Size Ramsey Numbers of
Stars versus 4-Chromatic Graphs

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Abstract

We investigate the asymptotics of the size Ramsey number $\hat{r}(K_{1,n}, F)$, where $K_{1,n}$ is the n-star and F is a fixed graph.

The author [11] has recently proved that $\hat{r}(K_{1,n}, F) = (1 + o(1))n^2$ for any F with chromatic number $\chi(F) = 3$. Here we show that

$$\hat{r}(K_{1,n}, F) \le \frac{\chi(F)(\chi(F) - 2)}{2} n^2 + o(n^2) \quad \text{if } \chi(F) \ge 4$$

and conjecture that this is sharp. We prove the case $\chi(F) = 4$ of the conjecture, that is, that $\hat{r}(K_{1,n}, F) = (4 + o(1))n^2$ for any 4-chromatic graph F. Also, some general lower bounds are obtained.

1 Introduction

Given two graphs F_1 and F_2, we say that a graph G *arrows* the pair (F_1, F_2), denoted by $G \to (F_1, F_2)$, if for any blue-red colouring of the edge set of G we necessarily have either a blue copy of F_1 or a red copy of F_2 (or both). The *size Ramsey number* $\hat{r}(F_1, F_2)$ is the minimum number of edges of a graph G such that $G \to (F_1, F_2)$.

Here we investigate the size Ramsey number $\hat{r}(K_{1,n}, F)$, the n-star $K_{1,n}$ versus a fixed graph F, as n tends to infinity. Problems of this type were first considered by Erdős, Faudree, Rousseau and Schelp [5] who studied the

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asymptotics of $\hat{r}(K_{1,n}, K_k)$ as n tends to infinity and showed that $\hat{r}(K_{1,n}, K_k) \geq \lfloor (k-2)^2/4 \rfloor n^2/2 + o(n^2)$, where K_k denotes the complete graph of order k.

Erdős [3] observed that $K_{n+1} + \overline{K}_n \to \hat{r}(K_{1,n}, K_3)$ and conjectured that the corresponding upper bound on $\hat{r}(K_{1,n}, K_3)$ is sharp. Faudree and Sheehan [8, Conjecture 1] made the more general conjecture that

$$\hat{r}(K_{1,n}, K_k) = \binom{(k-1)n+1}{2} - \binom{n}{2}, \quad n \geq 1, \; k \geq 3, \tag{1}$$

where the upper bound follows from the consideration of $K_{(k-2)n+1} + \overline{K}_n \to (K_{1,n}, K_k)$, see Section 2.

Some problems motivated by (1) were studied in [8, 4, 2]. But until recently there had not been any progress on the original conjecture (1). It can well be that there exists a simple argument proving (1) because the construction giving the upper bound is so simple. Unfortunately, it has evaded us so far.

Let $\mathcal{B}_k(n)$ be the family of all graphs G such that for any k-partition $V(G) = A_1 \cup \ldots \cup A_k$ there is $i \in [k] := \{1, \ldots, k\}$ such that $\Delta(G[A_i]) \geq n$, where $\Delta(G[A_i])$ denotes the maximum degree of the subgraph of G spanned by A_i. Let $b_k(n)$ be the minimum size of $G \in \mathcal{B}_k(n)$. The problem of computing $b_2(n)$ appears in [3]. The main motivation for this definition is that $b_k(n)$ is clearly a lower bound on $\hat{r}(K_{1,n}, F)$ for any F with $\chi(F) = k+1$. But the function b_k is also of interest on its own.

Erdős' conjecture (i.e. the case $k = 3$ of (1)) has been recently disproved by the author [11] who showed that $\hat{r}(K_{1,n}, F_n) = (1 + o(1))n^2$ if $\chi(F_n) = 3$ and $v(F_n) = o(\log n)$. Also, more precise estimates were obtained in special cases:

$$n^2 + (0.557 + o(1))\, n^{3/2} < b_2(n) \leq \hat{r}(K_{1,n}, K_3) < n^2 + \sqrt{2}\, n^{3/2} + n.$$

Here we show that for any fixed graph F with $\chi(F) \geq 4$ we have

$$\hat{r}(K_{1,n}, F) \leq \frac{\chi(F)(\chi(F) - 2)}{2}\, n^2 + o(n^2)$$

and conjecture that this is sharp:

Conjecture 1 *For any fixed graph F of chromatic number $k \geq 4$ we have*

$$\hat{r}(K_{1,n}, F) = \frac{k(k-2)}{2}\, n^2 + o(n^2). \tag{2}$$

Note that (2) does not hold for $k = 3$. If Conjecture 1 is true, then this would mean that the case $\chi(F) = 3$ is exceptional.

