Constructing Designs Straightforwardly: Worst Arising Cases

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Abstract

Suppose that we consecutively remove edges from some k-graph of order n in which every t vertices are covered by at least λ edges to obtain a minimal such k-graph. What can be said about the size of the eventual k-graph? While, by the result of Rödl [9], the minimum is $\lambda\binom{n}{t}/\binom{k}{t} + o(n^t)$, we show that the maximum is $\lambda\binom{n}{t} + o(n^t)$. Also, some partial results are obtained about possible size of a maximal k-graph covering every t-set by at most λ edges.

1 Introduction

Relaxing the definition of a t-(n, k, λ)-design we obtain the following notions. A *packing* (given t, n, k, λ) is a *k-graph* H (that is, a k-uniform set system) on $[n] = \{1, \dots, n\}$ such that every t-subset of $[n]$ is covered by at most λ edges of H. Analogously, the definition of a *covering* requires that any $T \in [n]^{(t)}$ is covered by at least λ H-edges.

One can try to construct a t-(n, k, λ)-design in one of the two following ways. The first possibility is to construct a maximal packing, that is, a k-graph H such that H is a packing but the addition of any new k-edge violates this. This can be achieved by starting with the empty graph and, consecutively and as long as

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possible, adding k-edges. The second possibility is to consider minimal coverings which are easy to construct by removing edges from some covering.

These two approaches are essentially equivalent, which can be seen by considering the complement of H and replacing λ by $\binom{n-t}{k-t} - \lambda$. However, we are interested in the case when t, k, λ are fixed integers whilst n is sufficiently large.

For these settings, Rödl [9] showed that there always exist nearly optimal packings and coverings, that is, having size

$$\lambda \binom{n}{t} \binom{k}{t}^{-1} + o(n^t). \tag{1}$$

The error term in (1) was estimated by Gordon, Kuperberg, Patashnik and Spencer [5] who showed it to be $O(g_{t,k}(n))$, where $g_{t,k}(n) = n^t/(\log n)^{1/D}$ with $D = \binom{k}{t} - 1$.

On the other hand, we ask ourselves how bad the resulting packings and coverings could be, that is, what are the values of

$$\begin{aligned} p_{t,k,\lambda}(n) &= \min\{e(H) : H \text{ is a maximal } (t,k,\lambda)\text{-packing of order } n\}, \\ c_{t,k,\lambda}(n) &= \max\{e(H) : H \text{ is a minimal } (t,k,\lambda)\text{-covering of order } n\}. \end{aligned}$$

Concerning the function $c_{t,k,\lambda}(n)$, we compute it exactly for $t = 1$ (except some small n), asymptotically for $t = 2$, and establish some connections with Turán numbers for $t \geq 3$. The *Turán number* $\alpha(n,t,k)$ is the smallest size of an (n,t,k)-*Turán graph*, that is, a t-graph on $[n]$ such that any k set contains at least one edge. For example, $\alpha(n,2,k) = \sum_{i=1}^{k-1} \binom{\lfloor \frac{n+i-1}{k-1} \rfloor}{2}$.

The function $p_{t,k,\lambda}$ turned out to be easier to compute; we determine it with an $O(n^{t-2})$ error term.

In many cases we can obtain more precise information by working harder. But trying to keep this paper short, the author had to omit less important results.

2 Packings

The problem of computing $p_{t,k,\lambda}(n)$ is an instance of the more general sat-problem; see Bollobás [2, Section 3] for a survey of the latter. From the saturation point of view, Kászonyi and Tuza [6] determined $p_{1,2,\lambda}(n)$ for any λ, n and the author [8] computed $p_{2,3,\lambda}(n)$ asymptotically for $n \to \infty$.

$t = 1$

The following theorem gives the exact answer in almost every case, except for some small n when we have only a lower bound.

Theorem 1 *Given $\lambda \geq 1$ and $k \geq 2$, define v by $\lambda/k \in [\binom{v-1}{k-1}, \binom{v}{k-1}]$. Then $p_{1,k,\lambda} \geq \binom{v}{k} + \lceil \frac{\lambda(n-v)}{k} \rceil$. If, furthermore, $\lambda \leq \binom{n-v-1}{k-1}$, then we have in fact equality.*

Proof. Given any maximal packing H, let $S \subset V(H)$ consist of all vertices contained in at most $\lambda - 1$ edges of H. Clearly, S must span the complete k-graph. Thus $e(H) \geq \min_{s \in [0,n]} f(s)$, where $f(s) = \binom{s}{k} + \frac{\lambda(n-s)}{k}$, which implies the lower bound.

