Operators Extending (Pseudo-)Metrics

Oleg Pikhurko*
Department of Mechanics and Mathematics
L'viv State Ivan Franko University
L'viv 290602, Ukraine

This paper appeared in the *Proccedings of All-Ukrainian Conference of Young Researchers (Mathematics)*, **N 1302**, Part I (1994) 10–16.
This version (minor corrections): December 12, 2000

Abstract

We introduce a general method of extending (pseudo-)metrics from X to FX, where F is a normal functor on the category of metrizable compacta. For many concrete instances of F, our method specializes to the known constructions.

1 Introduction

Consider the category of all compact metrizable spaces which will be referred to as $MComp$. All functors are expected to be *normal* (for the definition and properties see [2, page 165] or [3]) and to have $MComp$ as both the domain and the codomain. For a normal functor F, every space X is naturally embeddable in FX, so further in this work X is considered to be a subspace of FX.

By an *operator* $u : C(-) \to C(F(-))$ we mean a family of maps

$$(u_X : C(X) \to C(FX))_{X \in MComp},$$

where $C(X)$ denotes the set of all continuous mappings from X to $\mathbb{R}$. Considering different topologies on this set, one can speak about *operators continuous in the pointwise topology, in the uniform topology,* etc. An operator is called a *functorial operator* if for every $i : Y \to X$ the following identity holds:

$$u_Y \circ i_* = (F(i))_* \circ u_X. \tag{1}$$

Here, for $i : Y \to X$, the mapping $i_* : C(X) \to C(Y)$ corresponds ϕ to $\phi \circ i$.

For $f, g \in C(X)$ we write $f \geq g$ to denote the poinwise inequality: $f(x) \geq g(x)$ for all $x \in X$. An operator u is an *extension operator* if $u_X(\phi)|_X = \phi$;

*Current address: DPMMS, Cambridge University, Cambridge CB3 0WB, England. E-mail: `O.Pikhurko@dpmms.cam.ac.uk`

monotonous if $\phi \geq \psi$ implies $u_X(\phi) \geq u_X(\psi)$; *semiadditive* if $u_X(\phi+\psi) \leq u_X(\phi)+u_X(\psi)$; *positive* if $\phi \geq 0$ implies $u_X(\phi) \geq 0$, all X and $\phi, \psi \in C(X)$.

Here we investigate a general method for extending (pseudo-)metrics from a metrizable compact X to FX, where F is a normal functor. For many concrete instances of F, our method specializes to the known constructions.

2 Definition and properties of the new operator

Suppose that we have a normal functor F and an operator $u : C(-) \to C(F(-))$. For $a, b \in FX$, $\langle a, b\rangle$ denotes the set

$$\{c \in F(X \times X) \mid Fpr_1(c) = a,\ Fpr_2(c) = b\} = (F(pr_1), F(pr_2))^{-1}(a, b).$$

It is not empty since any normal functor is bicommutative. Also, we will use some other notation:

$$\Delta_X : X \to X \times X, \qquad \Delta(x) = (x, x); \tag{2}$$

$$\nabla_X : X \times X \to X \times X, \qquad \nabla_X(x, y) = (y, x) \tag{3}$$

If no confusion arises, we simply write Δ or ∇.

For any real-valued function p on X^2, we may define a function $\tilde{p}$ on $(FX)^2$ by the following formula:

$$\tilde{p}(a, b) = \inf\{u_{X\times X}(p)(c) \mid c \in \langle a, b\rangle\}, \quad a, b \in FX \tag{4}$$

The formula (4) gives the promised operator $\tilde{}$. Of course, to define it, one needs an operator u first, so it seems that do not gain much. But, for many functors F, there is usually a natural and obvious definition of u, while it is typically not clear how to define a (pseudo-)metric on FX should we have one on X.