We show that $b_3(n) = (4+o(1))n^2$ which settles the case $k = 4$ of Conjecture 1. Unfortunately, Conjecture 1 remains open for $k \geq 5$, although we have rather tight lower bounds on $b_k(n)$.

For $\chi(F) = 2$ we observe a different phenomenon: the size Ramsey number $\hat{r}(K_{1,n}, F)$ is bounded from above by a function linear in n. The more general results in [10] imply that the limit $\lim_{n\to\infty} \hat{r}(K_{1,n}, F)/n$ exists for any fixed bipartite graph F. Unfortunately, the limiting value is known only for very few (non-trivial) instances of F: $K_{2,m}$ (see [7]); more generally, $K_{s,t}$ for any $2 \le s \le t$ (see [10]); a path with at most 6 edges (see [9]). The case $\chi(F) = 2$ is not treated here.

2 Upper Bounds

Let $K_k(t)$ denote the complete k-partite graph with each part having t vertices. The following lemma clearly establishes the upper bound in Conjecture 1. We do not try to optimize the constants.

Lemma 2 *Fix $k \ge 3$. Let $\varepsilon > 0$ be arbitrary. Then there exist constants $c > 0$ and n_0 such that $K_m + \overline{K}_n \to (K_{1,n}, K_k(t))$ for all $n \ge n_0$, where $m = \lceil (k-2+\varepsilon)n \rceil$ and $t = \lfloor c \log n \rfloor$.*

Proof. Let n be sufficiently large. Assume $\varepsilon < 1/2$. Let the vertex set of $G = K_m + \overline{K}_n$ be $A \cup B$, where $|A| = m$, $|B| = n$ and $G[B]$ is the empty graph. Given a blue-red colouring of $E(G)$ without a blue $K_{1,n}$, let G' be the red subgraph.

The set A spans at most $(n-1)|A|/2$ blue edges, so the edge density of $G'[A]$ is at least $1 - \frac{1}{k-2+\varepsilon}$. By the Erdős-Stone theorem [6], $G'[A]$ contains a $K_{k-1}(s)$-subgraph K with $s = \Theta(\log n)$.

Each vertex in $V = V(K)$ sends at least $(k-2+\varepsilon)n - (k-1)s$ red edges to $\overline{V}$, where we denote $\overline{V} = V(G) \setminus V$ should the graph G be clear from the context. Hence, the number of red edges connecting V to $\overline{V}$ is at least $s(k-1)(k-2+\varepsilon)n + O(s^2)$. Let U consist of those vertices of $\overline{V}$ which send at least $(k-2+\varepsilon/k)s$ red edges to V. Of course, each vertex of U sends at most $|V| = (k-1)s$ red edges to V. Thus counting red edges between V and $\overline{V}$ we obtain

$$(k-1)s \cdot |U| + (k-2+\varepsilon/k)s \cdot |\overline{U}| + O(s^2) \ge (k-2+\varepsilon)n \cdot (k-1)s,$$

which implies that $|U| = \Theta(n)$. Let $t = \lfloor ds \rfloor$ with $d \le \varepsilon/k$. In G' each vertex of U covers at least one $K_{k-1}(t)$-subgraph of K. Hence, some such subgraph appears at least $|U|/\binom{s}{t}^{k-1}$ times, which is at least t if d is small enough. This gives us a red $K_k(t)$, as required. ∎

Finally, let us demonstrate that

$$K_{(k-2)n+1} + \overline{K}_n \;\to\; (K_{1,n}, K_k), \quad k \ge 2,\; n \ge 1. \tag{3}$$

The case $k = 2$ is trivial. Assume $k \geq 3$ and take any blue-red colouring without a blue $K_{1,n}$. A vertex x of degree $(k-1)n$ is incident to at least $(k-2)n+1$ red edges and the subgraph spanned by the red neighbourhood of x contains $K_{(k-3)n+1} + \overline{K}_n$. Now deduce (3) by induction.

We believe that $K_{(k-1)n+1} + \overline{K}_n \in \mathcal{B}_k(n)$ is extremal, that is,

$$b_k(n) = e(G) = \binom{kn+1}{2} - \binom{n}{2}. \tag{4}$$

Of course, (4), if true, would imply (1).

