Combinatorics, Probability and Computing (2004) 00, 000–000. © 2004 Cambridge University Press
DOI: 10.1017/S0000000000000000 Printed in the United Kingdom

On triple systems with independent neighborhoods

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Let $\mathcal{H}$ be a 3-uniform hypergraph on an n-element vertex set V such that the neighborhood $N(ab) := \{z : abz \in E(\mathcal{H})\}$ is independent for every pair $a, b \in V$. If the minimum degree of $\mathcal{H}$ exceeds $(\frac{4}{9} - \frac{1}{125})\binom{n}{2}$, and $n > n_0$, then the vertices of $\mathcal{H}$ can be split into two parts so that every triple meets both parts.

This is, in fact, a Turán type result. It solves a problem of Erdős and T. Sós, and verifies a conjecture of Mubayi and Rödl that

$$\mathrm{ex}(n, \mathbb{F}_{3,2}) = \max_{\alpha} (n - \alpha)\binom{\alpha}{2} = \frac{4}{9}\binom{n}{3} + O(n^2),$$

where $\mathbb{F}_{3,2}$ is the hypergraph on the vertices a, b, c, d, e having 4 triples abc, abd, abe, and cde. Moreover $e(\mathcal{H}) = \mathrm{ex}(n, \mathbb{F}_{3,2})$ is possible only if $V(\mathcal{H})$ can be partitioned into two sets A and B so that each triple of $\mathcal{H}$ intersects A in exactly two vertices and B in one.

Dedicated to the 60th birthday of Béla Bollobás

† Research supported in part by the Hungarian National Science Foundation grants OTKA T 032452, T 037846 and by a National Science Foundation grant DMS 0140692.

‡ Research supported in part by the Hungarian National Science Foundation grants OTKA T 026069, T 038210, and T 0234702.

1. Independent neighborhoods

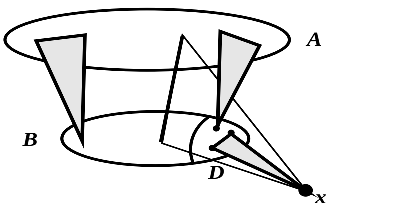
Consider a 3-uniform hypergraph $\mathcal{H}$, and let a and b be two distinct vertices. The **neighborhood**, $N(ab)$, of the pair ab consists of the vertices z for which $\{a, b, z\} \in E(\mathcal{H})$. Its size, $\mu(ab) := |N(ab)|$, is called the **codegree** of ab. We say that the neighborhoods of $\mathcal{H}$ are **independent** if $N(ab)$ contains no triple from $E(\mathcal{H})$ for any pair ab. The **link graph** G_a of a in $\mathcal{H}$ is defined as an ordinary graph on $V(\mathcal{H})$ consisting of the pairs bc for which $\{a, b, c\} \in E(\mathcal{H})$. The **degree** of a, $\deg_{\mathcal{H}}(a) = |E(G_a)|$. We use $d_{\min}(\mathcal{H})$ for the minimum degree and $\mathsf{maxcodeg}(\mathcal{H})$ or $\mu^\star$ for the $\max \mu(ab)$.

When it is possible, we shall use simplified notations, discarding parentheses and commas, e.g., we shall often abreviate a triple $\{a, b, c\}$ to abc. As usual, $G[A]$ denotes the subgraph (subhypergraph) of G induced by the vertices of A. For a graph G, $G[A, B]$ denotes the bipartite subgraph defined by the edges joining A to B.

Construction 1.1 ($(2,1)$-colorings and $\mathcal{H}_{2,1}(A, B)$). *A hypergraph $\mathcal{H}$ has a $(2,1)$-* **coloring** *if there exists a partition $V = A \cup B$ such that each triple in $E(\mathcal{H})$ meets A in exactly two vertices, and meets B in one vertex. Then the neighborhoods of $\mathcal{H}$ are independent. Denote by $\mathcal{H}_{2,1}(A, B)$ the hypergraph consisting of all $(2,1)$-colored triples.*

Let $\mathbb{H}_n$ denote the class of n-vertex $(2,1)$-colorable hypergraphs with maximum number of edges. Then $\left||A| - \frac{2}{3}n\right| < 1$, $e(\mathcal{H}) = \frac{4}{9}\binom{n}{3} + O(n^2)$ and every degree is $\frac{4}{9}\binom{n}{2} + O(n)$. For $n = 3k + 2$ the choices $|A| = 2k + 1$ and $2k + 2$ give the same edge-number, $\mathbb{H}_n$ has 2 members. $\mathbb{H}_n$ consists of a single hypergraph for $n = 3k$, $3k + 1$.

Construction 1.2 (A hypergraph without $(2,1)$-colorings). *Let $V = A \cup B \cup D \cup \{x\}$, $|V| = n$, $|A| + |B| + |D| = n - 1$, $|A| = \lfloor \frac{2}{3}n \rfloor$, $|B| > 0$ and $|D| \geq 2$. Define $E(\mathcal{H})$ by taking the edges of $\mathcal{H}_{2,1}(A, B \cup D)$ and having the link graph of x as follows:*



A hypergraph without a $(2,1)$-coloring

$E(G_x) := \{ab : a \in A,\ b \in B\} \cup \{b_1 b_2 : b_1, b_2 \in D\}$. Then the neighborhoods of $\mathcal{H}$ are independent but $\mathcal{H}$ has no $(2,1)$-coloring, although (in case of $|D| = O(1)$) each vertex has degree at least $\frac{4}{9}\binom{n}{2} - O(n)$.

The vertex x can be replaced by a set X to get a hypergraph containing all triples of types AAB, AAD, ABX and DDX.

The notion of "independent neighborhoods" can be considered as one of the hypergraph extensions of triangle-free graphs. Andrásfai, Erdős and T. Sós [1] showed that if an n-vertex graph is triangle-free and its minimum degree exceeds $\frac{2}{5}n$, then it is bipartite. The main result of this paper is the following analog of this important theorem. A hypergraph is **bipartite** if there exists a partition $A \cup B$ of the vertices for which every hyperedge meets both parts.

Theorem 1.1. *Let $\gamma \leq 1/125$ be fixed and $n > n_0$. Let $\mathcal{H}$ be an n-vertex 3-uniform* *hypergraph. Suppose that the neighborhoods of $\mathcal{H}$ are independent and*

$$d_{\min}(\mathcal{H}) > \left(\frac{4}{9} - \gamma\right)\binom{n}{2}. \tag{1.1}$$

Then $\mathcal{H}$ is bipartite.

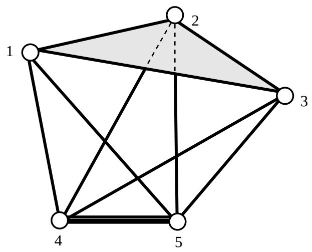
Construction 1.2 shows that the above min-degree condition does not imply that $\mathcal{H}$ must have a (2,1)-coloring. The next example shows that $\gamma \geq 5/72 = .069\ldots$ cannot be chosen in Theorem 1.1.

Construction 1.3 (A non-bipartite triple system). *Let V be an n-element set, $V = A \cup B$, with $|A| = (3/4)n + O(1)$, $|B| = n/4 + O(1)$, and let $E_0 = \{z_1, z_2, z_3\} \subset B$. Split the pairs of A into 3 almost equal parts: $\cup_{1\leq i\leq 3}\mathcal{E}_i = \{ab : a, b \in A\}$, $|\mathcal{E}_i| \sim \frac{1}{3}\binom{|A|}{2}$. Let $E(\mathcal{H})$ consist of E_0, the hyperedges of $\mathcal{H}_{2,1}(A, B \setminus E_0)$, and all triples of the form $\{abz_i : a, b \in A,\ z_i \in E_0, \text{ and } ab \in \mathcal{E}_j, 1 \leq j \neq i \leq 3\}$. Then $\mathcal{H}$ is not bipartite, its neighborhoods are independent, and $d_{\min}(\mathcal{H}) = \frac{3}{8}\binom{n}{2} + O(n)$.*

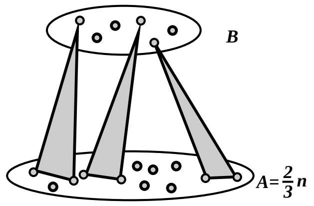
Conjecture 1.4. *Theorem 1.1 holds for every $\gamma < 5/72$.*

2. Turán's problem

Let $\mathbb{F}_{3,2}$ be the hypergraph on the vertices $1, 2, 3, 4, 5$ having 4 triples $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{2, 4, 5\}$ and $\{3, 4, 5\}$. In an ordinary graph the neighborhoods are independent if and only if it is triangle-free. Similarly, in a 3-uniform hypergraph the neighborhoods of pairs are independent if and only if it is $\mathbb{F}_{3,2}$-free.



