# Size Ramsey Numbers of Stars versus 3-Chromatic Graphs

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**Abstract**

Let $K_{1,n}$ be the star with $n$ edges, $K_3$ be the triangle, and $\mathcal{C}_{\mathrm{odd}}$ be the family of odd cycles. We establish the following bounds on the corresponding size Ramsey numbers.

$$n^2 + (0.577 + o(1))\, n^{3/2} < \hat{r}(K_{1,n}, \mathcal{C}_{\mathrm{odd}}) \le \hat{r}(K_{1,n}, K_3) < n^2 + \sqrt{2}\, n^{3/2} + n.$$

The upper (constructive) bound disproves a conjecture of Erdős.

Also we show that $\hat{r}(K_{1,n}, F_n) = (1 + o(1))n^2$ provided $F_n$ is an odd cycle of length $o(n)$ or $F_n$ is a 3-chromatic graph of order $o(\log n)$.

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# 1 Introduction

Given two graphs $F_1$ and $F_2$, we say that a graph $G$ *arrows* the pair $(F_1, F_2)$, denoted by $G \to (F_1, F_2)$, if for any blue-red colouring of the edge set of $G$ we necessarily have either a blue copy of $F_1$ or a red copy of $F_2$ (or both). The *size Ramsey number* $\hat{r}(F_1, F_2)$ is the minimal number of edges of a graph $G$ such that $G \to (F_1, F_2)$.

For example, it is easy to verify that $P_{n+1,n} \to (K_{1,n}, K_3)$, where $P_{m,n} = K_m + E_n$ has $m+n$ vertices of which $m$ vertices are connected to every other vertex. Erdős [3], see also e.g. [2, 4], conjectured that $P_{n+1,n}$ is a minimum such graph, that is, $\hat{r}(K_{1,n}, K_3) = e(P_{n+1,n}) = \binom{2n+1}{2} - \binom{n}{2}$.

In fact, Erdős [3] made a stronger conjecture which states that any graph with $\binom{2n+1}{2} - \binom{n}{2} - 1$ edges is a union of a bipartite graph and a graph with maximum degree less than $n$, $n \geq 3$. In the size Ramsey terminology, this is equivalent to the statement that $\hat{r}(K_{1,n}, \mathcal{C}_{\text{odd}}) = \binom{2n+1}{2} - \binom{n}{2}$, where $\mathcal{C}_{\text{odd}}$ denotes the family of odd cycles.

We show, however, that both these size Ramsey numbers grow as $n^2$ plus a term of order $n^{3/2}$, so that the conjecture fails for all $n \geq 5$. More precisely, we establish the following bounds. (Note that trivially $\hat{r}(K_{1,n}, K_3) \geq \hat{r}(K_{1,n}, \mathcal{C}_{\text{odd}})$.)

**Theorem 1**

$$\hat{r}(K_{1,n}, K_3) < n^2 + \sqrt{2}\, n^{3/2} + n, \quad \textit{for } n \geq 1, \tag{1}$$

$$\hat{r}(K_{1,n}, \mathcal{C}_{\text{odd}}) > n^2 + 0.577\, n^{3/2}, \qquad \textit{for sufficiently large } n. \tag{2}$$

By modifying our construction we demonstrate, for any given constant $\varepsilon > 0$, a graph $G$ of size at most $(1+\varepsilon)n^2$ arrowing $(K_{1,n}, C_m)$ for any $m$ between 3 and $cn$, where $c = c(\varepsilon) > 0$ does not depend on $n$. This implies by (2) that $\hat{r}(K_{1,n}, C_{2m+1}) = (1+o(1))n^2$ provided $m = o(n)$.

Of course, many questions remain open. One is to find more precise estimates of the error term in the formulas above. Also it is not clear how $\hat{r}(K_{1,n}, C_{2m+1})$ behaves for $m \geq cn$. A related direction of research is to view triangle as a complete graph and investigate $\hat{r}(K_{1,n}, K_m)$ for small fixed $m$.

# 2 Upper Bound

*Proof of* (1). We provide an explicit construction of a $(K_{1,n}, K_3)$-arrowing graph $G$.

