Maximum Number of Colorings of $(2k, k^2)$-Graphs

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Abstract

Let $\mathcal{F}_{2k,k^2}$ consist of all simple graphs on $2k$ vertices and k^2 edges. For a simple graph G and a positive integer λ, let $P_G(\lambda)$ denote the number of proper vertex colorings of G in at most λ colors, and let $f(2k, k^2, \lambda) = \max\{P_G(\lambda) : G \in \mathcal{F}_{2k,k^2}\}$. We prove that $f(2k, k^2, 3) = P_{K_{k,k}}(3)$ and $K_{k,k}$ is the only extremal graph. We also prove that $f(2k, k^2, 4) = (6 + o(1))4^k$ as $k \to \infty$.

1 Introduction

All graphs in this paper are finite and undirected, and have neither loops nor multiple edges. For all missing definitions and facts which are mentioned but not proved, we refer the reader to Bollobás [2].

For a graph G, let $V = V(G)$ and $E = E(G)$ denote the vertex set of G and the edge set of G, respectively. Let $|A|$ denote the cardinality of a set A. Let $n = v(G) = |V(G)|$ and $m = e(G) = |E(G)|$ denote the numbers of vertices (the *order*) of G and number of edges (the *size*) of G, respectively. An edge $\{x, y\}$ of G will also be denoted by xy, or yx. For $A \subset V(G)$, let $G[A]$ denote the subgraph of G *induced by* A, which means that $V(G[A]) = A$, and $E(G[A])$ consists of all edges xy of G with both x and y in A. For two disjoint subsets $A, B \subset V(G)$, by $G[A, B]$ we denote the bipartite subgraph of G induced by A and B, which means that $V(G[A, B]) = A \cup B$, and $E(G[A, B])$ consists of all edges of G with one end-vertex in A and the other in B.

Let $\mathcal{F}_{n,m}$ consist of all (n, m)*-graphs*, that is, graphs of order n and size m. For a positive integer λ, let $[\lambda] = \{1, 2, \ldots, \lambda\}$. A function $c : V(G) \to [\lambda]$ such that $c(x) \neq c(y)$ for every edge xy of G is called a *proper vertex coloring* of G in at most λ colors, or simply a λ*-coloring* of G. The set $[\lambda]$ is often referred to as the *set of colors*. It was shown by Birkhoff [1] that the number of all λ-colorings of G is a polynomial in λ of degree n. It is called the *chromatic polynomial* of G, denoted by P_G.

The following problem was posed by Wilf (see [5] for motivation):

Problem 1 *Compute $f(n, m, \lambda) = \max\{P_G(\lambda) : G \in \mathcal{F}_{n,m}\}$, that is, the largest number of λ-colorings that an (n, m)-graph can have.*

We say that an (n, m)-graph G is (n, m, λ)*-extremal* if $P_G(\lambda) = f(n, m, \lambda)$.

For $\lambda = 2$ this problem was solved by Lazebnik in [5], where also all extremal graphs were described. For $\lambda \geq 3$ various bounds or partial exact results have been obtained by Lazebnik [5, 6, 7], Liu [8], and Byer [3]. (See Chen [4] for a minor correction in [5].)

Let $T_r(n)$ denote the Turán graph, i.e., the r-partite graph of order n with partition cardinalities as close to equal as possible. Let $t_r(n) = e(T_r(n))$. It was shown in [7] that for $(n, m) = (rk, t_r(rk)) = (rk, \binom{r}{2}k^2)$, $T_r(n)$ is the only (n, m, λ)-extremal graph for large λ. Though some lower bounds for such λ were provided, namely $m^{5/2}$ for $r = 2$ and $2\binom{m}{3}$ for $r > 2$, they are definitely far from the best ones. In fact, the first author conjectured in 1987 (unpublished) that the correct lower bound in this case is r:

Conjecture 2 *For integers $k \geq 1$, $r \geq 2$, let $n = rk$, and $m = t_r(n) = \binom{r}{2}k^2$. Then $T_r(n)$ is the only $f(n, m, \lambda)$-extremal graph for all $\lambda \geq r$.*

The validity and sharpness of this conjecture for $\lambda = r$ follow immediately from Turán's theorem [10], which implies that $T_r(n)$ is the only graph in $\mathcal{F}_{n,m}$ which can be properly colored in r, but no fewer, colors.

Given the ostensible difficulties of resolving the conjecture, one may wish to concentrate on the special case when $r = 2$, in which case $n = 2k$ and $m = t_2(2k) = k^2$. Here $K_{k,k}$ is the unique bipartite graph in the family $\mathcal{F}_{n,m}$, and there may actually exist a nice, transparent proof that $K_{k,k}$ is indeed extremal. Hopefully, the techniques developed for $(2k, k^2)$-graphs may apply to a wider range of pairs (n, m).