The excluded hypergraph $\mathbb{F}_{3,2}$



The extremal hypergraph

More generally, given a 3-uniform hypergraph $\mathcal{F}$, let $\mathrm{ex}(n, \mathcal{F})$ denote the maximum possible size of a 3-uniform hypergraph of order n that does not contain any sub-hypergraph isomorphic to $\mathcal{F}$. An averaging argument shows that the ratio $\mathrm{ex}(n, \mathcal{F})/\binom{n}{3}$ is a non-increasing sequence [15], therefore $\pi(\mathcal{F}) := \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{3}$ always exists.

Erdős and T. Sós, in connection with Ramsey-Turán problems for hypergraphs, investigated $\mathrm{ex}(n, \mathbb{F}_{3,2})$. (In [7] $\mathbb{F}_{3,2}$ is denoted by $G^{(3)}(5, 4)$, and in [8], in Theorem 2, a more general class of hypergraphs is considered.) Mubayi and Rödl [20] showed that $4/9 \leq \pi(\mathbb{F}_{3,2}) \leq 1/2$ and conjectured that

$$\mathrm{ex}(n, \mathbb{F}_{3,2}) = \frac{4}{9}\binom{n}{3} + o(n^3). \tag{2.1}$$

They also conjectured that $\mathcal{H}_{2,1}(A, B)$ is the extremal hypergraph. (2.1) was verified by the present authors:

Theorem 2.1 (Turán density [13]). $\pi(\{abc, ade, bde, cde\}) = 4/9.$

Here we will give a new proof. The new method also leads to structure theorems for $\mathbb{F}_{3,2}$-free hypergraphs: to the min-degree stability theorem (Theorem 1.1) stated in the previous section, and to the following further refinements and exact solutions.

Theorem 2.2 (The finer structure). *Let $\mathcal{H}$ be an n-vertex 3-uniform hypergraph not containing $\mathbb{F}_{3,2}$ and suppose that it satisfies the min-degree condition (1.1) and that n is sufficiently large, $n > n_0(\gamma)$. Let A be a maximum independent set of vertices, denote its complement by B. Then*

(i) *B is also independent, (A, B) is a 2-coloring,*

(ii) $\left||A| - \frac{2}{3}n\right| < \sqrt{\gamma}n$, $\left||B| - \frac{1}{3}n\right| < \sqrt{\gamma}n$,

(iii) *the structure of $\mathcal{H}$ is very close to the extremal one, all but at most $\sqrt{\gamma}n^3$ hyperedges have type AAB, the rest is of type ABB.*

(iv) *Consider any two-coloring (A_1, B_1) of $\mathcal{H}$ with $|A_1| \geq |B_1|$, the existence of it was stated in Theorem 1.1. If $|A| \geq .65n$ then $A_1 \subset A$ and $|A \setminus A_1| \leq \sqrt{\gamma}n$.*

Theorem 2.3 (Extremal). *If $\mathcal{H}$ is an n-vertex 3-uniform hypergraph not containing $\mathbb{F}_{3,2}$ and n is sufficiently large, $n > n_0$, then $e(\mathcal{H}) \leq \max_\alpha (n-\alpha)\binom{\alpha}{2}$. In case of equality $\mathcal{H} \in \mathbb{H}_n$.*

We show that if $e(\mathcal{H})$ is sufficiently large, then the structure of $\mathcal{H}$ is close to that of $\mathcal{H}_{2,1}(A, B)$.

Theorem 2.4 (Global stability). *If $\mathcal{H}$ is an n-vertex 3-uniform hypergraph not containing $\mathbb{F}_{3,2}$ and $e(\mathcal{H}) > (\frac{4}{9} - c)\binom{n}{3}$, then one can delete $O(c^{1/3})n^3$ triples from $\mathcal{H}$ so that the remaining hypergraph has a $(2,1)$-coloring (A, B) with $\left||A| - \frac{2}{3}n\right| < O(c^{1/3})n$ and $\left||B| - \frac{1}{3}n\right| < O(c^{1/3})n$.*

Taking $|A| = (\frac{2}{3} - \frac{1}{2}c)n$, $|B| = (\frac{1}{3} - \frac{1}{2}c)n$, $|D| = |X| = \frac{1}{2}cn$ in Construction 1.2, one can obtain a hypergraph satisfying the constraint of Theorem 2.4, however one needs to remove at least $\Omega(c^3)n^3$ triples from it to make it $(2,1)$-colorable.

Theorem 2.5 (Codegree stability). *For every $\varepsilon > 0$ there exists a $c > 0$ such that the following holds. If $\mathcal{H}$ is an n-vertex 3-uniform hypergraph not containing an $\mathbb{F}_{3,2}$ and satisfying*

$$\left|\mathsf{maxcodeg}(\mathcal{H}) - \frac{2}{3}n\right| > \varepsilon n,$$

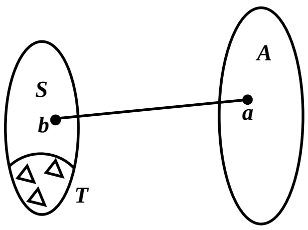
then

$$e(\mathcal{H}) \leq \left(\frac{4}{9} - c\right)\binom{n}{3}.$$

Theorem 1.1 is not the first Turán type stability result concerning hypergraphs. Based on earlier works of Füredi and Kündgen [12], de Caen and Füredi proved [5] that $\pi(\mathcal{L}_7) = \frac{3}{4}$, where $\mathcal{L}_7$ is the hypergraph formed by the 7 lines of a Fano plane. This result was sharpened in [14] to a min-degree result of type of Theorem 1.1. A slightly weaker form of this was also proved independently by Keevash and Sudakov [17]. Further, Keevash and Mubayi obtained in [16] a min-degree version of a Turán type result of Bollobás [2]. Keevash and Sudakov [18] also obtained a stability result, improving an extremal result of Frankl [9] on the hypergraph-triangle problem. In general, stability may not hold (see the constructions of W. G. Brown [4], and Kostochka [19] for $K_4^{(3)}$.) As far as we know, at present these are the only hypergraph results of *this* type.

3. Proofs

3.1. The partition



The partition. A and S are independent. The line (a,b) indicates that we shall sum multiplicities $\mu(a,b)$.

Let $\mathcal{H}$ be an $\mathbb{F}_{3,2}$-free triple system on the vertex set V, $|V| = n$, satisfying (1.1), the assumption of Theorem 1.1. Let α denote the maximum size of an independent (i.e, hyperedge-free) subset of V. Let A be a maximal independent set with $|A| = \alpha$ and denote its complement by B, $B = V(\mathcal{H}) \setminus A$. Take as many independent (i.e., pairwise disjoint) triples in B as possible, denote their number by ν. That is, $\mathcal{M} := \{E^1, E^2, \dots, E^\nu\} \subseteq E(\mathcal{H})$, $E^i \cap E^j = \emptyset$, $E^i \cap A = \emptyset$. Their vertices form T, the remaining part is S. So V is partitioned into $A \cup S \cup T$, where A and S are independent, and $|T| = 3\nu$.