Conversely, let $n' = n - v$. It is not hard to construct a k-graph H on $[n']$ such that every vertex has degree λ except a set D of at most $k-1$ vertices of degree $\lambda - 1$. (A possible proof: let the vertices $1, \ldots, n'$ form a regular n'-gon, as long as possible add a missing k-set and all its circle rotations, sparing $[k]$ for the very end when we add certain rotations of it.)

Add to H the complete k-graph on $S = [n'+1, n]$ plus, if $D = \{x \in [n'] : d(x) < \lambda\} \neq \emptyset$, an edge E intersecting $[n']$ in the set D. Note that the resulting graph is a packing (if $\binom{v-1}{k-1} \geq \lambda$, then we obtain the contradiction $f(v) \geq \lambda n/k > f(k-1)$) and it is clearly a maximal one. □

$t = 2$

Theorem 2 *Given $\lambda \geq 2$ and $k \geq 3$, let $\mu = \lambda/\binom{k}{2}$. Then*

$$\mu\alpha(n, 2, k) \leq p_{2,k,\lambda}(n) \leq \mu\alpha(n, 2, k) + O(n) = \mu\frac{n^2}{2(k-1)} + O(n), \tag{2}$$

where the lower bound holds for any $n \geq \max(k + \mu - 1, k\mu^{1/(k-2)})$.

Proof. Given a maximal packing H, we define on the same vertex set the 2-graph G so that $\{i, j\} \in E(G)$ iff there are λ H-edges containing $\{i, j\}$. Clearly, any k-set independent in G must be an edge of H. This implies that

$$e(H) \geq L(G) = k_k^2(\overline{G}) + \mu e(G), \tag{3}$$

where $k_k^2(\overline{G})$ denotes the number of K_k^2-subgraphs of $\overline{G}$, the complement of G. We want to find, for which 2-graphs G, the right-hand side of (3) is minimized. By a theorem of Bollobás [1] (for some extensions see Schelp and Thomason [10]), this happens if $\overline{G}$ is a complete multipartite 2-graph (that is, if G is a disjoint union of complete graphs). If the parts are of sizes $n_1 \geq n_2 \geq \ldots \geq n_l$, then we have to minimize

$$L(G) = k_k^2(\overline{G}) + \mu e(G) = \sum_{A \in [l]^{(k)}} \prod_{i \in A} n_i + \mu \sum_{i=1}^{l} \binom{n_i}{2}, \tag{4}$$

given the condition $\sum_{i=1}^{l} n_i = n$.

Suppose that $l \geq k$. Let G' be obtained from G by merging the smallest two parts together. This adds $n_{l-1}n_l$ extra edges to G, but this eliminates all K_k^2-subgraphs of $\overline{G}$ intersecting both of the affected parts, that is,

$$k_k^2(\overline{G}) - k_k^2(\overline{G'}) = n_{l-1}n_l \sum_{A\in[l-2]^{(k-2)}} \prod_{i\in A} n_i. \tag{5}$$

We claim that $\sum_{A\in[l-2]^{(k-2)}} \prod_{i\in A} n_i \geq \mu$. As n_l and n_{l-1} are two smallest parts, it is enough to verify the inequality for $n_2 = \ldots = n_{l-2} = n_{l-1} = n_l = x$ in which case it reduces to

$$g(x) = \binom{l-3}{k-2} x^{k-2} + \binom{l-3}{k-3} (n-(l-1)x)x^{k-3} \geq \mu. \tag{6}$$

The routine algebraic work shows that (6) holds for any real x with $1 \leq x \leq n/l$, provided n satisfies our assumptions.

Thus $L(G) \geq L(G')$, so we may assume that $l \leq k-1$. But then $k_k^2(G) = 0$ and $e(G)$ is minimal if we have exactly $k-1$ parts of nearly equal sizes and the lower bound follows.