So far, even this case is open. As we already mentioned, it was shown in [7] that $K_{k,k}$ is the unique extremal graph when $\lambda = 2$ or $\lambda \geq k^5$. More careful estimates would probably reduce k^5, but an argument working for every $\lambda \geq 2$ seems beyond reach at the present time.

The main results of this paper are the following Theorems 3 and 4. In Theorem 3 we prove the conjecture for $r = 2$ and $\lambda = 3$:

Theorem 3 *Let k be a positive integer. Then $f(2k, k^2, 3) = 6(2^k - 1)$ and $K_{k,k}$ is the only $(2k, k^2, 3)$-extremal graph.*

We are still unable to prove Conjecture 2 for $r = 2$ and $\lambda = 4$. It is easy to see that

$$P_{K_{k,k}}(4) = 6 \cdot 4^k + 8 \cdot 3^k - 24 \cdot 2^k + 12,$$

which gives a lower bound on $f(2k, k^2, 4)$.

The best known upper bound on $f(2k, k^2, 4)$ is roughly $(3^{\sqrt{2}})^k 4^k$, which follows from a general upper bound on $f(n, m, \lambda)$, see (2.1) of [6]. The following Theorem 4 represents a substantial improvement of this by showing that the lower bound provided by graph $K_{k,k}$ is asymptotically correct. Its proof is much longer than that of Theorem 3, though, curiously, the main idea behind the proof of the latter re-emerges at some point in the proof of the former.

Theorem 4 $f(2k, k^2, 4) = (6 + o(1))\, 4^k,\ k \to \infty.$

2 Proof of Theorem 3

Let us apply induction on k. The claim is trivial for $k = 1$.

Let $k \geq 2$ and $G \in \mathcal{F}_{2k,k^2}$. If $G = K_{k,k}$, then the number of 3-colorings which use exactly two colors is $2\binom{3}{2} = 6$ and the number of 3-colorings which use all three colors is $6(2^k - 2)$. This gives

$$P_{K_{k,k}}(3) = 6 + 6(2^k - 2) = 6(2^k - 1).$$

Let $G \neq K_{k,k}$. We will show that such a graph cannot be $(2k, k^2, 3)$-extremal. First, we establish the following simple lemma. Let us call a component of G *non-trivial* if it has more than one vertex.

Lemma 5 *For arbitrary positive integers n, m, λ there is an (n, m)-graph H with $P_H(\lambda) = f(n, m, \lambda)$ such that H has at most one non-trivial component.*

Proof. Let H be an extremal graph. If H has two non-trivial components, C_1 and C_2, then let us identify (glue together) some $x_1 \in C_1$ and $x_2 \in C_2$ but add one extra isolated vertex to H. Call the obtained graph H'. Clearly, $v(H) = v(H')$, $e(H) = e(H')$, and $P_H(\lambda) = P_{H'}(\lambda)$ for all $\lambda \geq 1$. ∎

Lemma 5 allows us to assume that G has only one non-trivial component (and that still $G \neq K_{k,k}$). By Turán's Theorem, G must contain a triangle. We consider the following two cases:

Case 1 There is an edge $uv \in E(G)$ which lies in a triangle, say on $\{u, v, w\}$, such that $d(u) + d(v) \leq 2k$.

Let $G' = G - u - v$. Then $v(G') = 2k - 2 = 2(k - 1)$ and

$$e(G') \geq e(G) - 2k + 1 = (k - 1)^2.$$

Let G'' be any spanning subgraph of G' with exactly $(k - 1)^2$ edges. By induction, there are at most $6(2^{k-1} - 1)$ 3-colorings of G''. Each such coloring can be extended to $\{u, v\}$ in at most two ways because u and v have a common neighbor $w \in G'$. Thus

$$P_G(3) \leq 2 \cdot P_{G'}(3) \leq 2 \cdot P_{G''}(3) \leq 2 \cdot (6(2^{k-1} - 1)) = 6(2^k - 2) < P_{K_{k,k}}(3),$$

that is, G is not an extremal graph.

Case 2 For every $uv \in E(G)$ which is in a triangle of G, we have $d(u) + d(v) \geq 2k + 1$.

