Exact Computation of the Hypergraph Turán Function for Expanded Complete 2-Graphs

Oleg Pikhurko

Department of Mathematical Sciences

Carnegie Mellon University

Pittsburgh, PA 15213-3890

Web: `http://www.math.cmu.edu/~pikhurko`

June 15, 2005

Abstract

Let $l > k \geq 3$. Let the k-graph $H_l^{(k)}$ be obtained from the complete 2-graph $K_l^{(2)}$ by enlarging each edge with a new set of $k-2$ vertices. Mubayi ["A hypergraph extension of Turán's theorem", to appear in *J. Combin. Th. (B)*] computed asymptotically the Turán function $\mathrm{ex}(n, H_l^{(k)})$. Here we determine the exact value of $\mathrm{ex}(n, H_l^{(k)})$ for all sufficiently large n, settling a conjecture of Mubayi.

1 Introduction

For $k, l \geq 2$ let $\mathcal{K}_l^{(k)}$ be the family of all k-graphs F with at most $\binom{l}{2}$ edges such that for some l-set L (called the *core*) every pair $x, y \in L$ is covered by an edge of F. Let the k-graph $H_l^{(k)} \in \mathcal{K}_l^{(k)}$ be obtained from the complete 2-graph $K_l^{(2)}$ by enlarging each edge with a new set of $k-2$ vertices.

These k-graphs were recently studied by Mubayi [13] in the context of the *Turán* ex-*function* which is defined as follows. Let $\mathcal{F}$ be a family of k-graphs. We say that a k-graph G is *$\mathcal{F}$-free* if no $F \in \mathcal{F}$ is a subgraph of G. (Here, when we talk about subgraphs, we do not require them to be induced.) Now, the *Turán function* $\mathrm{ex}(n, \mathcal{F})$ is the maximum size of an $\mathcal{F}$-free k-graph G on n vertices. Also, let

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{k}}.$$

(The limit is known to exist, see Katona, Nemetz, and Simonovits [9].)

To obtain the k-graph $T^{(k)}(n, l)$, $l \geq k$, partition $[n] = \{1, \ldots, n\}$ into l almost equal parts (that is, of sizes $\lfloor \frac{n}{l} \rfloor$ and $\lceil \frac{n}{l} \rceil$) and take those edges which intersect every part in at most one vertex. Let us, for notational convenience, identify k-graphs with their edge sets and, for a k-graph F, write $\mathrm{ex}(n, F)$ for $\mathrm{ex}(n, \{F\})$, etc.

Mubayi [13, Theorem 1] proved the following result.

Theorem 1 (Mubayi) *Let $n \geq l \geq k \geq 3$. Then $\mathrm{ex}(n, \mathcal{K}_{l+1}^{(k)}) = |T^{(k)}(n, l)|$, and $T^{(k)}(n, l)$ is the unique maximum $\mathcal{K}_{l+1}^{(k)}$-free k-graph of order n.* ∎

It follows from Theorem 1 and the super-saturation technique of Erdős and Simonovits [3] that $\pi(H_l^{(k)}) = \pi(\mathcal{K}_l^{(k)})$, see [13, Theorem 2]. This gave us the first example of a non-degenerate k-graph with known Turán's density for every k. (Previously, Frankl [5] did this for all even k.) Settling a conjecture posed in [13], we prove that the Turán functions of $H_{l+1}^{(k)}$ and $\mathcal{K}_{l+1}^{(k)}$ coincide for all large n.

Theorem 2 *For any $l \geq k \geq 3$ there is $n_0(l, k)$ such that for any $n \geq n_0(l, k)$ we have $\mathrm{ex}(n, H_{l+1}^{(k)}) = |T^{(k)}(n, l)|$, and $T^{(k)}(n, l)$ is the unique maximum $H_{l+1}^{(k)}$-free k-graph of order n.* ∎

Remark. Theorem 2 is true for $k = 2$ by the Turán theorem [21]. If $k \geq 3$ and $3 \leq l < k$, then Theorem 2 is false: $\mathrm{ex}(n, \mathcal{K}_{l+1}^{(k)}) = 0$ while $\mathrm{ex}(n, H_{l+1}^{(k)}) > 0$.