3 Easy Lower Bound

Let us turn to proving lower bounds on $b_k(n)$, which would give us lower bounds on $\hat{r}(K_{1,n}, F)$ for any F with $\chi(F) = k+1$.

A simple (but very useful) lemma first.

Lemma 3 *Let $k \geq 1$ and $n \geq 1$. If $G \in \mathcal{B}_k(n)$, then $\Delta(G) \geq kn$.*

Proof. Consider a partition $V(G) = A_1 \cup \ldots \cup A_k$ which minimizes $s = \sum_{i=1}^k e(G[A_i])$. For some $i \in [k]$, there exists $x \in G[A_i]$ of degree at least n. If we move x to any other part, then s does not decrease, so x sends at least n edges to each part. Hence, $d(x) \geq kn$. ∎

Theorem 4 *For any $k \geq 2$ and $n \geq 1$, we have $b_k(n) \geq \binom{k}{2} n^2$.*

Proof. Given $G \in \mathcal{B}_k(n)$, let $A_1 = V(G)$. Repeat the following for $i = 1, 2, \ldots, k-1$. Given a set A_i such that $G[A_i] \in \mathcal{B}_{k-i+1}(n)$, let $B_i \subset A_i$ be a set of size n minimizing $e(G[A_{i+1}])$, where $A_{i+1} = A_i \setminus B_i$. Clearly, $G[A_{i+1}] \in \mathcal{B}_{k-i}(n)$. So, if $i \leq k-2$, we can repeat the step with the next i.

Let $i \in [k-1]$. By Lemma 3, $G[A_{i+1}]$ contains a vertex x of degree at least $(k-i)n$. Now, every $y \in B_i$ sends at least $(k-i)n$ edges to A_{i+1}, because the exchange of x and y does not decrease $e(G[A_{i+1}])$.

Hence, $e(G) \geq \sum_{i=1}^{k-1} (k-i)n^2 = \binom{k}{2} n^2$, as required. ∎

The bound of Theorem 4 is quite good, which becomes clear when we compare it with the upper bound that follows from (3):

$$\frac{k^2-k}{2} n^2 \leq b_k(n) \leq \binom{kn+1}{2} - \binom{n}{2} = \frac{k^2-1}{2} n^2 + \frac{k+1}{2} n. \tag{5}$$

In the remainder of the paper we will work on improving the lower bound. This, however, requires some new notions.

4 Related Problem

As we have already mentioned, the function $b_2(n)$ (or the case $\chi(F) = 3$ of (2)) seems rather exceptional: while the right-hand side of (2) is $(1.5+o(1))n^2$, $\mathcal{B}_2(n)$-graphs with only $(1+o(1))n^2$ edges were constructed in [11]. Apparently, this makes Conjecture 1 hard to prove.

However, the graphs in [11] have large maximum degree. And this is an intristic feature of the problem: if the maximum degree of a $\mathcal{B}_2(n)$-graph is not large, then we can guarantee more than $(1+o(1))n^2$ edges. This enables us to improve our lower bounds on $b_k(n)$ by controlling the maximum degree of certain subgraphs of $G \in \mathcal{B}_k(n)$.

So, given $d \geq 0$ and $n \geq 1$, let

$$\mathcal{B}_2(n,d) = \{G \in \mathcal{B}_2(n) : \Delta(G) \leq 2n + d\}$$

and let $b_2(n,d)$ be the mimimum size of $G \in \mathcal{B}_2(n,d)$. The function $b_2(n,d)$ was asymptotically computed by Pikhurko and Thomason [12]. However, we will need not only the value of $b_2(n,d)$ but also some information on the structure of graphs in $\mathcal{B}_2(n,d)$. This is provided by the following lemma.

Given a graph G and $A, B \subset V(G)$, let

$$e(A,B) = |\{\{x,y\} \in E(G) : x \in A,\ y \in B\}|,$$

be the number of edges connecting a vertex of A to a vertex of B. Note that each edge of $G[A \cap B]$ is counted only once, so for example $e(A,A) = e(G[A])$.

Lemma 5 *For any $\varepsilon > 0$ there is n_0 such that for any $d \geq 0$, any $n \geq n_0$, any $G \in \mathcal{B}_2(n,d)$ with $e(G) \leq n^2 \ln n$ and any $A \subset V(G)$ with $a := |A| \leq (1-\varepsilon)n$, we have*

$$\Delta(G - A) \geq n\left(1 + \frac{n-a}{n+d} - \varepsilon\right)$$

and there exists a set $C \subset \overline{A} := V(G) \setminus A$ such that $|C| \leq n - a - \varepsilon n$ and

$$e(C, \overline{A}) \geq (n-a)n + \frac{(n-a)^2 n}{2n+2d} - 3\varepsilon n^2.$$

Proof. Without loss of generality assume that $\varepsilon < 1/2$. Suppose that n is sufficiently large. Let A and G be as above.