Let $A = \{x, y, z\}$ be the set of vertices of a triangle of G. Then

$$d(x) + d(y) + d(z) = \tfrac{1}{2}\big((d(x) + d(y)) + (d(y) + d(z)) + (d(z) + d(x))\big) \geq \lceil \tfrac{3}{2}(2k + 1) \rceil = 3k + 2.$$

(Here, $\lceil t \rceil$ denotes the smallest integer m for which $m \geq t$.) Let $B = V(G) \setminus A$. For $v \in B$, let $d(v) = d_G(v)$ denote the degree of v in G, and let $d_A(v)$ denote the number of neighbors of v in G which are in A. Let us assume that no vertex of B is adjacent to all vertices of A (otherwise $P_G(3) = 0$ and we are done). Thus we are able to partition B into the following three classes:

$$\begin{aligned} B_0 &= \{v \in B : d(v) = 0\}, \\ B_1 &= \{v \in B : d(v) \geq 1 \text{ and } d_A(v) \leq 1\}, \\ B_2 &= \{v \in B : d_A(v) = 2\}. \end{aligned}$$

Let $b_i = |B_i|$, $i = 0, 1, 2$. Then

$$b_1 + 2b_2 \geq e(G[A,B]) \geq (3k+2) - 6 = 3k - 4. \tag{1}$$

Substituting $b_2 = (2k-3) - b_1 - b_0$ into (1) we obtain

$$b_1 \leq k - 2 - 2b_0. \tag{2}$$

By Lemma 5, the graph $G[A \cup B_1 \cup B_2]$ is connected. Hence, there is an ordering of B_1 such that each of its vertices is adjacent to a vertex of $A \cup B_2$, or to some preceding vertex of B_1. Let us generate all 3-colorings of G by coloring the vertices of G in the following order: first we color the vertices of A, then those of B_2, then the vertices of B_1 relative to the above ordering, and finally those of B_0. This gives

$$P_G(3) \leq 3! \cdot 1^{b_2} \cdot 2^{b_1} \cdot 3^{b_0} \leq 6 \cdot 2^{k-2-2b_0} \cdot 3^{b_0} = 6 \cdot 2^{k-2} \cdot (3/4)^{b_0} < 6(2^k - 1),$$

where the second inequality follows from (2). This completes the proof of the theorem. ∎

3 Toward a proof of Theorem 4

In this section we consider 4-colorings of $(2k, k^2)$-graphs.

Let us first examine $P_{K_{k,k}}(4)$. Counting the number of 4-colorings of $K_{k,k}$ which use precisely i colors ($i = 2, 3, 4$) and then adding, we obtain:

$$\begin{aligned} P_{K_{k,k}}(4) &= 4 \cdot 3 + 24\,(2^k - 2) + \Big(8\,(3^k - 3 \cdot 2^k + 3) + 6\,(2^k - 2)^2\Big) \\ &= 6 \cdot 4^k + 8 \cdot 3^k - 24 \cdot 2^k + 12. \end{aligned}$$

Notice that $P_{K_{k,k}}(4) = 6 \cdot 4^k + O(3^k) = (6 + o(1))\,4^k$, and the leading term $6 \cdot 4^k$ appears only in the enumeration of those 4-colorings where each partition of $K_{k,k}$ gets precisely two colors.

In proving Theorem 4 we shall use the following approach:

- We begin by establishing a weaker result, namely that if a graph is "close" to $K_{k,k}$ (in some standard sense soon to be defined), then the number of its 4-colorings is at most $(6 + o(1))4^k$. In other words, we first establish Theorem 4 for these special graphs (see Theorem 6 below).
- Thus, it suffices to consider graphs which are not "close" to $K_{k,k}$. We define a *kite* as a graph F isomorphic to K_4 with one edge deleted, i.e., consisting of two triangles sharing an edge. Since $\chi(F) = 3$, the Stability Theorem of Simonovits [9] implies that, for sufficiently large k, any $(2k, k^2)$-graph G not "close" to $K_{k,k}$ contains a subgraph isomorphic to F.

- We attempt to establish our bound by induction on k. If we can remove a pair of vertices occurring in a unique triangle of a kite so that at most $2k-2 < k^2-(k-1)^2$ edges are deleted, then we do so, decreasing the number of colorings by at least a factor of 4.

  Suppose we do not have such an edge. It follows that every kite is incident to many edges. We split all colorings into two classes depending on whether or not there is a kite with all four vertices having different colors. Using the familiar argument from Theorem 3 (and some extra work) we bound the sizes of both classes.

Let $\varepsilon > 0$ be a fixed (small) number. We say that two graphs F and H with the same set of n vertices are *ε-close* if

$$|E(F) \triangle E(H)| \leq \varepsilon \binom{n}{2}.$$

(Here, $X \triangle Y$ denotes the symmetric difference of X and Y.) We now prove Theorem 4 for graphs ε-close to $K_{k,k}$.