Remark. We do not compute an explicit upper bound on $n_0(l, k)$ as this would considerably lengthen the paper. (We would have to reproduce some proofs from [13] in order to calculate an explicit dependence between the constants there.)

2 Stability of $H_l^{(k)}$

Two k-graphs F and G of the same order are *m-close* if we can add or remove at most m edges from the first graph and make it isomorphic to the second; in other words, for some bijection $\sigma : V(F) \to V(G)$ the symmetric difference between $\sigma(F) = \{\sigma(D) : D \in F\}$ and G has at most m edges.

Mubayi [13, Theorem 5] proved that $\mathcal{K}_l^{(k)}$ is *stable*, meaning for the purpose of this article that for any $\varepsilon > 0$ there are $\delta > 0$ and n_0 such that any $\mathcal{K}_l^{(k)}$-free k-graph G of order $n \geq n_0$ and size at least $(\pi(\mathcal{K}_l^{(k)}) - \delta)\binom{n}{k}$ is $\varepsilon\binom{n}{k}$-close to $T^{(k)}(n, l-1)$. Here we prove the same statement for the single forbidden graph $H_l^{(k)}$, which we will need in the proof of Theorem 2.

Lemma 3 *For any $l > k \geq 3$ the k-graph $H_l^{(k)}$ is stable, that is, for any $\varepsilon > 0$ there are $\delta = \delta(k, l, \varepsilon) > 0$ and $n_0 = n_0(k, l, \varepsilon)$ such that any $H_l^{(k)}$-free k-graph G of order $n \geq n_0$ and size at least $(\pi(H_l^{(k)}) - \delta)\binom{n}{k}$ is $\varepsilon\binom{n}{k}$-close to $T^{(k)}(n, l-1)$.*

Proof. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ which establishes the stability of $\mathcal{K}_l^{(k)}$ with respect to $\frac{\varepsilon}{2}$. Assume that $\delta \leq \varepsilon$. Let n be large and G be an $H_l^{(k)}$-free k-graph on $[n]$ of size at least $(\pi(H_l^{(k)}) - \frac{\delta}{2})\binom{n}{k}$.

Let us call a pair $\{x, y\}$ of vertices *sparse* if it is covered by at most

$$m = \left(l + (k-2)\binom{l}{2}\right)\binom{n}{k-3}$$

edges of G. Let G' be obtained from G by removing all edges containing sparse pairs, at most $\binom{n}{2} \times m < \frac{\delta}{2}\binom{n}{k}$ edges.

Let us show that the k-graph G' is $\mathcal{K}_l^{(k)}$-free. Suppose on the contrary that every pair from some l-set L is covered by an edge of G'. It follows that every pair $\{x, y\} \subset L$ is not sparse with respect to G, that is, G has more than m edges containing $\{x, y\}$. This means that if we have a partial embedding of $H_l^{(k)}$ into G with core L, then we can always find a G-edge $D \ni x, y$ such that $D \setminus \{x, y\}$ is disjoint from the rest of the embedding. Thus G has an $H_l^{(k)}$-subgraph with the core L, a contradiction.

We have $|G'| \geq (\pi(H_l^{(k)}) - \delta)\binom{n}{k}$. By the stability of $\mathcal{K}_l^{(k)}$, G' is $\frac{\varepsilon}{2}\binom{n}{k}$-close to $T^{(k)}(n, l-1)$. The triangle inequality implies that G is $(\frac{\delta}{2} + \frac{\varepsilon}{2})\binom{n}{k}$-close to $T^{(k)}(n, l-1)$. As $\delta \leq \varepsilon$, this finishes the proof of the lemma. ∎

3 Exactness

Proof of Theorem 2. Let us choose, in this order, positive constants $c_1, \ldots, c_5$, each being sufficiently small depending on the previous constants. Finally, let n_0 be sufficiently large. In fact, we can take some simple explicit functions of k, l for $c_1, \ldots, c_5$. However, n_0 should also be at least as large as the function $n_0(k, l+1, c_5)$ given by Lemma 3.