We are proving the first claim. If $d > (1-\varepsilon)n/\varepsilon$, then we are home: $\frac{n-a}{n+d} \leq \frac{n}{n+d} < \varepsilon$ and $\Delta(G - A) \geq n$ because $G \in \mathcal{B}_2(n)$. So, let us assume that $d \leq (1-\varepsilon)n/\varepsilon$. Let $B = \overline{A}$ and

$$H = \{x \in V(G) : d(x) \geq n\}.$$

By our assumption on $e(G)$ we have $|H| \le 2\ln n$. Define

$$\delta = \varepsilon^2, \quad q = (1+\delta)\,\frac{n+d}{2n+d-a}, \quad p = 1-q.$$

We have $0 < q \le 1$.

Choose $Y \subset B$ by placing independently each vertex of B into Y with probability p. Let $Z = B \setminus Y$. Let $d_Y(x)$ denote the number of neighbours of x that lie in Y, etc. By Chernoff's bounds [1], in view of $|H| = O(\ln n)$, we have almost surely that $d_Y(x)$ (and so $d_Z(x)$) differs from its expected value $pd_B(x)$ (resp. $qd_B(x)$) by at most $\delta^2 n$ for all $x \in H$. So, there exists Y for which this condition is true. We have

$$\begin{aligned} d_{A\cup Y}(x) &\le d_A(x) + p(d(x) - d_A(x)) + \delta^2 n \;=\; (1-p)d_A(x) + pd(x) + \delta^2 n \\ &\le (1-p)a + p(2n+d) + \delta^2 n \;<\; n, \quad \text{for all } x \in H. \end{aligned}$$

In particular, $\Delta(G[A \cup Y]) < n$. Hence, $G[Z]$ contains a vertex x of degree at least n. Of course, $x \in H$ and we have

$$\begin{aligned} d_B(x) &\ge \frac{d_Z(x) - \delta^2 n}{q} \ge \frac{(1-\delta^2)n(2n+d-a)}{(1+\delta)(n+d)} \\ &= (1-\delta)n\left(1 + \frac{n-a}{n+d}\right) \ge n\left(1 + \frac{n-a}{n+d} - \varepsilon\right), \end{aligned}$$

which proves the first claim.

Let us prove the second claim. Let n be large. Given G and A, we start with $C = \emptyset$ and $B = \overline{A}$ and apply the following procedure until $|A \cup C| = \lfloor n(1-\varepsilon)\rfloor$: move a vertex x of maximum degree in $G[B]$ from B to C. The first claim (which we have already proved) implies that

$$d_B(x) \ge n\left(1 + \frac{n - |A\cup C|}{n+d} - \varepsilon\right).$$

Hence the total number of edges encountered during the procedure is at least

$$\sum_{i=a}^{\lfloor (1-\varepsilon)n\rfloor} n\left(1 + \frac{n-i}{n+d} - \varepsilon\right) \ge (n-a)n + \frac{(n-a)^2 n}{2n+2d} - 3\varepsilon n^2,$$

as required. ∎

In particular we reprove the following result from [12], which is obtained by taking $A = \emptyset$ in Lemma 5.

Corollary 6 *$b_2(n,d) \ge (1+o(1))n^2\left(1 + \frac{n}{2n+2d}\right)$ as $n \to \infty$.* ∎

Remark. The above inequality is in fact sharp; see [12] for a construction of $G \in \mathcal{B}_2(n, d)$ exhibiting the corresponding upper bound on $b_2(n, d)$.

Remark. One is tempted to define analogously $b_k(n, d)$ as the minimum size of $G \in \mathcal{B}_k(n)$ with $\Delta(G) \le kn + d$. However, if (4) is true, then the function $b_k(n, d)$ does not say anything new for $k \ge 3$ because an extremal graph $K_{(k-1)n+1} + \overline{K}_n \in \mathcal{B}_k(n)$ has maximum degree kn, while there is no $G \in \mathcal{B}_k(n)$ with $\Delta(G) < kn$ by Lemma 3.

5 Useful Lemma

The following simple lemma seems very useful for our task.