Theorem 6 *There exist constants $\varepsilon > 0$ and k_0 such that for every $k \geq k_0$ and every $(2k, k^2)$-graph G which is ε-close to $K_{k,k}$ we have*

$$P_G(4) \leq 6 \cdot 4^k + (4-\varepsilon)^k.$$

Proof. Let $n = 2k$ and $\delta = \sqrt{\varepsilon}$. Let $V(G) = A_1 \cup A_2$ be a partition of $V(G)$ with $|A_1| = |A_2| = k$ and $e(G[A_1, A_2]) \geq (1-\varepsilon)k^2$. (Such a partition exists because G is ε-close to $K_{k,k}$.)

Let $i = 1, 2$. Call a vertex $x \in A_i$ *good* if $d_{A_{3-i}}(x) \geq (1-\delta)k$, that is, if it has many neighbors in the other part A_{3-i}. Let G_i be the set of all good vertices of A_i, $g_i = |G_i|$, $B_i = A_i \setminus G_i$ and $b_i = |B_i|$. We call vertices of B_i *bad*.

Counting the edges in $\overline{G}[A_1, A_2]$, where $\overline{G}$ denotes the complementary graph of G, we obtain

$$b_i \delta k < e(\overline{G}[A_1, A_2]) = |A_1||A_2| - e(G[A_1, A_2]) \leq k^2 - (1-\varepsilon)k^2 = \varepsilon k^2.$$

This gives $b_i < \varepsilon k/\delta = \delta k$ and thus $g_i > (1-\delta)k$ for $i = 1, 2$, that is, we have very few bad vertices in each part.

In what follows, we estimate the number of 4-colorings of G of different types. Given a coloring of G, call a color *essential* in a set $X \subset V(G)$ if more than δk vertices of X have this color.

Since ε, and so δ, can be considered small, for each coloring of G there exists at least one essential color in each G_i. Moreover, if a color is essential in G_i then every vertex of G_{3-i} *sees* it (i.e., is adjacent to a vertex of this color). In particular, no color can be essential in both G_1 and G_2.

Let us first consider the class $\mathcal{C}_1$ of colorings of G for which there is exactly one essential color in some G_i. All colorings from $\mathcal{C}_1$ can be constructed in the following manner: i) pick $i = 1, 2$ such that G_i has exactly one essential color, ii) pick a subset in G_i to receive the essential color and color it, iii) color G_{3-i} using the remaining 3 colors, iv) color the rest of G_i, v) color $B_1 \cup B_2$. Bounding from above the number of ways each of these steps can be achieved, we obtain

$$|\mathcal{C}_1| \leq 2 \cdot \binom{g_i}{(1-4\delta)k} \cdot 4 \cdot 3^{g_{3-i}} \cdot 3^{3\delta k} \cdot 4^{b_1+b_2} \leq 2 \cdot \binom{k}{4\delta k} \cdot 4 \cdot 3^{k+3\delta k} \cdot 4^{2\delta k}.$$

Since $\binom{k}{4\delta k} \leq 2^{h(\delta)\cdot k}$ with $h(\delta) \to 0$ when $\delta \to 0$, we have

$$|\mathcal{C}_1| \leq 8 \cdot \left(2^{h(\delta)} \cdot 3^{1+3\delta} \cdot 4^{2\delta}\right)^k = o((4-\varepsilon)^k) \tag{3}$$

as $k \to \infty$, provided ε (and so δ) is sufficiently small.

It remains to bound the class $\mathcal{C}_2 = \mathcal{C} \backslash \mathcal{C}_1$, which consists of those colorings from $\mathcal{C}$ in which each of G_1 and G_2 has exactly two essential colors. The colorings in this class are far more numerous than those in class $\mathcal{C}_1$, and consequently our analysis is longer. Our proof will iteratively change G; in particular, the partition $V(G) = A_1 \cup A_2$ may become unbalanced. To provide for these more general settings, we begin with a few additional definitions.

Let $\mathcal{G} = \mathcal{G}(\varepsilon, \delta, k)$ consist of all triples (G, X_1, X_2) such that

- G is a $(2k, k^2)$-graph,
- X_1 and X_2 are disjoint subsets of $V(G)$, each of size at least $(1-\delta)k$,
- there is a partition $V(G) = A_1 \cup A_2$, not necessarily balanced, such that $A_i \supset X_i$ and $e(G[A_1, A_2]) \geq (1-\varepsilon)k^2$.

For $(G, X_1, X_2) \in \mathcal{G}$, let $\mathcal{P}(G, X_1, X_2)$ consist of all 4-colorings of G such that every vertex of X_1 has either color 1 or 2 and every vertex of X_2 has either color 3 or 4. Finally, let $p(G, X_1, X_2) = |\mathcal{P}(G, X_1, X_2)|$.