Let G be a maximum $H_{l+1}^{(k)}$-free graph on $[n]$ with $n \geq n_0$. We have

$$|G| \geq |T^{(k)}(n, l)| \geq \pi(H_{l+1}^{(k)})\binom{n}{k}, \tag{1}$$

where the first inequality follows from the fact that $T^{(k)}(n, l)$ is $H_{l+1}^{(k)}$-free while the second inequality can be shown directly.

Let $V_1 \cup \ldots \cup V_l$ be a partition of $[n]$ such that

$$f = \sum_{D \in G} \Big|\{i \in [l] : D \cap V_i \neq \emptyset\}\Big|$$

is maximum possible. Let T be the complete l-partite k-graph on $V_1 \cup \ldots \cup V_l$. Clearly, $f \geq k\,|T \cap G|$. As n is sufficiently large, Lemma 3 implies that G is $c_5\binom{n}{k}$-close to $T^{(k)}(n, l)$. (The value of $\delta > 0$ returned by Lemma 3 is not significant here because of the lower bound (1) on the size of G.) The choice of T implies that $f \geq k(|G| - c_5\binom{n}{k})$. On the other hand, $f \leq k|G| - |G \setminus T|$. It follows that

$$|G \setminus T| \leq c_5 k\binom{n}{k}. \tag{2}$$

Thus we have $|T| \geq |T^{(k)}(n, l)| - c_5 k\binom{n}{k}$. This bound on $|T|$ can be easily shown to imply (or, alternatively, see Claim 1 in [13, Proof of Theorem 5]) that for each $i \in [l]$ we have, for example,

$$|V_i| \geq \frac{n}{2l}. \tag{3}$$

Let us call the edges in $T \setminus G$ *missing* and the edges in $G \setminus T$ *bad*. As $|T| \le |T^{(k)}(n,l)|$ with equality if and only if T is isomorphic to $T^{(k)}(n,l)$, see [13, Equation (1)], the number of bad edges is at least the number of missing edges.

If $G \subset T$, then we are done, for example, by Theorem 1. Thus, let us assume that B is non-empty, where the 2-graph B consists of all *bad* pairs, that is, pairs of vertices which come from the same part V_i and are covered by an edge of G.

For vertices x, y coming from two different parts V_i, call the pair $\{x, y\}$ *sparse* if G has at most

$$m = \left(\binom{l+1}{2}(k-2) + l + 1\right)\binom{n}{k-3}$$

edges containing both x and y; otherwise $\{x, y\}$ is called *dense*.

Note that there are less than $c_4 n^2$ sparse pairs for otherwise we get a contradiction to (2): each sparse pair generates at least

$$\left(\frac{n}{2l}\right)^{k-2} - m \ge \frac{1}{2}\left(\frac{n}{2l}\right)^{k-2} \tag{4}$$

missing edges by (3) while each missing edge contains at most $\binom{k}{2}$ sparse pairs.

