BIASED POSITIONAL GAMES ON MATROIDS

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ABSTRACT. Maker and Breaker alternatively select 1 and q previously unclaimed elements of a given matroid M. Maker wins if he claims all elements of some circuit of M. We solve this game for any M and q, including the description of winning strategies. In a special case when the matroid M is defined by a submodular function f, we find the rank formula, which allows us to express our solution in terms of f. The result is applied to positional games on graphs in which, e.g., Maker tries to create a cycle or where Maker's aim is to obtain a subgraph of given integer density.

1. INTRODUCTION

Let E be a finite set and $\mathcal{H} \subseteq 2^E$. In the *Maker-Breaker game* $\mathfrak{G}(E, \mathcal{H}, 1, q)$ two players, Maker and Breaker, alternately select respectively 1 and q ($q \geq 0$) previously unclaimed elements of E until all the elements have been claimed. Maker wins if and only if in the final position there is $Y \in \mathcal{H}$ such that every element of Y has been selected by Maker.

The rules of the game $\mathfrak{G}^*(E, \mathcal{H}, q, 1)$ are the same except it is Breaker who starts the game. Throughout the paper we use the convention that a star in the notation of a game means that Breaker is the first player; the first and the second numerical parameters of the game describe the number of elements selected by, respectively, the first and the second player. For the clarity of language, we refer to Maker as "he" and to Breaker as "she".

We study Maker-Breaker games in which E consists of elements of given matroid M and $\mathcal{H}$ is the family of M-circuits. We recall basic definitions and facts related to matroids in Section 2.

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Games on matroids were first proposed, to the best of our knowledge, by Lehman [10] who solved the Shannon Switching Game by using the cycle matroid. His game is related but different from ours.

Hamidoune and Las Vergnas [8] observed that Lehman's strategies, when appropriately modified, solve unbiased (i.e. $q = 1$) games on matroids, in which Maker's aim is to claim a base of given matroid M. Let denote such a *base game* by $\mathfrak{B}(M, 1, 1)$. The authors formulated conditions on M, sufficient and necessary for existing a winning strategy for Breaker and defined effective strategies for the players.

They also pointed out that this solution implies the outcome of the unbiased *circuit game* $\mathfrak{C}(M, 1, 1)$, where Maker's aim is to claim a circuit of the matroid M. In fact, circuit and base games are in a sense dual to each other, as we explain in Section 3.

In Section 3 we present the solution of the biased circuit game $\mathfrak{C}(M, 1, q)$ where Breaker is allowed $q \geq 1$ edges per move. Also, we settle $\mathfrak{C}^*(M, q, 1)$, the version where Breaker starts the game. This solves the corresponding dual base games as well.

In Section 4 we study the matroid M_f defined by a submodular and increasing function $f : 2^E \to \mathbb{Z}$. Such matroids, introduced by Edmonds and Rota [5], have been systematically studied (Perfect and Pym [14], Nguyen [11, 12], and Dawson [3]), but rather under the additional assumption that $f(\emptyset) = 0$. However, quite a few interesting combinatorial games correspond to functions with $f(\emptyset) \neq 0$. Motivated by this, we establish some properties of M_f for such general f. Our general decomposition lemma, which seems to be a useful tool for studying such matroids, allowed us to determine in terms of f the threshold of the game $\mathfrak{C}(M, 1, q)$, i.e. the smallest q such that Breaker has a winning strategy.

In Section 5, we present some consequences of the results from Sections 3 and 4, applied to games played on graphs and hypergraphs. Among others, we consider the biased *cycle games*, where the players select edges of a graph G and Maker tries to build a cycle and the *density games* in which Maker wants to obtain a subgraph of given density.

Finally, we discuss the algorithmic aspect of the strategies given by our solution of the circuit games.

2. Matroids: Definitions and Notation

For an introduction to matroid theory we refer the reader to the texts by Welsh [17] and Oxley [13]. Here we point out our notation which follows that of Oxley [13]. Some other notions are introduced in the text as we go along.