Since the neighborhoods are independent, we have $\alpha \geq \mu^\star$. Clearly,

$$n \times d_{\min}(\mathcal{H}) \leq \sum_{x \in V} \deg_{\mathcal{H}}(x) = 3e(\mathcal{H}) = \sum_{a,b \in V} \mu(ab) \leq \binom{n}{2} \mu^\star.$$

Then (1.1) implies that

$$\alpha \geq \mu^\star \geq \frac{2d_{\min}}{n-1} > \left(\frac{4}{9} - \gamma\right) n > \frac{2}{5} n.$$

Define the rectangular domain

$$D_1 := \{(x,y) : 2/5 \leq x \leq 1,\ 0 \leq y \leq 1/5\}.$$

Then $\alpha \geq \frac{2}{5} n$ gives $\nu \leq \frac{1}{3}(n - \alpha) \leq \frac{1}{5} n$. Hence for all possible values of the pairs (α, ν)

$$\frac{1}{n}(\alpha, \nu) \in D_1. \tag{3.1}$$

3.2. Sketch of the proof of Theorem 1.1

We shall classify the triples according to their positions relative to A, S and T. The number of hyperedges of type XYZ is denoted by Δ_{XYZ}. Since A and S are independent sets,

$$e(\mathcal{H}) = \Delta_{AAS} + \Delta_{AAT} + \Delta_{ASS} + \Delta_{AST} + \Delta_{ATT} + \Delta_{SST} + \Delta_{STT} + \Delta_{TTT}.$$

In Section 3.3 we collect some inequalities for degree-3 polynomials in two variables we will use later. In Sections 3.4–3.9 we are going to give several upper bounds for different combinations of Δ's which hold for every $\mathbb{F}_{3,2}$-free hypergraph. In Section 3.10 we show that the min-degree condition (1.1), used for the vertices of $A \cup B$, implies bipartiteness. This will complete the proof of Theorem 1.1. Meanwhile in Section 3.6 we make a short digression to prove $\pi(\mathbb{F}_{3,2}) = 4/9$.

3.3. Three inequalities on polynomials

Define the polynomial

$$f(x,y) = \frac{3}{2}\left((1-x)x^2 - \frac{4}{27}\right) + y\left(\frac{2}{3} - 2x^2 + (1-x-y)(1-x-3y)\right).$$

Lemma 3.1. *$f(x,y) \leq 0$ for every point (x,y) in the rectangle D_1.*

Proof. Standard optimization. The determinant of the Hessian matrix of $f(x,y)$ is negative for every $(x,y) \in D_1$, (see Section 4.1). This implies that f does not have local minima or maxima inside D_1: the extrema must be on the boundary ∂D_1. Then the maximum can be found by checking the behaviour of $f(x,y)$ on the four boundary segments of ∂D_1. On each line $f(x,y)$ reduces to a third degree polynomial of one variable, whose extrema can be identified by the roots of its derivative, a second degree polynomial. The details of the calculations are postponed to the Appendix (Section 4.1). □

The very same optimization method establishes the next two lemmas, too. Details are postponed to Section 4. Define the domain $D_2 \subset D_1$ (a trapezium) by the lines $y = 0$, $y = 1/5$, $x = 2/5$, and $y = 5x - 3$ (the vertices are $(2/5, 0)$, $(2/5, 1/5)$, $(3/5, 0)$ and $(64/100, 2/5)$). Define the polynomial

$$F(x,y) := f(x,y) + \frac{1}{2}\gamma(1-3y).$$

Lemma 3.2. *$F(x,y) < -1/4000$ for every $(x,y) \in D_2$.*

Let D_3 be the open half-plane $\{(x,y) : y < 5x - 3\}$. Note that $D_2 \cap D_3 = \emptyset$. Define

$$g(x,y) := -7x^2 - xy - \frac{3}{2}y^2 + 7x + y - \frac{31}{18} + 5\gamma. \tag{3.2}$$

Lemma 3.3. *$g(x,y) < -1/2500$ for every $(x,y) \in D_3$.*

3.4. Estimating $e(\mathcal{H})$

1. Let $X \cup Y \cup Z = V(\mathcal{H})$ be a partition of the vertices. Add up the codegrees of the pairs (x,y) with $x \in X, y \in Y$. For every x, y we have $\mu(xy) \leq \mu^\star \leq \alpha$, so

$$\sum_{x \in X, y \in Y} \mu(xy) \leq |X||Y|\mu^\star \leq |X||Y|\alpha.$$

In the left hand side of this sum we count the hyperedges of type XXY and XYY exactly twice, the types XYZ occur only once. Therefore

$$2\Delta_{XXY} + 2\Delta_{XYY} + \Delta_{XYZ} \leq |X||Y|\alpha.$$

Apply this to $(X, Y, Z) = (A, S, T)$:

$$2\Delta_{AAS} + 2\Delta_{ASS} + \Delta_{AST} \le (n - \alpha - 3\nu)\alpha^2. \tag{3.3}$$

For $X = A$, $Y = B$ (and $Z = \emptyset$) we obtain

$$\Delta_{AAB} + \Delta_{ABB} \le \frac{1}{2}(n - \alpha)\alpha^2. \tag{3.4}$$

The main goals are in the rest of the proof to show that B is independent, and that $\mu^\star$ is strictly less than $|A|$ because (3.4) and $\Delta_{BBB} = 0$ imply the upper bound $e(\mathcal{H}) \le (2/27)n^3$. Furthermore $\mu^\star < \alpha$ will imply the exact upper bound for $\mathrm{ex}(n, \mathbb{F}_{3,2})$.

2. For each pair $a, b \in V$ and edge $E \in E(\mathcal{H})$ we have $|N(ab) \cap E| \le 2$, otherwise we would have an $\mathbb{F}_{3,2} \subseteq \mathcal{H}$. This implies that $|\{z \in T : abz \in E(\mathcal{H})\}| \le 2\nu$. Hence

$$\Delta_{AAT} = \sum_{\{a,b\} \subset A} |\{z \in T : abz \in E(\mathcal{H})\}| \le 2\nu \binom{\alpha}{2} \le \nu\alpha^2. \tag{3.5}$$

3. We claim that

$$\Delta_{SST} \le \frac{1}{2}\nu(n - \alpha - 3\nu)^2 + 2\nu. \tag{3.6}$$

Consider again the maximum independent family $\mathcal{M} = \{E^1, \ldots, E^\nu\}$. Define the multigraph G^i with vertex set S by $E(G^i) := \{E \setminus E^i : E \in E(\mathcal{H}),\ |E \cap E^i| = 1,\ |E \cap S| = 2\}$, where the multiplicity of ab is the number of E's with $E \setminus E^i = \{a, b\}$. Clearly, $\Delta_{SST} = \sum_{1 \le i \le \nu} e(G^i)$. We claim that

$$e(G^i) \le \frac{1}{2}|S|^2 + 2,$$

which implies (3.6). Indeed, every edge of G^i has multiplicity at most 2. If all the edges are of multiplicity 0 or 1, then $e(G^i) \le \binom{|S|}{2}$ and we are done. If, on the other hand, there exists a pair, s_1s_2 with multiplicity 2, say $x_1s_1s_2,\ x_2s_1s_2 \in E(\mathcal{H})$, $x_1x_2x_3 = E^i$, then every other edge of G^i must meet s_1s_2. If not, say $xs_3s_4 \in E(\mathcal{H})$ with $x \in E^i$, and $x \ne x_1$, then we can replace E^i in $\mathcal{M}$ by the triples $x_1s_1s_2$ and xs_3s_4, obtaining $\nu + 1$ triples in $S \cup T$, which contradicts the maximality of $\mathcal{M}$. Hence $|E(G^i)| \le \deg(s_1) + \deg(s_2) - 2 \le 2 \times 2(|S| - 1) - 2 \le |S|^2/2 + 2$. $\square$