Take any representation $n = k_1 + \cdots + k_m$ and let $G$ be the disjoint union of $P_{k_i,n}$, $i \in [m]$, plus a vertex $x$ connected to everything else. Consider any blue-red colouring of $E(G)$ without a blue $K_{1,n}$. Among $mn + n$ edges incident to $x$ there are at least $mn + 1$ red ones. By the pigeon-hole principle, $x$ sends at least $n+1$ red edges to some $P_{k_j,n}$, say $\{x, y_i\}$, $i \in [0, n]$, of which at least one must be incident to a vertex of $K_{k_j} \subset P_{k_j,n}$, say $y_0$. But of $n$ edges $\{y_0, y_i\} \in E(G)$, $i \in [n]$, one is necessarily red and creates a red triangle whose third vertex is $x$. Hence, $G \to (K_{1,n}, K_3)$.

We have $e(G) = (m+n+1)n + \sum_{i\in[m]} \binom{k_i}{2}$. To minimise $e(G)$ we take the $k_i$'s nearly equal; so they are essentially uniquely determined by $m$. Any value of $m$ we choose will give some upper bound for $\hat{r}(K_{1,n}, K_3)$. Choose $m$ so that $n = 2m^2 + r$, where $|r| \le 2m$. So, for example, when $n = 2m^2 - 2m$ we could choose either $m$ or $m-1$. We believe, though we do not prove, that such a choice of $m$ is optimal. (A supporting evidence is that the real relaxation of the function $e(G)$, where $k_i = n/m$, has its minimum at $m = \sqrt{n/2}$.)

The verification of (1) is now best split into four cases. For example, for $0 \le r \le m$, $k_i = 2m$ occurs $m - r$ times and $k_i = 2m+1$ occurs $r$ times. Routine simplifications show that

$$2n^3 - (e(G) - n^2 - n/2 - r/2)^2 = 3m^2r^2 + 2r^3 \ge 0,$$

which implies (1). The other cases can be verified similarly. □

One can check that the bound (1) gives strictly better values than $\binom{2n+1}{2} - \binom{n}{2}$ for all $n \ge 6$. In fact, Erdős' conjecture fails also for $n = 5$ when the representation $n = 2+3$ produces a graph with 44 edges, as opposed to the conjectured value of 45.

We do not know any example beating our construction, which therefore might be an extremal one, but we do not dare to make any conjecture yet. It is surprising that a counterexample was not found earlier. An explanation might be that $P_{n+1,n}$ is perhaps minimum among all $(K_{1,n}, \mathcal{C}_{\mathrm{odd}})$-arrowing graphs with few vertices; as shown by Faudree (see [5] for a proof) this is the case for graphs of order $2n+1$. Note that we can beat $P_{n+1,n}$ using $3n+1$ vertices for $n \ge 5$: take $m = 2$ in our construction.

# 3 Lower Bound

In this section we suppose on the contrary to (2) that there is a $(K_{1,n}, \mathcal{C}_{\mathrm{odd}})$-arrowing graph $G$ with at most $n^2 + 0.577\, n^{3/2}$ edges and try to derive a contradiction for large $n$.

Instead of 2-colourings of $E(G)$ we find it more convenient to operate with 2-partitions of $V(G)$. Thus our assumption on $G$ states that

$$\max\{\Delta(G[A]), \Delta(G[B])\} \ge n$$

for any partition $V(G) = A \cup B$, where e.g. $\Delta(G[A])$ denotes the maximal degree of the subgraph of $G$ spanned by $A$.

We refer to the following simple argument as the *greedy algorithm.*

**Lemma 1 (Greedy Algorithm)** *Let $A \subset V(G)$. Consecutively and as long as possible let $x_i$ be any vertex (if exists) of degree at least $n$ in $G[A \setminus \{x_1, \ldots, x_{i-1}\}]$. Let $X = \{x_1, \ldots, x_k\} \subset A$ be the obtained set.*

*Then $k \ge n - |\overline{A}| + 1$, where $\overline{A} = V(G) \setminus A$. In particular, $e(G[A]) \ge n(n - |\overline{A}| + 1)$.*

*Proof.* By definition, $\Delta(G[A \setminus X]) < n$. But then $\overline{A} \cup X$ contains at least $n+1$ vertices (to allow a vertex of degree $n$), and the claim follows. □

Taking $A = V(G)$ we obtain $e(G) \geq n^2 + n$. We will add an $n^{3/2}$-term to this trivial bound by using a probabilistic argument. But before we can apply it, we have to gain some structural information about $G$. This is achieved, roughly speaking, by choosing each $x_i$ in the greedy algorithm with some foresight.