Lemma 7 *Let G be any graph. For any set $S \subset V(G)$, there exists $T \subset V(G)$ such that $\Delta(G[\overline{T}]) < n$, each vertex of T sends at least n edges to $\overline{T}$ and T is incident to at least $n(|T| - |S|) + e(S, V(G))$ edges.*

Proof. Let $T = S$. Repeat the following consecutively and as long as possible. Either move to T a vertex $y \in \overline{T}$ with $d_{\overline{T}}(y) \ge n$ or move to $\overline{T}$ a vertex $y \in T$ with $d_{\overline{T}}(y) \le n - 1$.

Consider the function $f(T) = e(T, V(G)) - n|T|$. Clearly, neither of our operations decreases f, while moving a vertex to $\overline{T}$ we increase f by at least 1. In particular, this shows that the above procedure terminates at some point. Let T be the final set. The first two conditions are obviously satisfied (otherwise we can repeat the procedure). The third condition follows from the inequality $f(T) \ge f(S)$. ∎

Remark. Note that we do not impose any condition on G in Lemma 7 at all.

6 Key Step

Let $G \in \mathcal{B}_k(n)$ with $k \ge 3$. Let n be large. Assume that $e(G) < n^2 \ln n$.

Define $T_k = V(G)$. Do the following consecutively for $i = k-1, k-2, \ldots, 2$. By Lemma 7 we know that there exists $T_i \subset T_{i+1}$ such that $\Delta(G[T_{i+1} \setminus T_i]) < n$ while each $x \in T_i$ sends at least n edges to $T_{i+1} \setminus T_i$. Let T_i be such a set with the minimum number of vertices. We deduce that $G[T_i] \in \mathcal{B}_i(n)$ by induction. In particular, $|T_i| > in$ by Lemma 3; let $|T_i| = in + t_i$. Each vertex of the non-empty set T_i sends at least n edges to $T_{i+1} \setminus T_i$, so the latter set has at least n vertices, which implies that $t_i \le t_{i+1}$.

The following lemma constitutes the core of our proof.

Lemma 8 *Let $2 \le i \le k-1$ and let $A_{i-1} \subset T_i$ be any set of size $(i-1)n + a_{i-1}$ with $0 \le a_{i-1} \le t_i$. Then there exists an $(in+t_i)$-set $A_i \subset T_{i+1}$ such that*

$$e(A_i, T_{i+1}) \ge e(A_{i-1}, T_i) + in^2 + t_i n + \frac{n((n-a_{i-1})_+)^2}{2n+2t_i} + o(n^2), \tag{6}$$

where for $x \in \mathbb{R}$ we denote $x_+ = x$ if $x > 0$ and $x_+ = 0$ otherwise.

Proof. First, suppose that $a_{i-1} > n$. As each vertex of A_{i-1} sends at least n edges to $T_{i+1} \setminus T_i$, we have

$$e(A_{i-1}, T_{i+1}) \ge e(A_{i-1}, T_i) + ((i-1)n + a_{i-1})n.$$

Apply Lemma 7 to the set A_{i-1} with respect to $G[T_{i+1}]$. The obtained set has at least $in + t_i$ elements by the choice of t_i. Consider the moment when the current set had precisely $in + t_i$ elements; let it be A_i. From Lemma 7 we know that

$$\begin{aligned} e(A_i, T_{i+1}) &\ge n(|A_i| - |A_{i-1}|) + e(A_{i-1}, T_{i+1}) \\ &\ge (n + t_i - a_{i-1})n + e(A_{i-1}, T_i) + ((i-1)n + a_{i-1})n, \end{aligned}$$

which gives the required.

So, suppose that $a_{i-1} \le n$.

Starting with $B_i = \emptyset$, iterate the following as long as possible or until $|B_i| = n - a_{i-1}$: move to B_i a vertex of $T_i \setminus (A_{i-1} \cup B_i)$ which has at least $2n + t_i - a_{i-1} - |B_i|$ neighbours in $T_{i+1} \setminus (A_{i-1} \cup B_i)$. Let b_i be the number of elements in B_i when we stop. Also, let

$$S_i = T_{i+1} \setminus (A_{i-1} \cup B_i).$$

Note the inequality which we will need later:

$$e(B_i, T_{i+1} \setminus A_{i-1}) \ge \sum_{i=1}^{b_i} (2n + t_i - a_{i-1} - i + 1) = b_i(2n + t_i - a_{i-1}) - \binom{b_i}{2}. \tag{7}$$

Case 1 Suppose that $b_i < n - a_{i-1}$.