4. We show that

$$\Delta_{STT} \le 2\nu^2(n - \alpha - 3\nu) + \nu(2n - 4). \tag{3.7}$$

Indeed, pick two hyperedges of the maximum matching $E^i, E^j \in \mathcal{M}$. Consider the 6-vertex, bipartite link graph $F_z := G_z[E^i, E^j]$ for every $z \in S$. Let $\mathcal{F}^{i,j}$ be the multigraph formed by their union, the multiplicity of the edge ab (where $a \in E^i$, $b \in E^j$) is $|\{z : z \in S, abz \in E(\mathcal{H})\}| \le |S|$. Let $\mathcal{F}_3^{i,j}$ be the multigraph defined by the edges of multiplicities at least 3. Counting the edges with multiplicities, we have

$$e(\mathcal{F}^{i,j}) = e(\mathcal{F}_3^{i,j}) + (\text{the number of edges of multiplicities} \le 2) \le e(\mathcal{F}_3^{i,j}) + 18.$$

The graph $\mathcal{F}_3^{i,j}$ contains no 3 disjoint edges, otherwise they could be extended by vertices from S to 3 disjoint triples. Then we could replace E^i, E^j in $\mathcal{M}$ by these 3 triples, obtaining a larger matching: this would contradict the maximality of ν. The König-Hall theorem (applied to this 6-vertex graph) implies that the edges of the bipartite graph $\mathcal{F}_3^{i,j}$ can be covered by two vertices, say, a and b; every edge of $\mathcal{F}_3^{i,j}$ is adjacent to either a or b. In every F_z every degree is at most 2 (because $\mathbb{F}_{3,2} \not\subset \mathcal{H}$). Hence F_z has at most 2×2 edges in $\mathcal{F}_3$ (namely, those adjacent to either a or b). Thus $e(\mathcal{F}_3) = \sum_{z\in S} e(\mathcal{F}_3 \cap F_z) \le 4|S|$.

$$\begin{aligned}\Delta_{STT} &= \sum_{i\neq j} |\{abz \in E(\mathcal{H}) : a \in E^i, b \in E^j, z \in S\}| \\ &\quad + \sum_i |\{abz \in E(\mathcal{H}) : a, b \in E^i, z \in S\}| \\ &\le \binom{\nu}{2}(4|S| + 18) + 3\nu|S| \le 2\nu^2(n-\alpha-3\nu) + \nu(2n-4).\end{aligned}$$

In the last step we used $\alpha \ge n/3$ (see (3.1)) and wrote the upper bound in a form convenient to use later.

3.5. Estimating the degrees in $A \cup S$

Add up the degrees in A and S:

$$\begin{aligned}\sum_{x\in A} \deg_{\mathcal{H}}(x) &= 2\Delta_{AAB} + \Delta_{ABB}, \\ \sum_{a\in S} \deg_{\mathcal{H}}(a) &= 2\Delta_{ASS} + 2\Delta_{SST} + \Delta_{AAS} + \Delta_{AST} + \Delta_{STT}.\end{aligned}$$

Adding them up and using $\Delta_{AAB} = \Delta_{AAS} + \Delta_{AAT}$, we obtain

$$\begin{aligned}|A \cup S| \times d_{\min}(\mathcal{H}) &\le (\Delta_{AAB} + \Delta_{ABB}) + (2\Delta_{AAS} + 2\Delta_{ASS} + \Delta_{AST}) \\ &\quad + \Delta_{AAT} + \Delta_{STT} + 2\Delta_{SST}.\end{aligned}$$

Here the right hand side can be estimated by (3.4), (3.3), (3.5), (3.7), and (3.6).

$$\begin{aligned}(n-3\nu) \times d_{\min}(\mathcal{H}) &\le \frac{1}{2}(n-\alpha)\alpha^2 + (n-\alpha-3\nu)\alpha^2 \\ &\quad + \nu\alpha^2 + 2\nu^2(n-\alpha-3\nu) + 2\nu n + \nu(n-\alpha-3\nu)^2.\end{aligned}$$

Rearranging, we get

$$\begin{aligned}(n-3\nu) \times \left(d_{\min}(\mathcal{H}) - \frac{2}{9}n^2\right) &\le \frac{3}{2}\left((n-\alpha)\alpha^2 - \frac{4}{27}n^3\right) \\ &\quad + \nu\left(\frac{2}{3}n^2 - 2\alpha^2 + (n-\alpha-\nu)(n-\alpha-3\nu)\right) + 2\nu n \\ &= n^3 f\left(\frac{\alpha}{n}, \frac{\nu}{n}\right) + 2\nu n. \qquad (3.8)\end{aligned}$$

Using that $12\nu \le 7n$ (see (3.1)) and rearranging again,

$$\begin{aligned}(n-3\nu) \times \left(d_{\min}(\mathcal{H}) - \left(\frac{4}{9} - \gamma\right)\binom{n}{2}\right) &\le \frac{3}{2}\left((n-\alpha)\alpha^2 - \frac{4}{27}n^3\right) \\ + \nu\left(\frac{2}{3}n^2 - 2\alpha^2 + (n-\alpha-\nu)(n-\alpha-3\nu)\right) &+ \frac{1}{2}\gamma n^2(n-3\nu) + n^2 \\ &= n^3 F\left(\frac{\alpha}{n}, \frac{\nu}{n}\right) + n^2. \qquad (3.9)\end{aligned}$$

3.6. Detour: The asymptotic density

Here we prove Theorem 2.1. We have to prove only an upper bound.

Lemma 3.4. *Let $\mathcal{H}$ be an arbitrary $\mathbb{F}_{3,2}$-free hypergraph. Then*

$$d_{\min}(\mathcal{H}) \leq \frac{2}{9}n^2 + n.$$

Proof. If the min-degree condition (1.1) does not hold then there is nothing to prove. Otherwise $\frac{1}{n}(\alpha, \nu) \in D_1$ by (3.1). Then Lemma 3.1 gives that $f(\alpha/n, \nu/n) \leq 0$. Since $(n - 3\nu) \geq \frac{2}{5}n$, from (3.8) we get that

$$\frac{2}{5}n \times \left(d_{\min}(\mathcal{H}) - \frac{2}{9}n^2\right) \leq 2\nu n \leq \frac{2}{5}n^2. \qquad \square$$

Proof of Theorem 2.1. Since the above lemma gives that

$$\mathrm{ex}(n, \mathbb{F}_{3,2}) \leq \mathrm{ex}(n-1, \mathbb{F}_{3,2}) + \frac{2}{9}n^2 + n,$$

it follows that $\mathrm{ex}(n, \mathbb{F}_{3,2}) \leq \sum_{i \leq n}(\frac{2}{9}i^2 + i) = \frac{2}{27}n^3 + O(n^2)$. $\square$

Recall the following lemma from [10]. If $\mathcal{F}$ is a k-uniform hypergraph such that every pair of its vertices is contained in some edge, then for every n

$$\pi(\mathcal{F})\binom{n}{k} \leq \mathrm{ex}(n, \mathcal{F}) \leq \pi(\mathcal{F})\frac{n^k}{k!}.$$

This can be applied to $\mathbb{F}_{3,2}$. So $\pi(\mathbb{F}_{3,2}) = \frac{4}{9}$ gives

Corollary 3.5. $\mathrm{ex}(n, \mathbb{F}_{3,2}) \leq \frac{2}{27}n^3$ *holds for every n.* $\square$

3.7. Estimating the degrees in T

Corollary 3.5 gives

$$\Delta_{TTT} \leq \mathrm{ex}(3\nu, \mathbb{F}_{3,2}) \leq 2\nu^3. \tag{3.10}$$

Consider the link graphs $G_z[B]$ restricted to B. A linear combination of (3.6), (3.7) and (3.10) with a little calculation give that

$$\begin{aligned}\sum_{z \in T} e(G_z[B]) &= \Delta_{SST} + 2\Delta_{STT} + 3\Delta_{TTT} \\ &\leq \frac{1}{2}\nu(n - \alpha - 3\nu)^2 + 4\nu^2(n - \alpha - 3\nu) + 4\nu n + 6\nu^3.\end{aligned}$$