Lemma 7 *For any sufficiently small $\varepsilon > 0$, there exists k_0 such that for any triple $(G, X_1, X_2) \in \mathcal{G}(\varepsilon, \delta, k)$, where $\delta = \sqrt{\varepsilon}$ and $k > k_0$, we have*

$$p(G, X_1, X_2) \leq 4^k + (4-2\varepsilon)^k. \tag{4}$$

Proof. First observe by the following argument that it suffices to prove the lemma for those triples (G, X_1, X_2) for which the bipartite graph $G[X_1, X_2]$ is complete. Indeed, suppose $a \in X_1$ is not adjacent to $b \in X_2$ in G. Choose a partition $A_1 \cup A_2$ as required in the definition of

$\mathcal{G}(\varepsilon, \delta, k)$. As $e(G) = k^2$, there are adjacent vertices c, d, both in A_1 or both in A_2. Let G' be obtained from G by removing edge cd and adding edge ab. Clearly, $\mathcal{P}(G, X_1, X_2) \subset \mathcal{P}(G', X_1, X_2)$, so $p(G, X_1, X_2) \leq p(G', X_1, X_2)$. Also $(G', X_1, X_2) \in \mathcal{G}$, as is demonstrated by the same partition $V(G') = A_1 \cup A_2$. Now repeat until all edges between X_1 and X_2 are present.

So, we can (and do) redefine $\mathcal{G}$ by requiring additionally that $G[X_1, X_2]$ be a complete bipartite graph. Then the lemma will follow by iteratively applying the following claim:

Claim 1 Let $(G, X_1, X_2) \in \mathcal{G}$. Then either

$$p(G, X_1, X_2) \leq 4^k + \frac{1}{2}(4 - 2\varepsilon)^k \tag{5}$$

or there is $(G', X_1', X_2') \in \mathcal{G}$ such that $|X_1'| + |X_2'| > |X_1| + |X_2|$ and

$$p(G', X_1', X_2') \leq p(G, X_1, X_2) + \frac{(4 - 2\varepsilon)^k}{4\delta k}. \tag{6}$$

Proof of Claim. Suppose that (5) does not hold. Of all partitions $A_1 \cup A_2 = V(G)$ with $A_i \supset X_i$ choose one which maximizes $e(G[A_1, A_2])$. By the definition of $\mathcal{G}$, $e(G[A_1, A_2]) \geq (1 - \varepsilon)k^2$.

Let $x_i = |X_i|$, $T_i = A_i \setminus X_i$, and $t_i = |T_i|$. Note that $t_1 + t_2 \neq 0$ for otherwise $G \supset K_{l,2k-l}$ for some l, and $p(G, X_1, X_2) \leq 2^{2k}$, implying (5).

Case 1 For some $i = 1, 2$, there exists $x \in T_i$ such that $d_{A_{3-i}}(x) \geq \frac{1}{3}k$.

Assume that $i = 1$. Since δ is small, this means that x has, for example, at least $\frac{1}{4}k$ neighbors in X_2. Let G' be obtained from G by adding edges adjoining x to each vertex of X_2 and removing arbitrary edges from $G[A_1]$ or $G[A_2]$. Let $X_1' = X_1 \cup \{x\}$ and $X_2' = X_2$.

The same partition $A_1 \cup A_2$ can be used to show that $(G', X_1', X_2') \in \mathcal{G}$. Every coloring from the set $\mathcal{P}(G, X_1, X_2) \setminus \mathcal{P}(G', X_1', X_2')$ has the property that the set of all neighbors (in G) of x in X_2 is monochromatic. But the number of such colorings is at most

$$2 \cdot 2^{x_1} \cdot 2^{x_2 - k/4} \cdot 4^{2\delta k} \leq \frac{(4 - 2\varepsilon)^k}{4\delta k}.$$

In this case, we have found the required triple (G', X_1', X_2').

Case 2 For each $i = 1, 2$, and every $x \in T_i$, $d_{A_{3-i}}(x) < \frac{1}{3}k$.

Then $d_G(x) \leq 2k/3$ for every $x \in T_i$. Indeed, if this were not so, then $d_{A_i}(x) > \frac{1}{3}k$ and

$$e(G[A_i \setminus \{x\}, A_{3-i} \cup \{x\}]) > e(G[A_1, A_2]),$$

contradicting our choice of A_1 and A_2.