Let $R_i \subset A_{i-1} \cup B_i$ be an r_i-set minimizing $e(R_i, R_i \cup S_i)$, where $r_i = b_i + a_{i-1}$. Set $A'_{i-1} = (A_{i-1} \cup B_i) \setminus R_i$.

The graph $G_i = G[R_i \cup S_i]$ has the $\mathcal{B}_2(n)$-property: it is obtained by removing the $(i-1)n$-set A'_{i-1} from the $\mathcal{B}_{i+1}(n)$-graph $G[T_{i+1}]$. Let us estimate the maximum degree of G_i. For $x \in T_{i+1} \setminus T_i$, we have by the definition of $T_i \subset T_{i+1}$ that

$$d_{R_i \cup S_i}(x) \le d_{T_{i+1} \setminus T_i}(x) + |T_i \cap (R_i \cup S_i)| < n + (n + t_i) = 2n + t_i. \tag{8}$$

For $x \in T_i \setminus (A_{i-1} \cup B_i)$ we have by the definition of B_i that

$$d_{R_i \cup S_i}(x) \leq d_{S_i}(x) + |R_i| < (2n + t_i - a_{i-1} - b_i) + (b_i + a_{i-1}) = 2n + t_i. \quad (9)$$

It remains to consider vertices in R_i. Here we have two cases depending on the value of

$$f_i = \max\{d_{R_i \cup S_i}(x) : x \in R_i\} - 2n.$$

Case 1.1 Suppose that $f_i < t_i$.

We have $\Delta(G_i) < 2n + t_i$. Applying Lemma 5 to $R_i \subset V(G_i)$, we conclude that there is a set $C_i \subset S_i$ of size $n - r_i$ incident to at least $(n - r_i)n + \frac{n(n-r_i)^2}{2n+2t_i} + o(n^2)$ edges lying within S_i. Also, each vertex of the set $A_{i-1} \subset T_i$ sends at least n edges to $T_{i+1} \setminus T_i$. Apply Lemma 7 to $S_i' = A_{i-1} \cup B_i \cup C_i$ with respect to the graph $G[T_{i+1}]$ to construct a set T_i' with the three corresponding properties. By the extremality of t_i we have $|T_i'| \geq in + t_i$. Let A_i equal the current set T_i' at the moment when its size was precisely $in + t_i$. The number of edges of $G[T_{i+1}]$ incident to A_i is

$$\begin{aligned}
e &\geq n(|A_i| - |S_i'|) + e(S_i', T_{i+1}) \\
&\geq t_i n + e(A_{i-1}, T_i) + e(A_{i-1}, T_{i+1} \setminus T_i) + e(B_i, T_{i+1} \setminus A_{i-1}) + e(C_i, S_i) \\
&\geq t_i n + e(A_{i-1}, T_i) + ((i-1)n + a_{i-1})n + b_i(2n + t_i - a_{i-1}) - \binom{b_i}{2} \\
&+ (n - a_{i-1} - b_i)n + \frac{n(n - a_{i-1} - b_i)^2}{2n + 2t_i} + o(n^2).
\end{aligned}$$

Disregarding the error term, the obtained expression (as a function of b_i) is monotone increasing for $0 \leq b_i \leq n - a_{i-1}$; hence it is minimized for $b_i = 0$, which gives the required.

Case 1.2 Suppose that $f_i \geq t_i$.

The graph $G_i = G[R_i \cup S_i] \in \mathcal{B}_2(n)$ has maximum degree $2n + f_i$, so we can find an $(n - r_i)$-set $C_i \subset S_i$ incident to at least $(n - r_i)n + \frac{n(n-r_i)^2}{2n+2f_i} + o(n^2)$ edges of $G[S_i] = G_i - R_i$ by Lemma 5.

By the choice of R_i, each $x \in A_{i-1}'$ sends at least $2n + f_i$ edges to $T_{i+1} \setminus A_{i-1}'$. Hence,

$$d_{T_{i+1} \setminus T_i}(x) \geq 2n + f_i - |T_i \setminus A_{i-1}'| = n + f_i - t_i, \quad x \in A_{i-1}'. \quad (10)$$

Let $q_i = |B_i \cap R_i|$; thus $|A_{i-1} \cap R_i| = b_i + a_{i-1} - q_i$ and $|A_{i-1} \setminus R_i| = (i-1)n - b_i + q_i$.