Since T is the union of ν edges, in case of $\nu > 0$ we can divide the above inequality by ν. There exists an edge $E^\star \in \mathcal{M}$ meeting at most that many triples of type BBB. We obtain

$$\sum_{z \in E^\star} e(G_z[B]) \leq \frac{1}{2}(n - \alpha - \nu)^2 + 2\nu(n - \alpha - \nu) + 4n. \tag{3.11}$$

3.8. Common links in A

Consider an arbitrary $z \in B$. Since A is maximal, there exists an $abz \in E(\mathcal{H})$, $a, b \in A$. Since $\mathcal{H}$ is $\mathbb{F}_{3,2}$-free, no pair uv with $u \in A$, $v \in B$ belongs to all the three bipartite graphs $G_a[A, B]$, $G_b[A, B]$ and $G_z[A, B]$. We obtain that

$$e(G_a[A, B]) + e(G_b[A, B]) + e(G_z[A, B]) \leq 2|A||B| = 2\alpha(n - \alpha). \tag{3.12}$$

Note that the link graph G_a has no edges in A and its maximum degree is at most $\mu^\star$. Therefore

$$e(G_a[B]) + e(G_a) = \sum_{y \in B} \deg_{G_a}(y) \leq |B|\mu^\star. \tag{3.13}$$

Same holds for b,

$$e(G_b[B]) + e(G_b) \leq |B|\mu^\star. \tag{3.14}$$

Adding up (3.12), (3.13) and (3.14) and using $|B|\mu^\star = (n - \alpha)\mu^\star \leq (n - \alpha)\alpha$, we get

$$\begin{aligned} e(G_a[A, B]) + e(G_b[A, B]) + e(G_z[A, B]) + e(G_a[B]) + e(G_a) + e(G_b[B]) + e(G_b) \\ \leq 2\alpha(n - \alpha) + 2|B|\mu^\star \leq 4\alpha(n - \alpha). \end{aligned} \tag{3.15}$$

On the left hand side we can use

$$\deg(a) = e(G_a) = e(G_a[B]) + e(G_a[A, B]),$$

since $G_a[A]$ has no edge. Similarly, $\deg(b) = e(G_b[B]) + e(G_b[A, B])$. So the left hand side of (3.15) is $e(G_z[A, B]) + 2\deg(a) + 2\deg(b)$. This gives

$$e(G_z[A, B]) \leq 4\alpha(n - \alpha) - 4d_{\min}(\mathcal{H}) \quad \text{for every} \quad z \in B. \tag{3.16}$$

Lemma 3.6. *Suppose that $\mathcal{H}$ satisfies the min-degree condition* (1.1) *and $E^\star \subset B$ satisfies* (3.11). *Suppose further that $\frac{1}{n}(\alpha, \nu) \in D_3$, $n > n_0$. Then for every pair $z_1, z_2 \in E^\star$ there are at least 4α common edges of $G_{z_1}[A]$ and $G_{z_2}[A]$.*

Proof. Indeed, assuming the contrary, we would also have that $e(G_{z_1}[A]) + e(G_{z_2}[A]) < \binom{\alpha}{2} + 4\alpha$. Using this, (3.16) and (3.11) we get

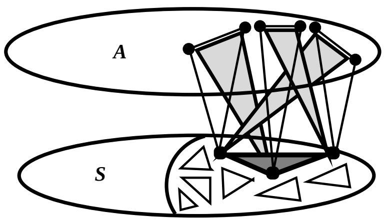
$$\begin{aligned} 2d_{\min} \quad &\leq \quad \deg_{\mathcal{H}}(z_1) + \deg_{\mathcal{H}}(z_2) = e(G_{z_1}[A]) + e(G_{z_2}[A]) \\ &\qquad + e(G_{z_1}[A, B]) + e(G_{z_2}[A, B]) + e(G_{z_1}[B]) + e(G_{z_2}[B]) \\ &\leq \quad \binom{\alpha}{2} + 4\alpha + 8\alpha(n - \alpha) - 8d_{\min} + \frac{1}{2}(n - \alpha - \nu)^2 + 2\nu(n - \alpha - \nu) + 4n. \end{aligned}$$

Rearranging, we get

$$0 \leq 5\gamma n^2 - \frac{31}{18}n^2 + 7\alpha(n - \alpha) + \nu(n - \alpha) - \frac{3}{2}\nu^2 + 10n = n^2 g\left(\frac{\alpha}{n}, \frac{\nu}{n}\right) + 10n.$$

By Lemma 3.3, $g \leq -1/2500$ on D_3. This is a contradiction for $n > 25000$. $\square$

3.9. Small substructures



The special configuration H_9

Suppose that $\mathcal{H}$ is an arbitrary $\mathbb{F}_{3,2}$-free hypergraph, A is a maximum independent set, $\alpha := |A|$, $B := V \setminus A$. Define a *well-positioned* $\mathbf{H}_9 \subseteq \mathcal{H}$ as follows. There are 6 vertices $x_1, \ldots, x_6 \in A$ and 3 vertices $\{z_1, z_2, z_3\} \subseteq B$ and the following 7 edges belong to $E(\mathcal{H})$: (z_1, z_2, z_3), $(z_i, x_{2i-1}x_{2i})$ and $(z_{i+1}, x_{2i-1}x_{2i})$ for $i = 1, 2, 3$ (where $z_4 = z_1$).

Lemma 3.7. *Suppose that $\alpha > \frac{3}{5}n$, $E^\star \subset B$ belongs to a well-positioned $\mathbf{H}_9$ and it satisfies* (3.11)*. Then $d_{\min}(\mathcal{H}) < .434\binom{n}{2}$ holds for $n > n_0$.*

Proof. We claim that

$$\frac{3}{2}\sum_{k=1,2,3} e(G_{z_k}[A,B]) + \sum_{1\le j\le 6} e(G_{x_j}[A,B]) \le 6\alpha(n-\alpha). \tag{3.17}$$

To see this, let us define the weight $w(ab)$ for $a \in A$, $b \in B$ as $w(ab) := \frac{3}{2}m_E(ab) + m_X(ab)$ where $m_E(ab)$ is the number of triples of the form abz_k and $m_X(ab)$ denotes the number of triples of the form abx_j ($1 \le k \le 3$, $1 \le j \le 6$). The left hand side of (3.17) equals to $\sum_{a\in A, b\in B} w(ab)$. We will show that $w(ab) \le 6$. This is certainly true if $m_E(ab) = 0$ since, obviously, $m_X(ab) \le 6$. For the case $m_E(ab) = 2$, i.e., if $N(ab)$ meets $E^\star$ in two vertices, observe that $\mathbb{F}_{3,2} \not\subset \mathcal{H}$ implies that $N(ab)$ contains at most one from each pair x_{2i-1}, x_{2i}, hence $m_X(ab) \le 3$. Finally, it is easy to check that $m_E(ab) = 1$ implies $m_X \le 4$, completing the proof of (3.17).