Let $e_i = e(G[X_i])$ and $t = t_1 + t_2 \le 2\delta k$. We have

$$e_1 + e_2 \ge e(G) - \frac{2k}{3}t - x_1x_2 \ge k^2 - \frac{2k}{3}t - \left(k - \frac{t}{2}\right)^2 \ge \frac{tk}{4}. \tag{7}$$

Observe that the number of 2-colorings of $G[X_i]$ is at most $2 \cdot 2^{x_i - 2\sqrt{e_i}}$. Indeed, as was shown in [5], given the order and size of $G[X_i]$, the number of 2-colorings is maximized in an almost balanced complete bipartite graph with perhaps some isolated vertices. This bipartite graph has order at least $2\sqrt{e_i}$ and admits precisely two distinct 2-colorings. Hence,

$$p(G, X_1, X_2) \le 4 \cdot 2^{2k-t-2\sqrt{e_1}-2\sqrt{e_2}} \cdot 4^t = 2^{2k+2+t-2\sqrt{e_1}-2\sqrt{e_2}}.$$

By (7), $\sqrt{e_1} + \sqrt{e_2} \ge \sqrt{tk/4}$. Now, the function $t - 2\sqrt{tk/4} = t - \sqrt{tk}$ is decreasing on the interval $[1, \delta k]$, hence attains a maximum of $1 - \sqrt{k}$. Thus we have

$$p(G, X_1, X_2) \le 2^{2k+3-\sqrt{k}} \le 4^k,$$

which implies (5). This completes the proof of Case 2, and so of Claim 1.

Lemma 7 now follows: Beginning with any $(G, X_1, X_2) \in \mathcal{G}$ we iteratively apply Claim 1, always doing at most $2\delta k$ iterations. When the process terminates, (5) is seen to hold, which implies (4) by (6). ∎

In order to finish our proof of Theorem 6, we observe that the triple (G, G_1, G_2) belongs to $\mathcal{G}(\varepsilon, \delta, k)$ and $|\mathcal{C}_2| \le \binom{4}{2} p(G, G_1, G_2)$. By Lemma 7, we have $p(G, G_1, G_2) \le 4^k + (4 - 2\varepsilon)^k$. Finally, Theorem 6 follows by noting that $P_G(4) = |\mathcal{C}_1| + |\mathcal{C}_2|$ and using bound (3). ∎

4 Proof of Theorem 4

Choose small positive constants $\varepsilon \gg \delta \gg 1/k_0$, that is, each being sufficiently small depending on the previous ones. Assume that ε and k_0 satisfy the statement of Theorem 6. We intend to show that

$$f(2l, l^2, 4) \le (6 + \varepsilon)4^l,$$

for all $l > k_0^2$.

Our proof employs an iterative procedure. Given an arbitrary $(2l, l^2)$-graph G, we initially let $k = l$ and $G_k = G$. If our current $(2k, m)$-graph G_k with $k > k_0$ and $m \ge k^2$ satisfies

$$P_{G_k}(4) \le (6 + \varepsilon)4^k, \tag{8}$$

then we stop. Otherwise, we show that one can find a graph G_{k-1} of order $2k - 2$ and size at least $e(G_k) + 2 - 2k$ such that

$$P_{G_k}(4) \le 4 \cdot P_{G_{k-1}}(4). \tag{9}$$

In the latter case, we decrease k by 1 and repeat the step. Iterating this procedure for any given $(2l, l^2)$-graph G_l yields a sequence of graphs $G_l, G_{l-1}, \ldots$, which must terminate before reaching G_{k_0}. Indeed, we would otherwise obtain the contradiction:

$$e(G_{k_0}) \geq l^2 + (2-2l) + (2-2(l-1)) + \cdots + (2-2(k_0+1)) = k_0^2 + l - k_0 > \binom{2k_0}{2}.$$

On the other hand, if some G_k with $k > k_0$ satisfies (8), then by consecutively unfolding (9) we obtain

$$P_{G_l}(4) \leq 4^{l-k} P_{G_k}(4) \leq (6+\varepsilon)4^l,$$

which completes the proof of the theorem.

So let $k > k_0$, and let $G = G_k$ be an arbitrary graph of order $n = 2k$ and size $m \geq k^2$. By Lemma 5 we may assume that G has a unique non-trivial component. If G has a $(2k, k^2)$-subgraph which is ε-close to a $K_{k,k}$, then G has at most as many 4-colorings as this subgraph, and we are done because (8) holds by Theorem 6. So we suppose otherwise. Then, according to the Simonovits Stability Theorem, G contains a kite (that is, K_4 with one edge deleted).

Case 1 There exist a kite F in G, with $V(F) = \{a, b, c, d\}$ and $E(F) = \{ab, ac, cb, bd, cd\}$, such that $d(a) + d(b) \leq 2k - 1$.