A *matroid* M is a pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a non-empty family of subsets of a finite set E, which which is *hereditary*, that is

$$A \in \mathcal{I}, B \subseteq A \;\Rightarrow\; B \in \mathcal{I}, \tag{1}$$

and satisfies the following *independence augmentation axiom*:

$$I, J \in \mathcal{I},\ |I| > |J| \;\Rightarrow\; \exists_{x \in I}\;\; J \cup \{x\} \in \mathcal{I}. \tag{2}$$

We refer to elements of E as *elements* of the matroid.

Sets in $\mathcal{I}$ are called *independent*; sets in $2^E \setminus \mathcal{I}$ are called *dependent*. Every one-element dependent set is called a *loop*. A *circuit* is a minimal (in the sense of inclusion) dependent set. The set of all circuits of a matroid M we denote by $\mathcal{C}(M)$. A *base* of the matroid is a maximal independent set. It follows from (2) that for any $X \subseteq E$ all maximal independent subsets of X have the same cardinality. This cardinality, called the *rank* of X, is denoted by $\mathrm{rank}(X)$. For $X \subseteq E$, the matroid

$$M \setminus X = (E \setminus X,\ \mathcal{I} \cap 2^{E \setminus X}),$$

is obtained by *deleting* X from M. The *matroid union* $\vee^k M$ is the matroid on the same set E such that $X \in 2^E$ is independent if and only if X is a union of k independent sets of M.

3. Circuit Games on Matroids

The problem of solving a biased game $\mathfrak{G}(E, \mathcal{H}, 1, q)$ can be approached in two equivalent ways. We can either fix q and ask what conditions on $\mathcal{H}$ assure the existence of a winning strategy for, say, Breaker, or for given $\mathcal{H}$ try to find q_0, the minimum q, such that Breaker has a winning strategy. We refer to q_0 as to the *threshold* of the game $\mathfrak{G}(E, \mathcal{H}, 1, q)$. To make the definition of the threshold complete, we assume that if $\mathcal{H}$ is empty then $q_0 = 0$ and if there is no q for which Breaker can win the game then we put $q_0 = \infty$.

Let us mention that the problem of finding the threshold for the positional games on graphs and hypergraphs originates from papers by Chvátal and Erdős [2] and Beck [1] and was extensively explored by the latter author afterwards.

We are going to compute the threshold for the circuit games $\mathfrak{C}(M, 1, q)$ and $\mathfrak{C}^*(M, q, 1)$. For that purpose we state the following lemma which for $d = 0$ specialises to a theorem of Edmonds [4].

Lemma 1. *We can cover all but at most d elements of a matroid $M = (E, \mathcal{I})$ by k independent sets if and only if*

$$k \times \mathrm{rank}(X) \geq |X| - d \quad \text{for every } X \subseteq E.$$

Proof. The required partial covering is possible if and only if the rank of $\vee^k M$ is at least $|E| - d$. By the union rank formula [13, Theorem 12.3.3] we have

$$\mathrm{rank}(\vee^k M) = \min \{k \times \mathrm{rank}(X) + |E| - |X| \, : \, X \subseteq E\}. \tag{3}$$

The claim easily follows. □

Let $\mathrm{ind}(M)$ be the smallest number of independent sets partitioning E. An immediate consequence of Lemma 1 is that

$$\mathrm{ind}(M) = \max_{\emptyset \subsetneq X \subseteq E} \left\lceil \frac{X}{\mathrm{rank}(X)} \right\rceil.$$

(If M has a loop then we put $\mathrm{ind}(M) = \infty$.)

Now we are ready to prove our main theorem.