Let us introduce some notation. By $d_A(x) = |A \cap N_G(x)|$ we denote the number of neighbours of $x$ lying in $A$, $x \in V(G)$, $A \subset V(G)$. Also let $L = \{x \in V(G) : d(x) \geq n\}$, $l = |L| - n$ and $e(G) = n^2 + c_g n^{3/2}$. Thus we assume that $c_g \leq 0.577$ and in fact, by adding edges to $G$, that $c_g = 0.577 + o(1)$.

**Lemma 2** $l \leq c_g n^{1/2} + O(1)$.

*Proof.* Apply a modified greedy algorithm. Set initially $A = C = \emptyset$ and $B = V(G)$. These three sets will always partition $V(G)$.

Repeat the following as long as possible or until $|A| = n+1$. Take a vertex $x \in B$ (if exists) with $d_B(x) \geq n$ and move it to $A$; colour amber all edges connecting $x$ to $B$. Then for every such $x$ do the *$n$-check*, that is, move to $C$ all vertices in $B \cap L$ whose $B \cup C$-degree is now smaller than $n$, that is, equals $n - 1$. (Thus before we proceed with another $x$ we ensure that a vertex $z \in L \setminus A$ belongs to $B$ if and only if $d_{B \cup C}(z) \geq n$.)

When we stop we have $a + c \geq n + 1$, where $a$, $b$, $c$ are the cardinalities of the eventual sets $A$, $B$, $C$. Indeed, if $a < n+1$ then $\Delta(G[B]) < n$ so $\Delta(G[A \cup C]) \geq n$ and the claim follows.

The number of amber edges is $e_a \geq an$. Call non-amber edges incident to $C$ cyan. Every vertex in $C$ is incident to exactly $n - 1$ cyan edges; hence we have $e_c \geq c(n-1) - \binom{c}{2}$ cyan edges.

By applying Lemma 1 (the greedy algorithm) to $B \cup C$ we obtain that there is a set $Y = \{y_1, \ldots, y_{n+1-a}\} \subset B \cup C$ such that each $y_i$ has at least $n$ neighbours in $(B \cup C) \setminus \{y_1, \ldots, y_{i-1}\}$. Clearly, $Y$ must be disjoint from $C$, that is, $Y \subset B$. We have $e_y \geq (n + 1 - a)n$ edges between $Y$ and $C \cup B$; colour all these edges yellow. (Some edges may be yellow and cyan simultaneously.) Finally, each vertex in $R = L \cap (B \setminus Y)$ has degree in $B \cup C$ at least $n$ (otherwise it would have been moved to $C$ earlier). Hence $R$ is incident to $e_r \geq r(n - |Y| - c) - \binom{r}{2}$ edges lying within $B \setminus Y$, where $r = |R|$; call them red edges.

We claim that $c = o(n)$. Suppose not. As $e_a + e_y > n^2$, the number of cyan-only edges is $o(n^2)$. Then, on average, $x \in C$ is incident to $o(n)$ cyan-only edges and, consequently, to $n + o(n)$ cyan-yellow edges; hence $|Y| \geq n + o(n)$. Now $|C| \geq |Y|$ because $a+c \geq n+1 = a+|Y|$, so $|C| \geq n+o(n)$. But $C \cup Y \subset L$ and $|L| \leq 2n+o(n)$ by the handshaking lemma. Therefore $c = n + o(n)$, $a = o(n)$, $r = o(n)$ and all but $o(n^2)$ edges lie between $C$ and $Y$. But consider partition $V(G) = V_1 \cup V_2$ obtained by placing in $V_1$ all of $A \cup R$, $\lfloor n/3 \rfloor$ vertices from $C$, $\lfloor n/3 \rfloor$ vertices from $Y$ and all ($= o(n)$) vertices from $C$ (and resp. from $Y$) which have in $G$ at least $n/6$ neighbours outside $Y$ (resp. outside $C$). As $|V_1| = 2n/3 + o(n)$ some $x \in V_2$ satisfies $d_{V_2}(x) \geq n$. But $x$ necessarily belongs to $Y \cup C$, say $x \in C$, and can have at most $|Y \cap V_2| + n/6 \le 5n/6 + o(n)$ $V_2$-neighbours, which is a contradiction proving $c = o(n)$.