Like in Case 1.1, by applying Lemma 7 to $S_i' = A_{i-1} \cup B_i \cup C_i$ with respect to the graph $G[T_{i+1}]$ we find a set $A_i \subset T_{i+1}$ of size precisely $in + t_i$ such that the number of edges of $G[T_{i+1}]$ incident to A_i is

$$\begin{aligned} e &\geq n(|A_i| - |S_i'|) + e(S_i', T_{i+1}) \;\geq\; t_i n + e(A_{i-1}, T_i) + e(A_{i-1} \setminus R_i, T_{i+1} \setminus T_i) \\ &+ e(A_{i-1} \cap R_i, T_{i+1} \setminus T_i) + e(B_i, T_{i+1} \setminus A_{i-1}) + e(C_i, S_i) \\ &\geq t_i n + e(A_{i-1}, T_i) + ((i-1)n - b_i + q_i)(n + f_i - t_i) + (b_i + a_{i-1} - q_i)n \\ &+ b_i(2n + t_i - a_{i-1}) - \binom{b_i}{2} + (n - a_{i-1} - b_i)n + \frac{n(n - a_{i-1} - b_i)^2}{2n + 2f_i} + o(n^2) \end{aligned}$$

The last expression (disregarding the error term) is monotone increasing in both q_i and f_i. (Recall that $i \geq 2$.) Hence it is at least its value for $q_i = 0$ and $f_i = t_i$, when we obtain precisely the lower bound from Case 1.1. Now the claim follows.

Case 2 Suppose that $b_i = n - a_{i-1}$.

The set $S_i' = A_i \cup B_i$ has now precisely in vertices. Apply Lemma 7 to S_i' with respect to $G[T_{i+1}]$ stopping the iteration when the current set has size $in + t_i$; call this set A_i. The number of $G[T_{i+1}]$-edges incident to A_i is

$$\begin{aligned} e &\geq n(|A_i| - |S_i'|) + e(S_i', T_{i+1}) \\ &\geq t_i n + e(A_{i-1}, T_i) + e(A_{i-1}, T_{i+1} \setminus T_i) + e(B_i, T_{i+1} \setminus A_{i-1}) \\ &\geq t_i n + e(A_{i-1}, T_i) + ((i-1)n + a_{i-1})n \\ &+ (n - a_{i-1})(2n + t_i - a_{i-1}) - \binom{n - a_{i-1}}{2} + o(n^2). \end{aligned}$$

Let consider the difference d between e and the required bound.

$$d \geq t_i n + \frac{n^2}{2} - a_{i-1}n - a_{i-1}t_i + \frac{a_{i-1}^2}{2} - \frac{n(n - a_{i-1})^2}{2n + 2t_i} + o(n^2)$$

It is routine to check that the obtained expression is monotone increasing in t_i. Hence (in view of $t_i \geq a_{i-1}$) we obtain.

$$d \geq \frac{n^2}{2} - \frac{a_{i-1}^2}{2} - \frac{n(n - a_{i-1})^2}{2n + 2a_{i-1}} + o(n^2).$$

The last expression (disregarding the error term) is non-negative for $0 \leq a_{i-1} \leq n$ which proves the claim and finishes the proof. ∎

7 Asymptotic Computation of $b_3(n)$

Now the b_3-case is a piece of cake.

Theorem 9 $b_3(n) = 4n^2 + o(n^2)$.

Proof. The upper bound follows from (3).

To prove the lower bound, let $G \in \mathcal{B}_3(n)$ and let T_i's and t_i's be as defined before Lemma 8. Define $A_1 \subset T_2$ be an n-set incident to as many as possible edges of $G' = G[T_2]$, which is at least $n^2(1 + \frac{n}{2n+2t_2} + o(1))$ by Lemma 5 because G' is a $\mathcal{B}_2(n)$-graph with $\Delta(G') < v(G') = 2n+t_2$. Apply Lemma 8 (with $a_1 = 0$) to find a set A_2 of size $2n + t_2$ with

$$\begin{aligned} e(G) \ &\geq\ e(A_2, T_3) \ \geq\ e(A_1, T_2) + 2n^2 + t_2 n + \frac{n^3}{2n + 2t_2} + o(n^2) \\ &\geq\ 3n^3 + t_2 n + \frac{n^3}{n + t_2} + o(n^2) \ \geq\ 4n^2 + o(n^2). \ \blacksquare \end{aligned}$$

Corollary 10 *Let* $(F_n)_{n\in\mathbb{N}}$ *be a sequence of* 4*-chromatic graphs such that* $v(F_n) = o(\ln n)$*. Then* $\hat{r}(K_{1,n}, F_n) = 4n^2 + o(n^2)$*.* $\blacksquare$

8 General Lower Bounds on $b_k(n)$

Unfortunately, despite the author's efforts, Conjecture 1 remains open for $k \geq 4$, although we are able to improve the bound of Theorem 4 by using Lemma 8.