Theorem 2. *Let $M = (E, \mathcal{I})$ be a matroid. For the circuit game $\mathfrak{C}(M, 1, q)$ the threshold is*

$$q_0 = \mathrm{ind}(M) - 1 = \max_{\emptyset \subsetneq X \subseteq E} \left\lceil \frac{|X|}{\mathrm{rank}(X)} \right\rceil - 1,$$

and the threshold for the circuit game $\mathfrak{C}^*(M, q, 1)$ is

$$q_0 = \max_{X \subseteq E} \left\lfloor \frac{|X|}{\mathrm{rank}(X) + 1} \right\rfloor.$$

Proof. Let us deal with $\mathfrak{C}(M, 1, q)$ first. We can assume that M has no loops for otherwise Maker wins in his first move and the claim is true. The claim also holds if $\mathrm{ind}(M) = 1$, i.e. M contains no circuit.

Let $k = \mathrm{ind}(M) \geq 2$ and suppose that $q \leq k - 2$. By Lemma 1 (with $d = 0$) there exists $X \subseteq E$ with

$$(q + 1) \times \mathrm{rank}(X) < |X|. \tag{4}$$

A simple winning strategy for Maker is to select elements of X as long as possible. He can do this at least $|X|/(q + 1)$ times. By (4) this is strictly bigger than $\mathrm{rank}(X)$. Hence, at the end of the game, Maker's set must be dependent. Thus $q_0 \geq k - 1$.

Suppose now that $q \geq k - 1$. We demonstrate a strategy of Breaker which prevents Maker from building a dependent set. By Lemma 1 we can find $I_1, \ldots, I_k \subseteq E$ such that

$$I_1, \ldots, I_k \text{ are independent sets covering } E. \tag{5}$$

We will show that the matroid M and these sets can be changed dynamically during the game so that (5) always holds as well as the fact that

$$\cap_{i=1}^{k} I_i \text{ contains all elements selected by Maker.} \tag{6}$$

Let Maker choose $x \in E$, say $x \in I_j$. The response of Breaker is to select, for each $i \in [k] \setminus \{j\}$ such that $I_i \cup \{x\}$ is dependent, some available element x_i in the (unique) circuit $C \subseteq I_i \cup \{x\}$. As the circuit C cannot be a subset of the independent set $I_j \ni x$, such x_i exists by (6).

Note that such a reply of Breaker requires at most $k-1 \leq q$ elements to be chosen. Let B stand for the set of all the elements x_i she has picked. For simplicity of description, we assume that Breaker can resign from choosing the remaining (if any) of $q - |B|$ possible elements. This assumption does not affect our thesis, since Breaker takes no advantage in selecting less elements than she is allowed to.

After this turn we modify M by deleting B, and changing the sets I_i into

$$I'_1 = (I_1 \cup \{x\}) \setminus B, \ \ldots, \ I'_k = (I_k \cup \{x\}) \setminus B.$$

It is not hard to see that the new matroid $M \setminus B$ and the new independent sets satisfy the required conditions (5) and (6).

Maker's set remains independent by (6) and the induction argument shows that Breaker wins $\mathfrak{C}(M, 1, q)$. Thereby we have proved that $q_0 = k - 1$.

Let us consider the game $\mathfrak{C}^*(M, q, 1)$. Recall that Breaker is the first player here. By the first part of the theorem, she has a winning strategy in $\mathfrak{C}^*(M, q, 1)$ if and only if it is possible to delete a set B of q (or less) elements from M so that the remaining elements of M can be covered by $q + 1$ independent sets. By Lemma 1, it holds if and only if

$$q \geq \max_{Y \subseteq E} \frac{|Y| - \operatorname{rank}(Y)}{\operatorname{rank}(Y) + 1},$$

which gives the required threshold. $\square$

By using Theorem 2 one can solve also the base game $\mathfrak{B}(M, q, 1)$, in which this is Maker who chooses q elements per move and his aim is to build a base of M. In that case, "to solve a game" means to compute q_0^*, the smallest q such that Maker has a winning strategy.

