is simply the trivial lower bound.

Consider $r \geq 4$. Suppose that the claim is not true, that is, we can find a proper partition $V(G) = V_1 \cup V_2$ such that we have at most $k = \min\{\delta(G_{\text{red}})-1,\ \kappa(G_{\text{red}})+r-4\}$ red edges across. Call these red edges F. Let $v_i = |V_i|$. We claim $v_i \geq \delta(G_{\text{red}}) + 1$, $i = 1, 2$. Indeed, there is a vertex in V_i whose F-degree is at most $\lfloor \frac{k}{v_i} \rfloor \leq \lfloor \frac{\delta(G_{\text{red}})-1}{v_i} \rfloor$. We have $v_i - 1 + \lfloor \frac{\delta(G_{\text{red}})-1}{v_i} \rfloor \geq \delta(G_{\text{red}})$, which routinely implies that $v_i \geq \delta(G_{\text{red}}) + 1$. Take $x_i \in V_i$ not incident to any edge of F, which is possible since $v_i \geq \delta(G_{\text{red}}) + 1 > |F|$.

Let X be a minimal set of vertices that cover the edge set F. Clearly X is a cutset of vertices for G_{red} ($x_i \notin X$ so $V_i - X \neq \emptyset$ for $i = 1, 2$) and therefore $|X| \geq \kappa(G_{\text{red}})$. By Kőnig's theorem for bipartite graphs, there is a matching $M \subset F$ of size $|X|$. Hence $|F| - |M| \leq k - \kappa(G_{\text{red}}) \leq r - 4$. As F consists of a matching along with no more than $r - 4$ additional edges, there is no red K_{r-1} that intersects both V_1 and V_2.

Construct G' from G as in the proof of Theorem 1. Namely, add a new vertex x. Color xx_1 and xx_2 blue. For any vertex $y \in V_i \setminus \{x_i\}$, color xy with the color of $x_i y$, $i = 1, 2$. Finally, recolor $x_1 x_2$ red. We have strictly increased the number of vertices and $G \in ERG(r, b)$ so we must have a large monochromatic clique in G'. First, suppose we have a red K_r in G', say on a set R. This set R must contain x because the only red edge added to G, namely $x_1 x_2$, cannot appear in a red K_r. But then $x_1, x_2 \notin R$. Moreover, $R - x$ lies entirely inside either V_1 or V_2. Suppose that $R - x \subset V_1$. But then $R - x + x_1$ is a red r-clique in G, a contradiction.

Next suppose that we have a blue $K_b \subset G'$, on a set B. We have $x \in B$ and at least one of x_1, x_2 does not belong to B. Suppose $x_1 \notin B$. But then $B - x + x_1$ spans a blue K_b in G (note that all edges between x_1 and V_2 are blue in G by the definition of x_1), a contradiction. ∎

4 Some Small Extremal Ramsey Graphs

In conclusion, we compare our lower bounds on connectivity with the actual connectivities for some small extremal Ramsey graphs.

(r,b)	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$	$(3,7)$	$(4,4)$
$R(r,b)$	6	9	14	18	23	18
$\|ERG(r,b)\|$	1	3	1	7	191	1
$\kappa(G_{\text{red}})$ lower bound	2	2	2	2	2	3
$\kappa(G_{\text{red}})$ actual	2	2	4	4	4,5,6	8
$\lambda(G_{\text{red}})$ lower bound	2	2	2	2	2	4
$\lambda(G_{\text{red}})$ actual	2	2	4	4	4,5,6	8
$\kappa(G_{\text{blue}})$ lower bound	2	3	4	5	6	3
$\kappa(G_{\text{blue}})$ actual	2	4	8	11	15	8
$\lambda(G_{\text{blue}})$ lower bound	2	4	6	8	10	4
$\lambda(G_{\text{blue}})$ actual	2	4	8	11	15	8

Table 1: Lower bounds for connectivity from Theorems 3 and 5
and the actual connectivity for small extremal Ramsey graphs

Let us explain how this table is constructed. Currently, there are 6 pairs (r, b) with $3 \leq r \leq b$ such that the graphs in $ERG(r, b)$ have been completely enumerated, see Brendan McKay's combinatorial data website [3]. Specifically, the extremal Ramsey graph catalog for $r = 3$ and $4 \leq b \leq 7$ and $r = b = 4$ (in addition to the pair $(r, b) = (3, 3)$) was obtained from [3]. This data (in the graph6 format) was processed by McKay's `showg` executable and subsequently checked and analyzed by the Combinatorica package for Mathematica.

For every such pair we checked all extremal (r, b)-graphs and wrote down the observed edge and vertex connectivities into the table rows marked 'actual'. As it turns out, the pair $(r, b) = (3, 7)$ is the only pair from our list where different extremal (r, b)-graphs may have different connectivities. For example, of the 191 graphs in $ERG(3, 7)$, 3 are red 4-connected, 178 are red 5-connected and 10 are red 6-connected.

For the ease of reference, we have also included our lower bounds, $\kappa(G_{\text{red}}) \geq r - 1$ and $\lambda(G_{\text{red}}) \geq 2r - 4$ of Theorem 3 and the comment immediately after Theorem 5, respectively. As expected, there is certainly room for improvement in the lower bounds for connectivity, even for these small graphs.

ACKNOWLEDGMENTS. We thank Tom Bohman and Alan Frieze for helpful discussions. We thank David Penman for useful comments.

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