Using the above lower bounds on $e_a, e_c, e_y$ and the inequality $a \ge n - c + 1$ we obtain

$$\begin{aligned} e(G) &= n^2 + c_g n^{3/2} \;\ge\; e_a + e_c + e_y - (n-a+1)c \\ &\ge n^2 + n - \tfrac{c^2+3c}{2} + ac \;\ge\; n^2 + n + \tfrac{-3c^2 + c(2n-1)}{2}. \end{aligned}$$

Solving this (quadratic in $c$) inequality we obtain that necessarily $c < c_g n^{1/2}$ for large $n$ as $c$ cannot be bigger than the larger root $2n/3 + o(n)$.

Writing $e(G) \ge e_a + e_c + e_y - (n-a+1)c + e_r$ and substituting $a \ge n - c + 1$ everywhere (as the total coefficent of $a$ is positive) we obtain

$$c_g n^{3/2} \ge \tfrac{-(r+c)^2 + (r+c)(2n+1)}{2} - r|Y| + O(n) \ge \tfrac{-(r+c)^2}{2} + (r+c)n + O(n + r\sqrt{n}). \tag{3}$$

The larger root of this (quadratic in $r+c$) inequality is $2n + o(n)$, but $r \le n + o(n)$ since $a = n + o(n)$ and $a + r \le |L|$. So we conclude that $l - 1 = c + r \le c_g n^{1/2} + O(1)$ as required. □

Now let us try to derive a final contradiction.

*Proof of* (2). Let $x_{\max}$ be a vertex of maximal degree $\Delta(G) = c_m n^{3/2}$. Lemma 1 (with $A = V(G)$ and $x_1 = x_{\max}$) shows that $e(G) \ge n^2 + \Delta(G)$, that is, $c_m \le c_g$. Let $c' = ((4 + c_g^2)^{1/2} - c_g)/2$ and $c_f = 1.732$.

We apply a version of the greedy algorithm. Set initially $A = C = \emptyset$ and $B = V(G)$.

At Stage 1 move to $A$, one by one and as long as possible, a vertex $x \in B$ with $d_{B\setminus L}(x) \ge n - l$ and $d_{B\cup C}(x) \ge n$. After $x$ was moved do the $n$-check, that is, move to $C$ all vertices $y \in B \cap L$ with $d_{B\cup C}(y) < n$. We may assume that we were selecting $x \in B$ so that $d_G(x)$ was non-increasing. Let $A_1$ be the set of vertices moved to $A$ at Stage 1, $F = \{x \in A_1 : d_G(x) \ge n + c_f n^{1/2}\}$ and $a_f = |F|/n$. By Lemma 2 we have $l \le c_g n^{1/2} + o(1)$, so the number of edges incident to $F$ is at least

$$\sum_{x\in F} d(x) - \sum_{x \in F} \frac{d(x) - n + l}{2} \ge a_f n^2 + a_f \frac{c_f - c_g}{2} n^{3/2} + o(n^{3/2}).$$

At Stage 2 move to $A$, one by one and as long as possible, any vertex $x \in B$ having at least $n + c' n^{1/2}$ neighbours in $B \cup C$ and for every such $x$ do the $n$-check as in Stage 1.

At Stage 3 we repeat the following until $B \cap L = \emptyset$. Take $x \in B \cap L$. As long as $d_{B\cup C}(x) \ge n$ move to $A$ some $x$-neighbour $y \in B \cap L$ (note that $d_{B\cup C}(y) \ge n$) and perform the $n$-check. Such $y$ necessarily exists as $x$ has fewer than $n - l$ neighbours in $B \setminus L$ while $|C| \le l$. (The latter inequality is true because if $|C| > l$ at some moment then continuing with the standard greedy algorithm (Lemma 1) applied to $B \cup C$ we find at least $n - |A| + 1$ vertices in $B \cap L$ which contradicts $|L| = n + l$.) Of course, the last $n$-check moves $x$ itself to $C$.