Take any bad pair $\{x_0, x_1\}$, where, for example, $x_0, x_1 \in V_1$ are covered by $D \in G$. The number of vertices in $H_{l+1}^{(k)}$ is $\binom{l+1}{2}(k-2) + l + 1$; consecutively, if we have a partial embedding of $H_{l+1}^{(k)}$ into G such that a pair of vertices x, y from the core is dense, then we can find a G-edge containing both x, y and disjoint from the rest of the embedding. It follows that for any choice of $(x_2, \ldots, x_l)$, where $x_i \in V_i \setminus D$ for $2 \le i \le l$, at least one pair $\{x_i, x_j\}$ with $\{i, j\} \ne \{0, 1\}$ is sparse. Since x_0 and x_1 are fixed, each such sparse pair $\{x_i, x_j\}$ is counted, very roughly, at most n^{l-3} times if $\{x_i, x_j\} \cap \{x_0, x_1\} = \emptyset$, and at most n^{l-2} times if $\{x_i, x_j\} \cap \{x_0, x_1\} \ne \emptyset$.

Since we have at most $c_4 n^2$ sparse pairs, the number of times the former alternative occurs is at most

$$c_4 n^2 \times n^{l-3} \le \frac{1}{2}\left(\frac{n}{2l} - k\right)^{l-1}.$$

That is, by (3), for at least half of the choices of $(x_2, \ldots, x_l)$, the obtained sparse pair intersects $\{x_0, x_1\}$. Let A consist of those $z \in V(G)$ which are incident to at least $c_1 n$ sparse pairs. Since $\frac{1}{4}(\frac{n}{2l} - k)^{l-1}/n^{l-2} \ge c_1 n$, at least one of x_0 and x_1 belongs to A. Thus, in summary, we have proved that every bad pair intersects A.

Considering the sparse pairs, we obtain by (4) at least

$$\frac{|A| \times c_1 n}{2} \times \frac{1}{2} \left(\frac{n}{2l}\right)^{k-2} \times \binom{k}{2}^{-1} \geq |A| \times c_2 n^{k-1}.$$

missing edges and, consecutively, at least $|A| \times c_2 n^{k-1}$ bad edges. Let $\mathcal{B}$ consist of the pairs $(D, \{x, y\})$, where $\{x, y\} \in B$, $D \in G$ and $x, y \in D$. (Thus D is a bad edge.) As each bad edge contains at least one bad pair, we conclude that $|\mathcal{B}| \geq |A| \times c_2 n^{k-1}$. For any $(D, \{x, y\}) \in \mathcal{B}$, we have $\{x, y\} \cap A \neq \emptyset$. Hence, some vertex $x \in A$, say $x \in V_1$, belongs to at least

$$\frac{|\mathcal{B}|}{k\,|A|} \geq \frac{c_2}{k} n^{k-1} \tag{5}$$

bad edges, each intersecting V_1 in another vertex y.

Let $Y \subset V_1$ be the neighborhood of x in the 2-graph B. We have

$$|Y| \geq \frac{c_2}{k} n^{k-1} \times \binom{n}{k-2}^{-1} \geq c_3 n.$$

For $j \in [2, l]$ let Z_j consist of those $z \in V_j$ for which $\{x, z\}$ is dense.

Suppose first that $|Z_j| \geq c_3 n$ for each $j \in [2, l]$. In this case we do the following. For every $y \in Y$, fix some $D_y \in G$ containing both x and y. Consider an $(l+1)$-tuple $L = (x, y, z_2, z_3, \ldots, z_l)$, where $y \in Y$ and $z_j \in Z_j \setminus D_y$ are arbitrary. We can find a partial embedding of $H_{l+1}^{(k)}$ with core L such that every pair containing x is covered: the pair $\{x, y\}$ is covered by D_y while each pair $\{x, z_i\}$ is dense. Since G is $H_{l+1}^{(k)}$-free, at least one pair from the set $\{y, z_2, \ldots, z_l\}$ is sparse. Since there are at least $(c_3 n - k)^l$ choices of L (note that x is fixed), this gives us at least $(c_3 n - k)^l / n^{l-2} > c_4 n^2$ sparse pairs, which is a contradiction as we already know.