Lemma 1 *If u is an extension operator, then the function $\tilde{p}$ extends p.*

Proof. The claim is obvious because, for any normal functor F and arbitrary $a, b \in X$, the set $\langle a, b\rangle \subset F(X \times X)$ consists of one point. ∎

Lemma 2 *If u is a positive, monotonous, semiadditive functorial operator, then for any pseudometric p on X the function $\tilde{p}$ is a pseudometric on FX.*

Proof. For any pair $(X \supset Y)$ and for every $\phi \in C(X)$ such that $\phi|_Y = 0$, we have $u_X(\phi)|_{FY} = 0$. This can be deduced from (1) by letting i be the identity map $Y \to X$. Here,

$$u_X(\phi)|_{FY} = (Fi)_*(u_X(\phi)) = u_Y(i_*(\phi)) = u_Y(0) = 0.$$

Now we can prove that, for any $a \in FX$, we have $\tilde{p}(a, a) = 0$. Since $p|_{\Delta(X)} = 0$ we have $u_{X\times X}(p)|_{F\Delta(X)} = 0$, and

$$0 \leq \tilde{p}(a, a) \leq u_{X\times X}(p)(F\Delta(a)) = 0.$$

The function $\tilde{p}$ is symmetric, as

$$F\nabla(\langle b,a\rangle) = \langle a,b\rangle, \quad \forall a,b \in FX$$

and

$$\begin{aligned}\tilde{p}(a,b) &= \inf(u_{X\times X}(p)(\langle a,b\rangle)) = \inf(u_{X\times X}(p)(F\nabla(\langle b,a\rangle)) \\ &= \inf(u_{X\times X}(\nabla_*(p))(\langle b,a\rangle)) = \inf(u_{X\times X}(p)(\langle b,a\rangle) = \tilde{p}(b,a).\end{aligned}$$

In this chain of equalities we used the symmetry of p (i.e. $\nabla_*(p) = p$), the functoriality of u (i.e. $u_{X\times X}(\nabla_*(p))(x) = F\nabla_* \circ u_{X\times X}(p)(x) = u_{X\times X}(p)(F\nabla(x))$) and the identity

$$F\nabla_* \circ u_{X\times X} = u_{X\times X} \circ \nabla_* : C(X\times X) \to C(F(X\times X)). \tag{5}$$

Let a, b and c be arbitrary points in FX. Choose $x_1 \in \langle a,b\rangle$ and $x_2 \in \langle b,c\rangle$, such that $\tilde{d}(a,b) = u_{X\times X}(x_1)$ and $\tilde{d}(b,c) = u_{X\times X}(d)(x_2)$. F is bicommutative so there exists $y \in F(X^3)$ such that $Fpr_{12}(y) = x_1$ and $Fpr_{23}(y) = x_2$. Let $x_3 = Fpr_{13}(y) \in \langle a,c\rangle$. Then

$$\begin{aligned}\tilde{d}(a,c) &\leq u_{X\times X}(d)(x_3) = u_{X\times X}(d)(Fpr_{13}(y)) \\ &= u_{X^3}(d\circ pr_{13})(y) \leq u_{X^3}(d\circ pr_{12} + d\circ pr_{23})(y) \\ &\leq u_{X\times X}(Fpr_{12}(y)) + u_{X\times X}(Fpr_{23}(y)) = \tilde{d}(a,b) + \tilde{d}(b,c).\end{aligned}$$

The lemma is proved. ∎

Lemma 3 *If u is continuous in the uniform topology, then so is the operator $\tilde{}$.*