Let $G_{k-1} = G - a - b$. Then $v(G_{k-1}) = 2k-2$ and $e(G_{k-1}) \geq e(G) + 2 - 2k$. Also, any 4-coloring of G_{k-1} extends in at most four different ways to $\{a, b\}$. Indeed, as b is adjacent to each of c and d it sees at least two distinct colors in G_{k-1}, while, after we have colored b, vertex a sees at least two colors by virtue of its adjacency to each end-vertex of $bc \in E(G)$. Thus G_{k-1} is the required graph satisfying (9).

Case 2 For every kite F in G, $d_G(x) + d_G(y) \geq 2k$ for all $x, y \in V(F)$ with $d_F(x) + d_F(y) = 5$.

We partition the set of 4-colorings of G into two classes as follows: Define $\mathcal{R}$ to consist of all 4-colorings of G for which there is a *rainbow* kite, that is, a kite in G whose vertices receive 4 distinct colors, and let $\mathcal{N}$ consist of the remaining colorings. We intend to show that, for example,

$$|\mathcal{R}| \leq 4^k, \tag{10}$$
$$|\mathcal{N}| \leq 4^k, \tag{11}$$

which gives, in fact, a stronger bound than (8).

Let us start by estimating $|\mathcal{R}|$. We fix a kite $F \subset G$ and prove that

$$|\mathcal{R}_F| \leq 4^k/(2k)^4, \tag{12}$$

where $\mathcal{R}_F$ consists of 4-colorings of G which make F rainbow. Then the desired estimate (10) will follow from the union bound since there are clearly fewer than $(2k)^4$ choices of F.

Set $A = V(F)$ and $B = V(G) \setminus A$. If some vertex of G is adjacent to everything in A, then $\mathcal{R}_F = \emptyset$ and (12) holds. So let us assume otherwise. Define $B_3 = \{x \in B : d_A(x) = 3\}$, and recall the definitions of B_0, B_1, B_2 from the proof of Theorem 3. We have $B = B_0 \cup B_1 \cup B_2 \cup B_3$. Let $b_i = |B_i|$. As we are not in Case 1, we conclude that

$$e(G[A, B]) \geq \sum_{v \in A} d(v) - 12 \geq 4k - 12.$$

Thus we have

$$\begin{aligned} b_0 + b_1 + b_2 + b_3 &= 2k - 4, &(13)\\ b_1 + 2b_2 + 3b_3 &\geq 4k - 12. &(14) \end{aligned}$$

We now construct all colorings from $\mathcal{R}_F$ by coloring the vertices of G in the following order: i) vertices of A, ii) those of B_3, iii) those of B_2, iv) those of B_1 (so that each such vertex has a previously colored neighbor), v) those of B_0. From this, we deduce:

$$|\mathcal{R}_F| \leq 4! \cdot 1^{b_3} \cdot 2^{b_2} \cdot 3^{b_1} \cdot 4^{b_0}. \tag{15}$$

In order to estimate the right-hand side of (15), consider the following chain of inequalities:

$$2b_0 + (\log_2 3)b_1 + b_2 + \frac{b_0 + b_1 + b_3}{100} \leq \frac{11b_0}{5} + \frac{8b_1}{5} + b_2 + \frac{2b_3}{5} \leq 2k - \frac{8}{5}. \tag{16}$$

The first inequality follows since for each b_i, its coefficient on the left is at most that on the right; the second inequality is obtained by subtracting (14) multiplied by $\frac{3}{5}$ from (13) multiplied by $\frac{11}{5}$. It directly follows from (15) and (16) that $|\mathcal{R}_F| = O(4^k)$, but we need a better estimate, namely (12).

Suppose that (12) is not true. It follows that $\frac{b_3+b_1+b_0}{100} = O(\log k)$; thus $b_2 = 2k - O(\log k)$ and so $e(G[B_2]) \geq (1 - \delta)k^2$.

Once we fix a rainbow coloring of F, there are only two available colors for each $x \in B_2$ by virtue of the fact that x is adjacent to two vertices of F. Partition B_2 into 6 parts $X_{ab}, X_{ac}, \ldots, X_{cd}$ depending on the adjacencies of its vertices to $V(F) = \{a, b, c, d\}$.

Suppose first that there is a part $X = X_{ij}$ such that $G[X]$ has at least δk^2 edges. By the previously mentioned result from [5], the number of 2-colorings of $G[X]$ is at most

$$2^{|X| - 2\sqrt{e(G[X])}} \leq 2^{|X|} \cdot 4^{-k\sqrt{\delta}}.$$

This allows us to improve on (15) by coloring the vertices of G in this order: A, X, $B_2 \setminus X$, and then the remaining vertices. We obtain

$$|\mathcal{R}_F| \leq 4! \cdot P_{G[X]}(2) \cdot 2^{b_2 - |X|} \cdot 4^{2k - b_2} \leq 4! \cdot 4^{-k\sqrt{\delta}} \cdot 2^{b_2} \cdot 4^{O(\log k)} \leq 4^k/(2k)^4,$$

as desired. So, let us assume that each part X_{ij} spans less than δk^2 edges.