Consider the identity

$$\begin{aligned}\frac{3}{2}\sum_{k=1,2,3} \deg_{\mathcal{H}}(z_k) \quad + \quad & 2\sum_{1\le j\le 6} \deg_{\mathcal{H}}(x_j) \\ = \quad & \frac{3}{2}\sum_{k=1,2,3} e(G_{z_k}[A]) + \\ & \frac{3}{2}\sum_{k=1,2,3} e(G_{z_k}[A,B]) + \sum_{1\le j\le 6} e(G_{x_j}[A,B]) + \\ & \sum_{1\le j\le 6}\big(e(G_{x_j}) + e(G_{x_j}[B])\big) + \\ & \frac{3}{2}\sum_{k=1,2,3} e(G_{z_k}[B]).\end{aligned}$$

Estimating the right hand side, in *its* first row we can use that no pair of vertices belongs to all the three $E(G_{z_i})$'s. In the second row we use (3.17), in the third row each term is at most $\alpha(n-\alpha)$ by (3.13), and for the last row we use (3.11). We obtain

$$\frac{33}{2}d_{\min} \le \frac{3}{2}\times 2\binom{\alpha}{2} + 12\alpha(n-\alpha) + \frac{3}{2}\left(\frac{1}{2}(n-\alpha-\nu)^2 + 2\nu(n-\alpha-\nu) + 4n\right).$$

Using that $\frac{3}{2}\left(\frac{1}{2}(x-\nu)^2+2\nu(x-\nu)\right)\le x^2$ holds for all x,ν, one gets

$$\frac{33}{2}d_{\min}\le\frac{3}{2}\alpha^2+12\alpha(n-\alpha)+(n-\alpha)^2+6n=n^2+10\alpha n-\frac{19}{2}\alpha^2+6n.$$

Here $10\alpha n-(19/2)\alpha^2$ is monotone decreasing for $\alpha>\frac{10}{19}n$. Substituting $\alpha=\frac{3}{5}n$ we get the upper bound $d_{\min}(\mathcal{H})\le(179/825)n^2+6n$, implying our claim. $\square$

3.10. $\mathcal{H}$ is bipartite

Here we prove Theorem 1.1. The above lemmas hold, except Lemma 3.7, for every $\mathbb{F}_{3,2}$-free hypergraph satisfying $\alpha\ge\frac{2}{5}n$. ¿From now on we will use that $\mathcal{H}$ satisfies the min-degree condition (1.1). Consider the partition $V=A\cup B=A\cup S\cup T$ defined in Section 3.1. We will prove that B is independent. (3.1) asserts $\frac{1}{n}(\alpha,\nu)\in D_1$. We will show that $\nu=0$.

Our estimates could not handle all cases of (α,ν) simultaneously, so we divide the domain D_1 into two parts by the line $y=5x-3$ and use different estimates for each of them. We chose this line to (somewhat) maximize the value of γ achievable with the method presented, and, simultaneously, to keep the required calculations minimal. Taking $y=5x-3.0053$ one can push γ up to 0.0084.

Consider first the case when $\frac{1}{n}(\alpha,\nu)\in D_2$. Then Lemma 3.2 gives that $F(\alpha/n,\nu/n)\le -1/4000$. Since $(n-3\nu)\ge\frac{2}{5}n$, we get from (3.9) that

$$\frac{2}{5}n\times\left(d_{\min}(\mathcal{H})-\left(\frac{4}{9}-\gamma\right)\binom{n}{2}\right)\le-\frac{1}{4000}n^3+n^2.$$

Here the left hand side is nonnegative, but the right hand side is negative for $n>4000$. This contradiction implies that $\frac{1}{n}(\alpha,\nu)\in D_3\cap D_1$, in particular $\alpha>\frac{3}{5}n$.

Suppose, on the contrary, that $\nu>0$. Then there exists an edge $E^\star=(z_1,z_2,z_3)$, $E^\star\subset B$ satisfying (3.11). Lemma 3.6 implies that $E^\star$ can be extended into a well-positioned $\mathbf{H}_9$. Therefore we can apply Lemma 3.7. This leads to the contradiction $d_{\min}(\mathcal{H})<.434\binom{n}{2}$. Thus only $\nu=0$ is possible and $\mathcal{H}$ is bipartite. $\square$

3.11. The finer structure of $\mathcal{H}$

In this part we prove Theorem 2.2 (i)–(iii). Let A be a maximum independent set. The proof of Theorem 1.1 implies that $\mathcal{H}$ is bipartite, B is independent. Then all triples have types AAB or ABB. So the min-degree condition (1.1) and (3.4) give

$$\frac{1}{3}n\left(\frac{4}{9}-\gamma\right)\binom{n}{2}\le e(\mathcal{H})\le\frac{1}{2}(n-\alpha)\alpha^2.$$

Multiplying by 6 and rearranging we obtain

$$\left(\frac{2}{3}n-\alpha\right)^2(n+3\alpha)\le\left(\gamma+\left(\frac{4}{9}-\gamma\right)\frac{1}{n}\right)n^3,$$

implying Theorem 2.2 (ii) for $n>2/\gamma$:

$$\left|\alpha-\frac{2}{3}n\right|<\sqrt{\gamma}\cdot n. \tag{3.18}$$

Proof of Theorem 2.2 (iii). Use that $\mathcal{H}$ is bipartite, Corollary 3.5, (1.1) and (3.18):

$$\begin{aligned}\Delta_{BBA} &= 2(\Delta_{BBA}+\Delta_{BAA})-(\Delta_{BBA}+2\Delta_{BAA})\\ &= 2e(\mathcal{H})-\sum_{x\in A}\deg_{\mathcal{H}}(x)\le 2\times\frac{2}{27}n^3-\alpha\left(\frac{4}{9}-\gamma\right)\binom{n}{2}\\ &= \frac{2}{9}\left(\frac{2}{3}n-\alpha\right)n^2+\gamma\alpha\binom{n}{2}+\frac{2}{9}n\alpha<\sqrt{\gamma}n^3.\end{aligned}$$

□

3.12. The case $|A|\ge .65n$

In this Section we suppose that $\mathcal{H}$ is an $\mathbb{F}_{3,2}$-free hypergraph satisfying the conditions of Theorem 2.2 and also suppose that $\alpha\ge .65n$.

Claim 3.8. *If C is an independent set of vertices with $|C\cap A|\ge\frac{1}{4}n$ then $C\subseteq A$.*

Proof. Assuming the contrary, we may fix a $z\in C\cap B$. Consider G_z. It has no edges in B (by Theorem 1.1), neither in $C\cap A$ and only very few edges joining A and B (by (3.16)). We get

$$d_{\min}\le e(G_z[A,B])+e(G_z[A])\le 4\alpha(n-\alpha)-4d_{\min}+\binom{\alpha}{2}-\binom{|A\cap C|}{2}.$$

This is a contradiction for $d_{\min}>\left(\frac{4}{9}-\gamma\right)\binom{n}{2}$, $\alpha\ge .65n$ and $|A\cap C|\ge .25n$. □

Claim 3.9. $\mu^\star<\alpha$.

Proof. Consider a pair $x,y\in V$ with $\mu(xy)=\mu^\star$, and let $C:=N(xy)$. If $|C|<0.65n\le|A|$, then there is nothing to prove. Otherwise $|C|>|B|+\frac{1}{4}n$, so it has at least $n/4$ common vertices with A. Since C is independent, too, Claim 3.8 implies that $C\subseteq A$. Thus $\mu^\star=\alpha$ is only possible if $C=A$ and $x,y\in B$.

Consider $E(G_x[A])$ and $E(G_y[A])$. If they have a common triangle, say abc, then the hyperedges xab, xac, xay and bcy form an $\mathbb{F}_{3,2}$. So we get

$$e(G_x[A])+e(G_y[A])\le\binom{|A|}{2}+|E(G_x[A])\cap E(G_y[A])|\le\binom{|A|}{2}+\lfloor\frac{1}{4}\alpha^2\rfloor.$$

Since B is independent (by Theorem 2.2) (i)) we get $e(G_x)=e(G_x[A])+e(G_x[A,B])$. Use this for x and y and apply (3.16).

$$2d_{\min}\le e(G_x[A])+e(G_y[A])+e(G_x[A,B])+e(G_y[A,B])\le\frac{3}{4}\alpha^2+8\alpha(n-\alpha)-8d_{\min}.$$

For $\alpha\ge .65n$ this gives $d_{\min}\le .43\binom{n}{2}$. This contradiction shows that $\alpha=\mu^\star$ is not possible. □

Proof of Theorem 2.2 (iv). Consider an arbitrary 2-coloring (A_1,B_1) of $V(\mathcal{H})$ with A_1 and B_1 being independent. Then one of the color classes meets A in at least $\frac{1}{4}n$ vertices, say $|A_1\cap A|\ge n/4$. Claim 3.8 states that $A_1\subseteq A$. We show that

$$|A\setminus A_1|=O(\sqrt{\gamma})n.$$

Indeed, every triple meeting $A \setminus A_1$ must meet both A_1 and B. Thus for $x \in A \setminus A_1$ we have $d_{\min} \le \deg_{\mathcal{H}}(x) \le |B||A_1|$. This is equivalent to $|B|\,(|A| - |A_1|) \le |B||A| - d_{\min} = \alpha(n-\alpha) - d_{\min}$. Rearranging we get

$$|B|\,(|A| - |A_1|) \le \left(\frac{2}{3}n - \alpha\right)\left(-\frac{1}{3}n + \alpha\right) + \left(\frac{4}{9}\binom{n}{2} - d_{\min}\right) + \frac{2}{9}n.$$