Here is our construction establishing the upper bound. Choose the maximal integer $v \leq \frac{n}{k-1}$ such that a 2-(v, k, λ)-design exists. By the fundamental result of Wilson [12], $v = \frac{n}{k-1} - O(1)$. Build such a design on $V_i = [v(i-1)+1, iv]$ for $i \in [k-1]$. Completing the union of these to a maximal packing H on $[n]$, we add only $O(n)$ extra edges. Indeed, each new edge shares at most one vertex with each V_i while each of the remaining $O(1)$ vertices can belong to at at most $\lambda\frac{n-1}{k-1}$ edges. Hence $e(H) = \mu\frac{n^2}{2(k-1)} + O(n)$ as required. □

Remark. The lower bound in (2) is sharp if there exist 2-$(\lfloor\frac{n}{k-1}\rfloor, k, \lambda)$- and 2-$(\lceil\frac{n}{k-1}\rceil, k, \lambda)$-designs, which is the case for a periodic series of values of n when n is large (see Wilson [12]).

$t \geq 3$

Finally, let us consider the general case $t \geq 3$. It seems that $p_{t,k,\lambda}(n)$ is related to $\alpha(n, t, k)$.

Theorem 3 *For any fixed t, k, λ, and large n*

$$p_{t,k,\lambda}(n) \geq (1-o(1))\lambda\alpha(n,t,k)\binom{k}{t}^{-1}. \tag{7}$$

Proof. Let H be a maximal packing. Let the t-graph G consist of all t-sets covered by λ edges of H. Similarly to the above, we note that any k-subset of $[n]$ not spanning an edge in G must belong to $E(H)$. Therefore,

$$e(H) \geq \lambda e(G) \binom{k}{t}^{-1} + k_k^t(\overline{G}). \tag{8}$$

If $e(G) \geq (1 - o(1))\alpha(n,t,k)$ then the first summand in the right-hand side of (8) itself gives the desired lower bound. Otherwise, the result of Erdős and Simonovits [4] implies that the second summand is $\Theta(n^k)$ which is far more than required. $\square$

We do not have many structural results related to the Turán problem for complete hypergraphs. Sidorenko [11] mentions the following conjectures.

$$\alpha(n,3,k) = \left(\frac{2}{k-1}\right)^2 \binom{n}{3} + o(n^3), \tag{9}$$

$$\alpha(n,4,5) = \frac{5}{16}\binom{n}{4} + o(n^4). \tag{10}$$

Theorem 4 *There exists a maximal $(3,4,\lambda)$-packing with $\frac{\lambda}{9}\binom{n}{3} + O(n^2)$ edges. There exists a maximal $(3,k,\lambda)$-packing with $\lambda(\frac{2}{k-1})^2\binom{n}{3}/\binom{k}{3} + O(g_{3,k}(n))$ edges if $k \geq 5$ is odd. In particular, if (9) is true, then these packings are asymptotically smallest possible.*

Proof. Consider $k = 4$. Let $m = \lfloor n/3 \rfloor$. Define $A_i = [(i-1)m+1, im]$, $i \in [3]$. The graph G on $[3m]$ consisting of all triples $\{x,y,z\}$ with $x, y \in A_i$ and $z \in A_i \cup A_{i+1}$, where $A_4 = A_1$, is a $(3m,4,3)$-Turán graph with $\frac{4}{9}\binom{n}{3} + O(n^2)$ edges.

Consider the graph H consisting of edges $E = \{w,x,y,z\}$ with $\{x,y,z\} \subset A_i$ and $w \in A_{i+1}$ (then all 3-subsets of E belong to $E(G)$), $i \in [3]$, such that $u+x+y+z$ is congruent modulo m to an element in $[\lambda]$. It is easy to see that each edge G, except $O(n^2)$ edges, is covered by exactly λ edges of the packing H. Therefore, completing H to a maximal packing on $[n]$ we add only $O(n^2)$ edges, as required.