Suppose next that for some two parts $X = X_{ij}$ and $Y = X_{ih}$, which share a common index, we have at least $4\delta k^2$ edges between them. Let

$$Z = \{z \in X : d_Y(z) \geq \delta k\}.$$

As, very roughly, $|Z| \cdot 2k + 2k \cdot \delta k \geq e(G[X, Y])$, we conclude that $|Z| \geq \delta k$. Once we have colored F rainbow, we have three available colors for $X \cup Y$. Let these colors be $1, 2, 3$ so that 1 can be used only on X and 2 only on Y. The number of colorings where all vertices of Z receive color 1 is at most $2^{|X|-\delta k+|Y|}$. On the other hand, if some vertex $z \in Z$ has color 3, then the color of any z-neighbor $y \in Y$ (at least δk vertices) must be 2 and, again, we obtain the extra factor $2^{-\delta k}$ in the upper bound. Hence, there are at most

$$2 \cdot 2^{|X|+|Y|} \cdot 2^{-\delta k} = o(2^{|X|+|Y|}/k^4)$$

ways to color $X \cup Y$ after F has been colored rainbow. This easily implies (12).

By the above, it remains to consider the case in which

$$e(G[X_{ab}, X_{cd}]) + e(G[X_{ac}, X_{bd}]) + e(G[X_{ad}, X_{cb}]) \geq (1 - (1 + 6 + 4 \cdot 12)\,\delta)k^2 = (1 - 55\,\delta)k^2,$$

that is, informally speaking, almost all edges of G connect the 'opposite' parts X_{ij}. But if three vertex-disjoint bipartite graphs on at most $2k$ vertices span at least $(1 - 55\,\delta)k^2$ edges, where $\delta \ll \varepsilon$, then two of them have order at most, say, $\frac{\varepsilon k}{8}$. The remaining bipartite graph must be $\frac{\varepsilon}{2}$-close to both $K_{k,k}$ and G. Thus G and $K_{k,k}$ are ε-close. But this contradicts the assumption on G we made immediately before Case 1. The estimate (10) has been completely proved.

So, it remains to prove (11). Recall that $\mathcal{N}$ consists of those 4-colorings of G which do not contain a rainbow kite. This means that for any kite F in G the two vertices of degree 2 in F get the same color.

Consider a graph H with $V(H) = \{a, b, c, d, e\}$ and $E(H) = \{ab, ac, cb, bd, cd, ce, de\}$, that is, H is a chain of three triangles glued along edges. Then $\chi(H) = 3$, and, invoking the Simonovits Stability Theorem again, G contains a subgraph isomorphic to H.

Take a copy of H in G on $\{a, b, c, d, e\}$. Let $A = \{b, c, d\}$. Each of the edges bc, bd, cd belongs to a copy $F \subset H$ such that the degrees in F of the end-vertices of each edge are 2 and 3. Because we are in Case 2, we conclude that the sum of the degrees in G of any two vertices of A is at least $2k$. Thus, letting $B = V(G) \setminus A$, we obtain

$$e(G[A, B]) \geq 3k - 6.$$

Assume that no vertex $x \in B$ is adjacent to all vertices from A for otherwise $A \cup \{x\}$ would always contain a rainbow copy of F. Defining B_0, B_1, B_2 and b_0, b_1, b_2 as before we obtain

$$b_0 + b_1 + b_2 = 2k - 3, \tag{17}$$

$$b_1 + 2b_2 \geq 3k - 6. \tag{18}$$

Substituting $b_2 = (2k-3) - b_1 - b_0$ into (18) we obtain that $b_1 \leq k - 2b_0$.

We have $|\mathcal{N}| \leq 4! \cdot 1^{b_2} \cdot 3^{b_1} \cdot 4^{b_0}$. Note that here we have a factor 1^{b_2} rather than 2^{b_2} as we had earlier. The reason is as follows: For $x \in B_2$, suppose without loss of generality that x is adjacent to b and c. When we assign a color to x we are not allowed to use the colors of its neighbors b and c. Of the remaining two colors, we must use the color of d because otherwise we would get a rainbow kite on $\{b, c, d, x\}$. Thus we have only one choice of color for each such x.

Using our bound on b_1, we obtain

$$|\mathcal{N}| \leq 24 \cdot 3^{k-2b_0} \cdot 4^{b_0} \leq 24 \cdot 3^k \leq 4^k.$$

This finishes Case 2 and completes the proof of Theorem 4. ∎

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