Hence, assume that, for example, $|Z_2| < c_3 n$. This means that all but at most $c_3 n$ pairs $\{x, z\}$ with $z \in V_2$ are sparse, that is, there are at most

$$c_3 n \times \binom{n}{k-2} + n \times m \leq c_3 n^{k-1} \tag{6}$$

G-edges containing x and intersecting V_2. Let us contemplate moving x from V_1 to V_2. Some edges of G may decrease their contribution to f by 1. But each such edge must contain x and intersect V_2 so the corresponding total decrease is at most $c_3 n^{k-1}$ by (6). On the other hand, the number of edges of G containing x, intersecting $V_1 \setminus \{x\}$, and disjoint from V_2 is at least $\frac{c_2}{k}n^{k-1} - c_3 n^{k-1}$ by (5) and (6). As c_3 is much smaller than c_2, we strictly increase f by moving x from V_1 to V_2, a contradiction to the choice of the parts V_i. The theorem is proved. ▮

4 Concluding Remarks

Lemma 3 also follows from the following more general Lemma 4. In order to state the latter result, we need some further definitions.

Let us call a family $\mathcal{F}$ of k-graphs *s-stable* if for any $\varepsilon > 0$ there are $\delta > 0$ and n_0 such that for arbitrary $\mathcal{F}$-free k-graphs $G_1, \dots, G_{s+1}$ of the same order $n \geq n_0$, each of size at least $(\pi(\mathcal{F}) - \delta)\binom{n}{k}$, some two are $\varepsilon\binom{n}{k}$-close. Please note that if $\mathcal{F}$ is s-stable for some s then it is also t-stable for any $t > s$. Lemma 3 implies that $H_l^{(k)}$ is 1-stable. Let $F[t]$ denote the *t-blowup* of a k-graph F, where each vertex x is replaced by t new vertices and each edge is replaced by the corresponding complete k-partite k-graph. Clearly, $|F[t]| = t^k\,|F|$.

Lemma 4 *Let $t \in \mathbb{N}$. Let $\mathcal{F}$ be a finite family of k-graphs which is s-stable. Let $\mathcal{H}$ be another (possibly infinite) k-graph family such that for each $F \in \mathcal{F}$ there is $H \in \mathcal{H}$ such that $H \subset F[t]$. If $\pi(\mathcal{H}) \geq \pi(\mathcal{F})$, then $\pi(\mathcal{H}) = \pi(\mathcal{F})$ and $\mathcal{H}$ is s-stable.*

Proof. Our proof uses the following theorem of Rödl and Skokan [18, Theorem 7.1] which in turn relies on the Hypergraph Regularity Lemma of Rödl and Skokan [19] and the Counting Lemma of Nagle, Rödl, and Schacht [16] (see also Gowers [8]).

Theorem 5 (Rödl and Skokan) *For all integers $l > k \geq 2$ and a real $\varepsilon > 0$ there exist $\mu = \mu(k, l, \varepsilon) > 0$ and $n_1 = n_1(k, l, \varepsilon) \in \mathbb{N}$ such that the following statement holds.*

Given a k-graph F with $v \leq l$ vertices, suppose that a k-graph G with $n > n_1$ vertices contains at most μn^v copies of F as a subgraph. Then one can delete at most $\varepsilon\binom{n}{k}$ edges of G to make it F-free. ▮

Let $\varepsilon > 0$ be arbitrary. Let $\delta > 0$ and n_0 be constants satisfying the s-stability assumptions for $\mathcal{F}$ and $\frac{\varepsilon}{3}$. Assume that $\delta \leq \varepsilon$. Let l be the maximum order of a k-graph in $\mathcal{F}$ and $m = |\mathcal{F}|$. Let $\mu = \mu(k, l, \frac{\delta}{3m})$ and $n_1 = n_1(k, l, \frac{\delta}{3m})$ be given by Theorem 5. Also, assume that n_2 is so large that for every $F \in \mathcal{F}$ any $F[t]$-free k-graph of order $n \geq n_2$ contains at most $\mu n^{v(F)}$ copies of F, where $v(F)$ denotes the number of vertices in F. Such n_2 exists because any $F[t]$-free k-graph G of order n has at most $o(n^{v(F)})$ copies of F, which follows from a theorem of Erdős [4]. Let $n_3 = \max(n_0, n_1, n_2)$.