Let $a_i n$ (resp. $c_i n^{1/2}$) be the number of vertices moved to $A$ (resp. to $C$) at the $i$th stage. As eventually $\Delta(G[B \cup C]) < n$ we conclude that $a_1 + a_2 + a_3 > 1$. Also $a_3 \leq c' c_3$ as for every $x$ moved to $C$ at Stage 3 we moved at most $c' n^{1/2}$ vertices to $A$.

Note that the first vertex moved at Stage 1 may be assumed to have degree $\Delta(G) = c_m n^{3/2}$ unless $\Delta(G) = O(n)$. So our algorithm produces the following lower bound on the size of $G$:

$$e(G) \geq n^2 + \left(c_m + a_f \tfrac{c_f - c_g}{2} + a_2 c' + c_3(1 - a_3) + o(1)\right) n^{3/2}. \tag{4}$$

The term $c_3 n^{3/2}$ comes from Stage 3 and counts $n - 1$ edges incident to each vertex moved to $C$; however, we have to exclude all edges incident to $A$ (at most $c_3 n^{1/2} \cdot a_3 n$ edges) which might be included in the $n^2$ term. Also, we could have added $(c_1 + c_2) n^{3/2}$ to the right-hand side of (4), but this does not seem to give any improvement on (2).

Now using the inequalities $a_3 \leq c' c_3$ (twice) and $0 \leq c_3 \leq c_g + o(1)$ (by Lemma 2 we have $c_3 \leq |C| n^{-1/2} \leq c_g + o(1)$) we obtain from (4) that

$$\begin{aligned} a_2 + a_3 &\leq \frac{c_g - a_f(c_f - c_g)/2 - c_m - c_3 + c' c_3^2}{c'} + c' c_3 + o(1) \\ &\leq \frac{c_g - a_f(c_f - c_g)/2 - c_m}{c'} + \max\left(0, c_g^2 + c' c_g - c_g/c'\right) + o(1). \end{aligned} \tag{5}$$

But our $c'$ satisfies $c_g^2 + c' c_g = c_g / c'$ so the second term disappears.

Choose a set $Y \subset \overline{L}$ by placing each vertex of $\overline{L}$ into $Y$ independently with probability $p = (c_f + 2\varepsilon) n^{-1/2}$, where $\varepsilon > 0$ denotes a small constant.

The number of $Y$-neighbours of any $x \in L$ has a binomial distribution with expectation at most $p c_m n^{3/2} = (c_f + 2\varepsilon) c_m n$. Hence the probability that say $d_Y(x) > (c_f c_m + 3\varepsilon) n$ is exponentially small in $n$ by Chernoff's bounds [1].

Similarly the expected value of $d_Y(x)$, $x \in A_1$, is at least $p(n - l) \approx (c_f + 2\varepsilon) n^{1/2}$. So, $d_Y(x) < (c_f + \varepsilon) n^{1/2}$ with probability at most $\exp(-c n^{1/2})$ for some constant $c > 0$.

Hence, there exists $Y$ (in fact, almost every choice would do) such that $d_Y(x) \leq (c_f c_m + 3\varepsilon) n$ for every $x \in L$ and $d_Y(y) \geq (c_f + \varepsilon) n^{1/2}$ for every $y \in A_1$.

Now consider the partition $V(G) = V_1 \cup V_2$, where $V_1 = (\overline{L} \setminus Y) \cup (A_1 \setminus F)$. Any $x \in A_1 \setminus F$ has at least $(c_f + \varepsilon) n^{1/2} > d(x) - n$ neighbours in $Y$, so $d_{V_1}(x) < n$. But then $d_{V_2}(x) \geq n$ for some $x \in L \cap V_2$. Hence,

$$n \leq |V_2 \setminus Y| + d_Y(x) \leq n + l - |A_1| + |F| + (c_f c_m + 3\varepsilon) n,$$

or equivalently

$$a_2 + a_3 + a_f + c_f c_m \geq 1 + \text{error term}, \tag{6}$$

where the error term can be made arbitrarily small by choosing the constant $\varepsilon$ small.