Then (3.18) implies $|A| - |A_1| = O(\sqrt{\gamma})n$. $\square$

3.13. The extremal hypergraph

Here we prove Theorem 2.3. Suppose that $\mathcal{H}$ is an n-vertex $\mathbb{F}_{3,2}$-free triple system of maximum cardinality,

$$e(\mathcal{H}) \ge \max_a \frac{1}{2}(n-a)a(a-1) := e(n).$$

The degrees of any two vertices of $\mathcal{H}$ differ by at most $n-2$. Otherwise one can delete the vertex of smaller degree and duplicate the other, thus increasing the size of $\mathcal{H}$. Thus we may suppose that (for $n > n_0$) $d_{\min}(\mathcal{H}) > (\frac{4}{9} - 10^{-4})\binom{n}{2}$. Apply Theorem 1.1 and 2.2 (i). We obtain that $\mathcal{H}$ has a 2-coloring (A, B) where $|A|$ is the maximal independent set, $|A| = \alpha$. Then Theorem 2.2 (ii) implies that $\alpha > .65n$. Then Claim 3.9 gives $\mu^\star \le \alpha - 1$. We obtain the desired upper bound

$$2e(\mathcal{H}) = \sum_{\substack{a \in A\\ b \in B}} \mu(ab) \le \alpha(n-\alpha)\mu^\star \le 2(n-\alpha)\binom{\alpha}{2} \le 2e(n).$$

Moreover, equality can hold only if $\mu(ab) = \alpha - 1$ for every crossing pair $a \in A$, $b \in B$. Then $|N(ab)|$ is large, so by Claim 3.8 it is contained in A. Thus all hyperedges must be of type AAB. $\square$

3.14. Reduction to minimum degree

Here we prove Theorems 2.4 and 2.5. Both deal with hypergraphs satisfying

$$e(\mathcal{H}) > \left(\frac{4}{9} - c\right)\binom{n}{3}. \tag{3.19}$$

Lemma 3.10. *Let $\gamma = c^{2/3}$. Then for $n > n_0(c)$ one can find a subset $V_1 \subseteq V$, $|V_1| = n_1 > (1 - c^{1/3})n$, such that for $\mathcal{H}_1 := \mathcal{H}[V_1]$,*

$$\deg_{\mathcal{H}_1}(x) > \left(\frac{4}{9} - \gamma\right)\binom{n_1}{2} \tag{3.20}$$

holds for every $x \in V_1$.

Proof. Delete a vertex from V if its degree is at most $(\frac{4}{9} - \gamma)\binom{|V|}{2}$. Repeat this if we can find another vertex of small degree. This way the average degree slightly goes up, but it cannot go too high. A routine counting shows that the process stops within $c^{1/3}n$ steps. We omit the details. $\square$

To prove Theorem 2.4 consider the hypergraph $\mathcal{H}$. Using Lemma 3.10, deleting at most $c^{1/3}n$ vertices (and $O(c^{1/3}n^3)$ edges) we get the hypergraph $\mathcal{H}_1$ satisfying (3.20). Apply Theorem 1.1 to $\mathcal{H}_1$ to obtain a bipartition (A, B). Apply Theorem 2.2 (iii) to $\mathcal{H}_1$ to obtain a $(2,1)$-colorable hypergraph after deleting another $O(\sqrt{\gamma}n^3) = O(c^{1/3}n^3)$ edges. $\square$

To prove Theorem 2.5 consider a triple system $\mathcal{H}$ satisfying (3.19). Apply Lemma 3.10 to get $\mathcal{H}_1$. Then Theorem 2.2 (ii), more exactly (3.18), implies that $\alpha(\mathcal{H}_1)$ is about $\frac{2}{3}|V_1|$. Since $|V \setminus V_1|$ is small this gives an upper bound for the maximum codegree,

$$\mathsf{maxcodeg}(\mathcal{H}_1) \le \mathsf{maxcodeg}(\mathcal{H}) \le \alpha(\mathcal{H}) \le \alpha(\mathcal{H}_1) + |V \setminus V_1| \le \frac{2}{3}n + O(\sqrt{\gamma})n.$$

Finaly, since B_1 is independent, (3.4) gives

$$e(\mathcal{H}_1) \le \frac{1}{2}\alpha(\mathcal{H}_1)(|V_1| - \alpha(\mathcal{H}_1))\mu^\star(\mathcal{H}_1).$$

Then (3.19), and $|V_1| \sim n$, $\alpha(\mathcal{H}_1) \sim \frac{2}{3}n$ give the lower bound for $\mathsf{maxcodeg}(\mathcal{H}_1)$. $\square$

References

[1] Andrásfai, B., Erdős P. and T. Sós, V. (1974) On the connection between chromatic number, maximal clique and minimal degree of a graph. *Discrete Math.* **8** 205-218.

[2] Bollobás, B. (1974) Three-graphs without two triples whose symmetric difference is contained in a third. *Discrete Math.* **8** 21–24.

[3] Bollobás, B. (1978) *Extremal Graph Theory*, Academic Press, London.

[4] Brown, W. G. (1983) On an open problem of Paul Turán concerning 3-graphs. *Studies in Pure Mathematics*, 91–93, Birkhauser, Basel.

[5] de Caen, D. and Füredi, Z. (2000) The maximum size of 3-uniform hypergraphs not containing a Fano plane. *J. Combin. Theory Ser. B* **78** 274–276.

[6] Erdős, P. and Simonovits, M. (1973) On a valence problem in extremal graph theory. *Discrete Math.* **5** 323–334.

[7] Erdős, P. and T. Sós, V. (1980) Problems and results on Ramsey-Turán type theorems (preliminary report). *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing* (Humboldt State Univ., Arcata, Calif., 1979), *Congress. Numer. XXVI*, pp. 17–23, Utilitas Math., Winnipeg, Manitoba.

[8] Erdős, P. and T. Sós, V. (1982) On Ramsey-Turán type theorems for hypergraphs. *Combinatorica* **2** 289–295.

[9] Frankl, P. (1990) Asymptotic solution of a Turán-type problem. *Graphs Combin.* **6** 223–227.

[10] Frankl, P. and Füredi, Z. (1988) Extremal problems and the Lagrange function for hypergraphs. *Bulletin of the Institute of Mathematics, Academia Sinica* **16** 305–313.

[11] Füredi, Z. (1991) Turán type problems, pp. 253–300, *in:* "Surveys in Combinatorics, 1991", edited by A. D. Keedwell, Cambridge University Press.

[12] Füredi, Z. and Kündgen, A. (2002) Turán problems for integer-weighted graphs. *J. Graph Theory* **40** 195–225.

[13] Füredi, Z., Pikhurko, O. and Simonovits, M. (2003) The Turán density of the hypergraph $\{abc, ade, bde, cde\}$. *Electronic J. Combinatorics* **10** # R18, 7 pp. (electronic, `http://www.combinatorics.org/`).