Proof. For any $a,b \in FX$, we have

$$\begin{aligned}\|u_{X\times X}(d_1) - u_{X\times X}(d_2)\|_\infty &\geq u_{X\times X}(d_1)(x_2,y_2) - u_{X\times X}(d_2)(x_2,y_2) \\ \geq \tilde{d}_1(a,b) - \tilde{d}_2(a,b) &\geq u_{X\times X}(d_1)(x_1,y_1) - u_{X\times X}(d_2)(x_1,y_1) \\ &\geq -\|u_{X\times X}(d_1) - u_{X\times X}(d_2)\|_\infty,\end{aligned}$$

where $\tilde{d}_i(a,b) = u_{X\times X}(d_i)(x_i,y_i)$, $i = 1,2$. Hence

$$\|\tilde{d}_1 - \tilde{d}_2\|_\infty \leq \|u_{X\times X}(d_1) - u_{X\times X}(d_2)\|_\infty$$

and the operator $\tilde{}$ is continuous in the uniform topology. ∎

Lemma 4 *If the mapping*

$$H_X = (Fpr_1, Fpr_2) : F(X\times X) \to FX \times FX$$

is open for any $X \in MComp$, then $\tilde{d} : FX\times FX \to \mathbb{R}$ is continuous.

Proof. In fact, $\langle a,b\rangle = H_X^{-1}(a,b)$. Mapping H_X is both open and closed as $dom(H_X), codom(H_X) \in MComp$. So the mapping

$$H_X^{-1} : FX \times FX \to \exp(F(X \times X))$$

is continuous. Also, for any fixed $f \in C(X)$. the infimum map $\inf_f : \exp(X) \to \mathbb{R}$, defined by $\inf_f(A) = \inf f(A)$, is continuous.

Putting this all together we obtain the required. ∎

The direct consequence of Lemmas 1–4 is the following.

Theorem 5 *If u_X is a positive, monotone, semiadditive functorial operator extending functions from X to FX, then the operator $\tilde{}$ defined by formula* (4) *extends pseudometrics from X to FX. Moreover, if u is continuous in the uniform topology, then so is the operator $\tilde{}$; if H_X is an open mapping for all $X \in MComp$, then the pseudometric $\tilde{d}$ is continuous for every continuous pseudometric d.* ∎

A remarkable fact about the above defined opeartor $\tilde{}$ is that in many cases it coincides with the well-known constructions, as we are going to demonstrate now.

3 Case $F = \exp$

Let $F = \exp$ (the functor of all closed subsets equipped with the Vietoris topology, see [2, page 139].) We define $u : C(-) \to C(\exp(-))$ by the formula $u_X(\phi)(A) = \sup(\phi(A))$, $\phi \in C(X)$, $A \in \exp(X)$.

Theorem 6 *For every metric d on X, we have $\tilde{d} = d_H$ (Hausdorff metric).*

Proof. Let $A, B \in \exp(X)$,

$$M = d_H(A,B) = \inf\{\epsilon > 0 \mid A_\epsilon \supset B, B_\epsilon \supset A\},$$

where, for example, $A_\epsilon = \{x \in X \mid d(x,A) \le \epsilon\}$.

Then either there is $b \in B$ with $d(b,A) = M$ or there is $a \in A$ with $d(a,B) = M$. Since $pr_1(C) = A$ and $pr_2(C) = B$ for every $C \in \langle a,b\rangle$, we have $u_{X\times X}(d)(C) \ge M$, which implies $\tilde{d} \ge d_H$.

On the other hand, define

$$C = \{(a,b) \in A \times B \mid d(a,B) = d(a,b) \text{ or } d(A,b) = d(a,b)\}.$$

It is easy to prove that $C \in \langle a,b\rangle$ and $u_{X\times X}(d)(C) = M$. Thus, we obtain that $\tilde{d} = d_H$. ∎

4 Case $F = (-)^n$.

To define an operator u one has to assign a certain number, given a real-valued function ϕ on X and a sequence $x_1, \dots, x_n \in X$. It may be done in many ways but the following definitions are most interesting:

$$\begin{aligned} u_{X\times X}(\phi)(x_1, \dots, x_n) &= \left(\sum_{i=1}^{n} \phi(x_i)^p\right)^{1/p}, \quad p \geq 1; \\ u_{X\times X}(\phi)(x_1, \dots, x_n) &= \max_i(\phi(x_i)). \end{aligned}$$