Let $n \geq n_3$ and let $G_1, \ldots, G_{s+1}$ be arbitrary $\mathcal{H}$-free k-graphs each having n vertices and at least $(\pi(\mathcal{F}) - \frac{\delta}{2})\binom{n}{k}$ edges. By Theorem 5, for each $F \in \mathcal{F}$ each G_i can be made F-free by removing at most $\frac{\delta}{3m}\binom{n}{k}$ edges. Hence, we can transform G_i into an $\mathcal{F}$-free k-graph $G_i' \subset G_i$ by removing at most $|\mathcal{F}| \frac{\delta}{3m}\binom{n}{k} \leq \frac{\delta}{3}\binom{n}{k}$ edges.

We conclude that $\pi(\mathcal{F}) \geq \pi(\mathcal{H}) - \frac{\varepsilon}{3}$. As $\varepsilon > 0$ was arbitrary, we have $\pi(\mathcal{F}) = \pi(\mathcal{H})$. Thus the density of each G_i' is at least $\pi(\mathcal{H}) - \frac{\delta}{2} - \frac{\delta}{3} > \pi(\mathcal{F}) - \delta$. By the s-stability of $\mathcal{F}$, some two of these graphs, for example, G_i' and G_j', are $\frac{\varepsilon}{3}\binom{n}{2}$-close. It follows that G_i and G_j are $\varepsilon\binom{n}{k}$-close. Thus the constants $\frac{\delta}{2}$ and n_3 demonstrate the s-stability of $\mathcal{H}$, proving Lemma 4. ∎

The line of argument we used in this article might be useful for computing the exact value of $\mathrm{ex}(n, F)$ for other forbidden k-graphs F. The approach in general could be the following.

1. Find a suitable k-graph family $\mathcal{F} \ni F$ for which we can compute $\pi(\mathcal{F})$ and prove the stability of $\mathcal{F}$.

2. Deduce from Lemma 4 that $\pi(F) = \pi(\mathcal{F})$ and F is stable too.

3. Using the stability, obtain the exact value of $\mathrm{ex}(n, F)$. (The fact that stability often helps in proving exact results for the hypergraph Turán problem was observed and used by Füredi and Simonovits [7], Keevash and Sudakov [12, 11], and others.)

Extending the results by Sidorenko [20], the author [17] has successfully applied the above approach to computing the exact value of $\mathrm{ex}(n, T^{(4)})$ for $n \geq n_0$, where the k-graph $T^{(k)}$ consists of the following three edges: $[k]$, $[2, k+1]$, and $\{1\} \cup [k+1, 2k-1]$. The exact value of $\mathrm{ex}(n, T^{(3)})$ was previously computed by Frankl and Füredi [6] (see also Bollobás [2], Keevash and Mubayi [10]).

Lemma 3 has an interesting application. Namely, the method of Mubayi and the author [14] (combined with Lemma 3) shows that the pair $(H^{(k)}_{k+2}, K^{(k)}_{k+1})$ is *non-principal* for any $k \geq 3$, that is,

$$\pi\left(\{H^{(k)}_{k+2}, K^{(k)}_{k+1}\}\right) < \min\left\{\pi(H^{(k)}_{k+2}), \pi(K^{(k)}_{k+1})\right\}, \tag{7}$$

where $K^{(k)}_m$ denotes the complete k-graph of order m. This completely answers a question of Mubayi and Rödl [15] (cf. also Balogh [1]). We refer the Reader to [14] for further details.

Acknowledgments

The author is grateful to Vojta Rödl and Mathias Schacht for providing the manuscripts [16, 18] before their publication and to the anonymous referees for the very useful and detailed comments.

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