[14] Füredi, Z. and Simonovits, M. Triple systems not containing a Fano configuration. *Combinatorics, Probability and Computing*, to appear.
[15] Katona, Gy., Nemetz, T. and M. Simonovits, M. (1964) A new proof of a theorem of P. Turán and some remarks on a generalization of it. (In Hungarian), *Mat. Lapok* **15** 228-238.
[16] Keevash, P. and Mubayi, D. Stability theorems for cancellative hypergraphs. submitted
[17] Keevash, P. and Sudakov, B. The exact Turán number of the Fano plane. *Combinatorica*, to appear
[18] Keevash, P. and Sudakov, B., On a hypergraph Turán problem of Frankl. *Combinatorica*, to appear
[19] Kostochka, A. V. (1982) A class of constructions for Turán's (3, 4)-problem. *Combinatorica* **2** 187–192.
[20] Mubayi, D. and Rödl, V. (2002) On the Turán number of triple systems, *J. Combin. Theory Ser. A* **100** 135-152.
[21] Simonovits, M. (1997) Paul Erdős' influence on extremal graph theory. *in* The Mathematics of Paul Erdős II, (eds. R. L. Graham and J. Nešetřil) (1997), *Algorithms and Combinatorics* **14**, Springer Verlag, 148–192.
[22] Turán, P. (1961) Research problems. *MTA Mat. Kutató Int. Közl.* **6** 417-423.

4. Appendix: Proof of the lemmas on polynomials

4.1. Proof of Lemma 3.1

First, we show that the Hessian of f is indefinite on the open halfplane $D_4 := \{(x, y) : x > 1/3\}$, so the extrema of f on $D_1 \subset D_4$ must be on the four boundary line-segments of ∂D_1. Let f_{xx}, f_{xy}, f_{yy} denote the partial derivatives, $\mathbf{J}(x, y)$ the determinant of the Hessian. Then $f_{xx} = -9x - 2y + 3$, $f_{yy} = 8x + 18y - 8$, $f_{xy} = -2x + 8y - 2$ and

$$\mathbf{J}(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 88x - 76x^2 - 146xy - 28 + 102y - 100y^2.$$

We have that $\mathbf{J}(x, y) \neq 0$ for every $x > 1/3$, because solving $\mathbf{J}(x, y) = 0$ for y, the discriminant $-2271x^2 + 1354x - 199$ is negative for $x > 1/3$. Since there is at least one point where $\mathbf{J}$ is negative, e.g., $\mathbf{J}(1, 0) = -16 < 0$, continuity implies that $\mathbf{J}$ is negative for every point of D_4.

Let us check the behaviour of $f(x, y)$ on the boundary ∂D_1. On each segment it reduces to a third degree polynomial of one variable, whose extrema can be identified by the roots of its derivative, a second degree polynomial.

(1) First check the lower horizontal boundary, $I_1 := \{(x, 0) : 2/5 \leq x \leq 1\}$. Consider $f(x, 0) = \frac{3}{2}\left((1 - x)x^2 - \frac{4}{27}\right) = -\frac{3}{2}(\frac{2}{3} - x)^2(\frac{1}{3} + x)$. So $f \leq 0$ on I_1.

(2) Next check the left, vertical boundary line $x = 2/5$. Let $\varphi_2(y) := f(2/5, y) = (225y^3 - 180y^2 + 53y)/75 - 88/1125$ and $I_2 := \{(2/5, y) : 0 \leq y \leq 1/5\}$. The derivative φ_2' has no real roots. So φ_2 is strictly increasing and takes its maximum on $[0, 1/5]$ at $y = 1/5$. Then $\varphi_2(1/5) = -2/225$ implies $f < 0$ on I_2.

(3) The third part to be checked is the boundary segment on $x = 1$. Let $\varphi_3(y) := f(1, y) = (27y^3 - 12y - 2)/9$ and $I_3 := \{(1, y) : 0 \leq y \leq 1/5\}$. We have $\varphi_3'(y) = (27y^2 - 4)/3$, its roots are $\pm\frac{2}{9}\sqrt{3} \sim \pm.3849$. So φ_3' is negative on $[0, 1/5]$ and φ_3 is decreasing and takes its maximum at $y = 0$. Then $\varphi_3(0) = -2/9$ implies $f < 0$ on I_3.

(4) Finally, check the segment on $y = 1/5$. Let $\varphi_4(x) := f(x, 1/5) = (-75x^3 + 65x^2 - 12x)/50 + 28/1125$ and $I_4 := \{(x, 1/5) : 2/5 \le x \le 1\}$. The derivative of φ_4 is $(-225x^2 + 130x - 12)/50$ and it has one root $x_2 := (13 + \sqrt{61})/45 \sim .462$ in I_4. Hence φ_4' is positive on $[2/5, x_2)$ and negative on $(x_2, 1]$, and φ_4 has its maximum at x_2. We have $\varphi_4(x_2) = (61\sqrt{61} - 665)/30375 \sim -.0062$ so $f < 0$ on I_4. □

4.2. Proof of Lemma 3.2

It is enough to prove the next two lemmas for $\gamma = 1/125$.

Since $F(x, y)$ and $f(x, y)$ differ by a linear term, their Hessians coincide. So $\mathbf{J}$ is negative on $D_2 \subset D_1$, too. So the extrema of F must be on some of the four boundary segments of ∂D_2.

(5) Consider the lower horizontal boundary, $I_5 := \{(x, 0) : 2/5 \le x \le 3/5\}$. Define $\varphi_5(x) := F(x, 0) = \frac{3}{2}\left((1 - x)x^2 - \frac{4}{27}\right) + \frac{1}{2}\gamma$. Then $\varphi_5' = \frac{3}{2}(2x - 3x^2)$ and it is positive on I_5. So φ_5 is increasing and takes its maximum at $x = \frac{3}{5}$. Then $\varphi_5(3/5) = \frac{1}{2}\gamma - 7/1125 = -1/450$, implying $F < -1/4000$ on I_5.

(6) Check F on the left, vertical boundary line $x = 2/5$. Let $\varphi_6(y) := F(2/5, y) = (6750y^3 - 5400y^2 + 1563y - 167)/2250$ and $I_6 := \{(2/5, y) : 0 \le y \le 1/5\} = I_2$. Then φ_6' has no real roots, it is positive on $[0, 1/5]$. So φ_6 takes its maximum at $y = 1/5$. We have $\varphi_6(1/5) = -41/5625 \sim -.007289$, so $F < -1/4000$ on I_6.

(7) Consider the boundary segment on the tilted line $y = 5x - 3$. Let $\varphi_7(x) := F(x, 5x - 3) = (210825x^3 - 405225x^2 + 258873x - 54982)/450$ and $I_7 := \{(x, 5x - 3) : 3/5 \le x \le 0.64\}$. The derivative of φ_7 is $(210825x^2 - 270150x + 86291)/150$, and it has one root $x_1 = (9005 - \sqrt{235358})/14055 \sim .6061$ in I_7. Then φ_7' is positive on $[3/5, x_1)$ and negative on $(x_1, 0.64]$. So φ_7 takes its maximum at x_1. We have $\varphi_7(x_1) \sim -2.58 \times 10^{-4}$, so $F < -1/4000$ on I_7.

(8) Finally, check the upper boundary on $y = 1/5$. Let $\varphi_8(x) := F(x, 1/5) = \varphi_4(x) + 1/625$ and $I_8 := \{(x, 1/5) : 2/5 \le x \le 0.64\} \subset I_4$. Since $\varphi_4 < -.0062$, on I_4 we get $\varphi_4(x) + .0016 = \varphi_8(x) < -.0046$. So $F < -1/4000$ on I_8. □

4.3. Proof of Lemma 3.3

Consider the closed half-plane $\overline{D_3} := \{(x, y) : y \le 5x - 3\}$. We claim that

$$g(x, y) = -7x^2 - xy - \frac{3}{2}y^2 + 7x + y - \frac{31}{18} + 5\gamma,$$

defined in (3.2), takes its maximum on $\overline{D_3}$ at the point $(x_0, y_0) := (20/33, 1/33)$ on the boundary line and $g(x_0, y_0) = -4/99 + 5\gamma = -1/2475 < -1/2500$. Indeed, $g(x, y) = 0$ is an ellipse lying outside D_3, its center is $(20/41, 7/41)$. g has an absolute maximum at this center, it is a concave function so its maximum on $\overline{D_3}$ should be on the boundary. Finally, $g(x, 5x - 3) = -99x^2/2 + 60x - 4091/225$ and its maximum can easily be calculated. □