Chopping off some terms in (4) we obtain that $a_f$ lies between 0 and $2(c_g - c_m)/(c_f - c_g) + o(1)$. Hence, by (5),

$$\begin{aligned} a_2 + a_3 + a_f &\le \frac{c_g - a_f(c_f - c_g)/2 - c_m}{c'} + a_f + o(1) \\ &\le \max\left(\tfrac{c_g - c_m}{c'}, 2\tfrac{c_g - c_m}{c_f - c_g}\right) + o(1). \end{aligned} \tag{7}$$

Using the values of $c_g$ and $c_f$ we obtain from (6) and (7) that necessarily

$$\max(0.767 + 0.403\, c_m,\ 0.9992 + 0.0004\, c_m) \ge 1 + o(1),$$

which cannot be satisfied for $0 \le c_m \le 0.577$. $\Box$

**Remark.** The constant 0.577 can be improved, even with the present proof. For example, the optimal choice

$$c_f = \min\left(\sqrt{4 + c_g^2},\ c_g + \sqrt{2(c_g - c_m)/c_m}\right),$$

should give (with extra algebraic work) $c_g \ge 0.591$.

Also, after Stage 2 we could apply the algorithm of Lemma 2: we have identified at least $(c_m + a_2(c' - c_g) + a_f \frac{c_f - c_g}{2})n^{3/2}$ 'useless' (from the point of view of Lemma 2) edges, which should bring down the bound on $l$ there. We do not know how much gain this would have given (the calculations get rather messy) but we believe that we have reached a good compromise in the sense that the proof is not too long and the bound is not too bad.

# 4 Stars versus Tripartite Graphs

We demonstrate here, by modifying the construction of Section 2, that if we allow $(1 + \varepsilon)n^2$ edges, then we can witness much stronger arrowing properties.

In the proof below we introduce constants $c_1, c_2$, and so on. It should not be hard to check that we can always choose sufficiently small $c_i$ (depending on $c_1, \ldots, c_{i-1}$) satisfying all conditions set in the proof. We do not try to optimize the constants.

**Theorem 2** *For any fixed $\varepsilon > 0$, there are a constant $c = c(\varepsilon) > 0$ and a graph $G$ with at most $(1 + \varepsilon)n^2$ edges such that if $E(G)$ is coloured blue-red without a blue $K_{1,n}$, then the red subgraph contains cycles of all lengths (even and odd) up to $cn$ as well as a complete tripartite graph $K_{s,s,s}$ with $s \ge c \log n$.*

*Proof.* Choose integers

$$\begin{aligned} m &= \sqrt{n/2} + O(1), \\ k &= (\sqrt{2} + c_1)\sqrt{n} + O(1), \\ l &= n + c_1 n + O(1), \\ h &= c_1\sqrt{n} + O(1). \end{aligned}$$

Choose $k$-sets $K_1, \ldots, K_m$, $l$-sets $L_1, \ldots, L_m$, and an $h$-set $H$ (all disjoint). Let $G$ consist of all edges intersecting $H$ and of all edges intersecting $K_i$ and lying within $K_i \cup L_i$, $i \in [m]$, that is, $G = K_h + mP_{k,l}$. If $c_1 > 0$ is small, then $G$ has at most $(1+\varepsilon)n^2$ edges.

Consider any blue-red colouring of $E(G)$ without a blue $K_{1,n}$. Let $G' \subset G$ be the red subgraph, let $d'(x)$ be the red degree of $x \in V(G)$, and so on.

Define the bipartite graph $F$ with classes $H$ and $[m]$ as follows; $x \in H$ is connected to $i \in [m]$ if and only if $x$ sends at least $l + c_1\sqrt{n}/2$ red edges to $K_i \cup L_i$. Now, the inequality

$$(m - d_F(x))c_1\sqrt{n}/2 + d_F(x)k \geq mk - n + 1,$$

implies that each $x \in H$ has $\Theta(\sqrt{n})$ neighbours in $F$.

*Claim 1.* Suppose $i \in [m]$ is connected in $F$ to $x, y \in H$. There is a constant $c_2 > 0$ such that, for any $j \in [2, c_2\sqrt{n}]$, there exists a red path of length $j$ which connects $x$ to $y$ and lies within $K_i \cup L_i$.