The easy verification shows that corresponding operators ˜ have the following appearence:

$$\begin{aligned} \tilde{d}(x, y) &= \left(\sum_{i=1}^{n} d(x_i, y_i)^p\right)^{1/p}; \\ \tilde{d}(x, y) &= \max_i(d(x_i, y_i)). \end{aligned}$$

5 Case $F = P$

Let P denote the functor of probability measures, see [1]. The topology on the space PX can be defined by means of the metric

$$\bar{d}(\mu, \nu) = \inf\{\eta(d) \mid \eta \in P(X \times X), Ppr_1(\eta) = \mu, Ppr_2(\eta) = \nu\}, \quad \mu, \nu \in PX$$

Letting $u_X(\phi)(\mu) = \mu(\phi)$, $\mu \in PX$, $\phi \in C(X)$, one can see that the definitions of $\bar{d}$ and $\tilde{d}$ coincide.

6 Case of the free (free abelian) group functor

On the contrary to our default assumptions, here we suppose that the functor $G(-)$, the free group functor, is defined on the category of metrizable compacta with selected point. (The selected point plays the role of the identity in GX.)

The topology on the space GX may be defined in different ways. Among them are the constructions of Swierczkowski and Graev. To find distance between "words" $A, B \in GX$ one has to find all *proper representations* $A = \prod_{i=1}^{n}(a_i)^{\epsilon_i}$ and $B = \prod_{i=1}^{m}(b_i)^{\sigma_i}$, $a_i, b_i \in X$, $\epsilon_i, \sigma_i = \pm 1$, that is, representations which have the same number of letters and degrees coinciding exactly: $n = m$ and $\epsilon_i = \sigma_i$ for $1 \leq i \leq n$. Then

$$d_1(A, B) = \inf\left(\sum_{i=1}^{n} d(a_i, b_i)\right),$$

where the infimum is taken for all proper representations. This is Graev's construction. That of Swierczkowski (let us denote it by d_2) is nearly the same except we calculate the sum only for all *different* pairs (a_i, b_i). Obviously, $d_1 \geq d_2$.

It turned out that these metrics can also be represented in the form (4) for suitable u. Indeed, for $\phi \in C(X)$ and for $A = \prod_{i=1}^{n}(a_i)^{\epsilon_i} \in FX$ (written in the reduced form), let

$$u_X(\phi)(A) = \sum_i \phi(a_i),$$

but in the first case we take sum for all $i = 1, \ldots, n$ and in the second for all different a_i's. The points of the set $\langle A, B\rangle$ are in the bijective correspondence with the proper representations, which sends $C = \prod_{i=1}^{n}(c_i)^{\epsilon_i}$ (in the reduced form) to the representations $A = \prod_{i=1}^{n} pr_1(c_i)^{\epsilon_i}$ and $B = \prod_{i=1}^{n} pr_2(c_i)^{\epsilon_i}$. Since

$$u_{X\times X}(d)(C) = \sum_i d(pr_1(c), pr_2(c))$$

we get the claimed result.

The case $F = A$ (the free abelian group functor) is analogous. The interested reader should be able to transfer easily all results by himself.

7 Acknowledgements

I am very grateful to Michael Zarichnyi who suggested this construction for my Diploma Work at the L'viv State University and who kindly offered other help on this study.

References

[1] T. O. Banakh, On Extension of pseudometrics to the space of probability measures, *Notes L'viv Univ., Ser. Mech.-Math*, **30** (1988) 49–53 (in Ukrainian).

[2] V. V. Fedorchuk, V. V. Filippov, *General Topology. Principal Constructions*, Moscow Univ. Press, 1988 (in Russian).

[3] E. V. Shchepin, Functors and uncountable powers of compacta, *Uspekhi Mat. Nauk*, **36** (1981) 3–62 (in Russian). English translation: *Russ. Math. Surv.*, **36** (1981) 1-71.