For $k = 2l+1$, $l \geq 2$, an example of an $(n,3,k)$-Turán graph G attaining (9) is obtained by partitioning $[n] = A_1 \cup \ldots \cup A_l$ into nearly equal parts and letting G be the union of the complete graphs on A_i, $i \in [l]$. By (1) we can find an almost optimal packing on each A_i; let H be the union of these. Completing it to a maximal packing, we add only $O(g_{3,k}(n))$-extra edges, which proves the claim. $\square$

However, we do not know any matching construction for $t = 3$ and even $k \geq 6$ or for $t = 4$ and $k = 5$. We have to find a packing which covers almost every edge of a corresponding Turán graph exactly λ times and covers only $o(n^t)$ edges outside this graph. The known to the author constructions achieving (9) or (10) do not admit such a packing by some trivial edge-counting. This includes the constructions presented by Kostochka [7] and by de Caen, Kreher and Wiseman [3]. Unfortunately, the author has no likely guess what $p_{t,k,\lambda}$ could be then.

3 Coverings

Minimal coverings are easier to handle. Here is our main result.

Theorem 5 *Let $k > t \geq 1$ and $\lambda \geq 1$ be fixed integers and let n be large. Let v be the minimal integer such that $\binom{v}{k-t} \geq \lambda$. Then*

$$c_{t,k,\lambda}(n) = \lambda\binom{n}{t} - \lambda v\binom{n}{t-1} + O(n^{t-2}). \tag{11}$$

Proof. First we provide a construction of a minimal covering H, which gives the lower bound on $c_{t,k,\lambda}(n)$. Let $A = [v]$ and $B = [n] \setminus A$.

To construct H add, for every $X \in B^{(t)}$, any λ edges whose intersection with B equals X. If n is large, we can additionally require that every t-set intersecting A in at most $k - t$ vertices is covered by at least λ edges. Now, consecutively let $i = t, t-1, \ldots, k-t+1$. (If $k \geq 2t$, we do not do anything.) For each t-set X which is covered by $s < \lambda$ edges and intersects A in exactly i vertices, add any new $\lambda - s$ edges of the form $(A \cap X) \cup Y$, where Y is a $(k-i)$-subset of B. We claim that the obtained covering is minimal. Indeed, any edge E added at the first stage contains a t-subset, namely $E \cap B$, which is covered by exactly λ edges, because all later edges intersect B in at most $t-1$ vertices. Also, an edge E added for some i cannot be removed: when it was added than the corresponding $X \subset E$ was covered by exactly λ edges, while any edge added later cannot contain X because we took i decreasing.

The easy computation of $e(H)$ gives the required lower bound.

Let us prove the upper bound. Let H be a minimal (t, k, λ)-covering of the largest size. Let the t-graph G consist of all those t-sets which are covered by exactly λ edges of H. Count the number s of pairs $(K \supset T)$, with $K \in E(H)$ and $T \in E(G)$. As each edge of H contains at least one suitable T as a subset, we conclude that

$$e(H) + r \leq s \leq \lambda e(G) \leq \lambda\binom{n}{t}, \tag{12}$$

where r is the number of H edges which contribute at least 2 to s. By the already proved lower bound, we have $r = O(n^{t-1})$.

Let M be the set of (ordered) pairs (T_1, T_2) such that $T_1 \in E(G)$, $|T_2| = t$, $|T_1 \setminus T_2| = 1$, and there exists $K \in E(H)$ with $T_1 \cup T_2 \subset K$.

It is not hard to see that an edge $T_1 \in E(G)$ appears as the first coordinate in at least tv elements of M. On the other hand, let T_2 be any t-set. If, for some t-sets S_1, S_2 with $S_1 \setminus T_2 = S_2 \setminus T_2 = \{x\}$, we have $(S_1, T_2), (S_2, T_2) \in M$, then we have $K \in E(H)$ containing both $S_1, S_2 \in E(G)$. If $T_2 \in E(G)$ and $(T_1, T_2) \in M$ for some T_1, then we have an H-edge containing both $T_1, T_2 \in E(G)$. Hence we have, rather crudely,

$$tve(G) \leq |M| \leq (n-k)e(\overline{G}) + \binom{k}{t}^2 r.$$

This implies $e(G) \leq \binom{n}{t} - v\binom{n}{t-1} + O(n^{t-2})$, and the claim follows from the inequality $e(H) \leq \lambda e(G)$. $\square$

Remark. The proof (as it stands) gives that $c_{1,k,\lambda}(n) = \lambda(n-v)$ for $n \geq (v-k)(v-1) + \lambda k^2 v$.

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