Let $k \geq 4$ and let G be a minimum $\mathcal{B}_k(n)$-graph. Assume that n is large; in particular $e(G) < n^2 \ln n$. Let T_i's and t_i's be defined as before Lemma 8.

Define $A_1 \subset T_2$ to be an n-set incident to as many as possible edges of $G[T_2]$. By Lemma 5, $e(A_1, T_2) \geq n^2 + \frac{n^3}{2n+2t_2} + o(n^2)$. Consecutively for $i = 2, 3, \ldots, k-1$ apply Lemma 8 to $A_{i-1} \subset T_i$ to obtain an $(in + t_i)$-set $A_i \subset T_{i+1}$ satisfying (6). This gives the following lower bound on $e(G)$.

$$e(G) \geq 3n^2 + t_2 n^2 + \frac{n^3}{n + t_2} + \sum_{i=3}^{k-1} \left(in + t_i n + \frac{n((n - t_{i-1})_+)^2}{2n + 2t_i} \right) + o(n^2).$$

Thus,

$$b_k(n) \geq \binom{k}{2} n^2 + mn^2 + o(n^2), \tag{11}$$

where m is the minimum of

$$f(x_2, \ldots, x_{k-1}) = x_2 + \frac{1}{1 + x_2} + \sum_{i=3}^{k-1} \left(x_i + \frac{((1 - x_{i-1})_+)^2}{2 + 2x_i} \right).$$

over all reals $0 \leq x_2 \leq \cdots \leq x_{k-1}$.

Unfortunately, (11) does not give the sharp lower bound for $k \geq 5$. Numerical calculations show that we obtain, for example,

$$(7.477 + o(1))n^2 \ < \ b_4(n) \ \leq \ 7.5\, n^2 + 2.5\, n, \tag{12}$$

$$(11.944 + o(1))n^2 \ < \ b_5(n) \ \leq \ 12\, n^2 + 3\, n. \tag{13}$$

(We included here the upper bounds (5) for comparison.)

It seems that the lower bound (11) cannot be substantially simplified. So we leave (11) as it is, especially that we believe it can be improved.

Working harder on the case $k = 4$ we have improved the lower bound in (12) to

$$b_4(n) > (7.494 + o(1))n^2. \tag{14}$$

We give only a sketch of the proof.

Let $G \in \mathcal{B}_4(n)$ and T_i's be as usual. The graph $G[T_3]$ has the property that $\Delta(G - B) \geq 2n$ for any n-set $B \subset T_3$. One can show (similarly to the proof of Lemma 5; see also [12]) that $\Delta(G[T_3 \setminus A]) \geq 2n + \frac{2n(n-|A|)}{2n+t_3} + o(n)$ for any $A \subset T_3$ with $|A| \leq (1 - o(1))\, n$.

Start with $A = \emptyset$ and n times move to A a vertex of $G[T_3 \setminus A]$ of maximum degree. Let $f = \Delta(G[T_3 \setminus A]) - 2n$. By the way we defined A, each moved vertex was incident to least $2n + f$ edges outside A at that moment. Hence,

$$e(A, T_3) \geq o(n^2) + \sum_{i=1}^{n} \max\left(2n + f, 2n + \frac{2n(n-i)}{2n + t_3}\right). \tag{15}$$

By Lemma 5, applied to $G[T_3 \setminus A] \in \mathcal{B}_2(n, f)$, we can find an n-set $B \subset T_3 \setminus A$ with

$$e(B, T_3 \setminus A) \geq n^2 + \frac{n^3}{2n + 2f} + o(n^2). \tag{16}$$

Apply Lemma 7 to $A \cup B$ to find a set $D \subset T_3$ of size $2n + t_2$ incident to at least $e(A \cup B, T_3) + t_2 n$ edges. Now, applying Lemma 8 to D we obtain:

$$e(G) \geq e(A, T_3) + e(B, T_3 \setminus A) + t_2 n + 3n^2 + t_3 n + \frac{n((n - t_2)_+)^2}{2n + 2t_3} + o(n^2). \tag{17}$$

We believe that the inequalities $f \leq t_3$ and (15)–(17) imply (14), which involves rather messy calculations. We do not provide any further details because the last bound is not best possible anyway.

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