To prove the claim, consider $X = (N'(x) \cup N'(y)) \cap (K_i \cup L_i)$. The set $X \cap K_i$ has at least $c_1\sqrt{n}/2$ elements, each being incident in $G'$ to at least $c_1 n + O(\sqrt{n})$ elements of $N'(x) \cap N'(y)$. Clearly (cf. Erdős and Gallai [6]), we can find, within $X$, a red path $P = (z_1, \ldots, z_p)$ with $p \geq c_2\sqrt{n}$ and $z_1 \in N'(x) \cap N'(y)$. Now, either $(x, z_1, \ldots, z_{j-1}, y)$ or $(y, z_1, \ldots, z_{j-1}, x)$ is the required red path. The claim is proved.

The obvious modification of the above argument for the case $x = y$ shows that $G'$ contains cycles of all lengths between 3 and $c_2\sqrt{n}$.

Let us show how to find cycles of larger length. It is not hard to see that the bipartite graph $F$ (which has positive density) contains a cycle of length $2t = \Theta(\sqrt{n})$ with $2t < c_2\sqrt{n}$ for large $n$. Let the cycle go through vertices $x_1, i_1, \ldots, x_t, i_t, x_{t+1} = x_1$, where $x_j \in H$ for $j \in [t]$. By Claim 1 we can connect $x_j$ and $x_{j+1}$ by a red path though $K_{i_j} \cup L_{i_j}$ of any length up to $c_2\sqrt{n}$, $j \in [t]$. Joining these $t$ paths together we can produce a red cycle of any length between $2t$ and $2tc_2\sqrt{n}$, which proves the first claim of the theorem.

Let us show $G' \supset K_{s,s,s}$. Let $i \in [m]$ be a vertex of degree at least $c_3\sqrt{n}$ in $F$. Let $s' = \lfloor c_4 \log n \rfloor$. As each $F$-neighbour of $i$ sends at least $c_1\sqrt{n}/2$ red edges to $K_i$, there are $s'' \geq c_3\sqrt{n}\binom{c_1\sqrt{n}/2}{s'}$ pairs $(x, X)$ with $x \in H$ sending a red each to every vertex of an $s'$-set $X \subset K_i$. We conclude that some $X$ appears in at least

$$c_3\sqrt{n}\binom{c_1\sqrt{n}/2}{s'}\binom{k}{s'}^{-1} \approx c_3\sqrt{n}\left(\frac{c_1/2}{(\sqrt{2}+c_1)}\right)^{s'} > (1+o(1))s'$$

pairs. Hence, $G'[H \cup K_i]$ contains a $K_{s',s'}$-subgraph $T'$; let $S' = V(T)$.

Let $U = \{x \in L_i : d'_{S'}(x) \geq (1 + c_1/3)s'\}$. Obviously, the number of red edges connecting $S'$ to $L_i$ is at most $2s'|U| + (l - |U|)(1 + c_1/3)s'$. On the other hand this number is at least $s'n(1 + 2c_1 + o(1))$ because $d'_{L_i}(y) = (1 + c_1)n + O(\sqrt{n})$ for $y \in T \cap H$ and $d'_{L_i}(y) \geq c_1 n + O(\sqrt{n})$ for $y \in T \cap K_i$. This implies (assuming $c_1 < 1$) that $|U| \geq 2c_1 n/9$.

Finally, let $s = \lfloor c_5 \log n \rfloor$; assume $s \le c_1 s'/3$. Each $x \in U$ covers at least $\binom{s'}{s}$ red $K_{s,s}$-subgraphs of $T'$. Therefore, some subgraph is covered by at least $|U|\binom{s'}{s}/\binom{s'}{s}^{-2} \ge s$ vertices of $U$, which gives a red copy of $T_{s,s,s}$, as required. $\square$

**Remark.** Our graph $G$ has other arrowing properties. For example, $F$ necessarily contains a matching of size $t = \Theta(\sqrt{n})$ and we can find $t$ vertex-disjoint cycles in $G'$ each of any prescribed length between 3 and $c_4\sqrt{n}$.

Theorem 2 and formula (2) imply the following.

**Corollary 1** *Let $F_n = C_{2m+1}$ with $m = o(n)$ or let $F_n$ be any graph with $\chi(F_n) = 3$ and $v(F_n) = o(\log n)$. Then $\hat{r}(K_{1,n}, F_n) = (1 + o(1))n^2$.* $\square$

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