The base games are *dual* to the circuit games in the following sense. We have to recourse to the *dual matroid* M^* of M. It has the same element set E and, which can be taken as a possible definition, $I \subseteq E$ is independent in M^* if and only if $E \setminus I$ contains a base of M. Observe that claiming a base of M is the same as *preventing* the opponent from constructing a circuit of M^*. Thereby a winning strategy of Maker, resp. Breaker, in $\mathfrak{B}(M,q,1)$ is a winning strategy of Breaker, resp. Maker, in $\mathfrak{C}^*(M^*,q,1)$.

Theorem 3. *For any matroid $M=(E,\mathcal{I})$ the threshold for the base game $\mathfrak{B}(M,q,1)$ is*

$$q_0^* = \max_{X\subseteq E} \left\lfloor \frac{|X|}{|X| + \mathrm{rank}(E\setminus X) - \mathrm{rank}(E) + 1} \right\rfloor,$$

and the threshold for $\mathfrak{B}^*(M,1,q)$ is

$$q_0^* = \max_{\emptyset\subsetneq X\subseteq E} \left\lceil \frac{|X|}{|X| + \mathrm{rank}(E\setminus X) - \mathrm{rank}(E)} \right\rceil - 1.$$

Proof. The game $\mathfrak{B}(M,q,1)$ is dual to $\mathfrak{C}^*(M^*,q,1)$, and $\mathfrak{B}^*(M,q,1)$ is dual to $\mathfrak{C}(M^*,q,1)$, so the claim follows from Theorem 2 and the standard formula for the rank of the dual matroid M^*:

$$\mathrm{rank}_{M^*}(X) = |X| + \mathrm{rank}_M(E\setminus X) - \mathrm{rank}_M(E), \quad X\subseteq E.$$

□

Unfortunately, it seems that our methods, good for finding the solution of $\mathfrak{B}(M,q,1)$ and $\mathfrak{C}(M,1,q)$, do not apply in general to the biased games $\mathfrak{B}(M,1,q)$ and $\mathfrak{C}(M,q,1)$. For example, let us consider a special case of the game $\mathfrak{B}(M,1,q)$, when M is the *cycle matroid* of the complete graph K_n. (A subset I of edges of K_n is independent in M if and only if I is a forest.) Equivalently, Maker and Breaker select respectively 1 and q edges of K_n, and Maker wins if he claims all edges of a spanning tree. It is easy to compute the parameters of M, for example, $\mathrm{rank}(M)=n-1$, $\mathrm{ind}(M)=\lceil n/2\rceil$, but it is hard to relate them to the threshold q_0 which, according to the results by Chvátal and Erdős [2], is of order $n/\log n$.

4. Matroids Defined by Submodular Functions

The following construction, introduced by Edmonds and Rota [5], supplies us with many interesting matroids.

Let E be a nonempty finite set and let a function $f: 2^E \to \mathbb{Z}$ (into integers) be *increasing*, that is,

$$f(X) \le f(Y), \quad \text{for any } X\subseteq Y\subseteq E, \tag{7}$$

and *submodular*, that is,

$$f(X\cup Y) + f(X\cap Y) \le f(X) + f(Y), \quad \text{for any } X, Y\subseteq E. \tag{8}$$

Then we can define a matroid M_f on E so that independent sets are $\emptyset$ and those nonempty $X \subseteq E$ for which

$$|Y| \leq f(Y), \quad \text{for every non-empty } Y \subseteq X.$$

This indeed defines a matroid, see [13, Proposition 12.1.1] for the proof.

Every matroid M can be represented in the form M_f: take the rank function rank_{M_f} for f. In fact, the extra restriction $0 \leq f(X) \leq |X|$, for $\emptyset \subseteq X \subseteq E$, would ensure that f *equals* the rank function of M_f. One advantage of this construction is that there are matroids which can be represented as M_f for some simple transparent function f while their rank function is very complicated.

This construction was studied by Perfect and Pym [14], Nguyen [11, 12], and Dawson [3]. However, all these papers additionally require that the submodular function is *normalised*, that is, $f(\emptyset) = 0$. (Oxley [13, Section 12.1] does not assume that $f(\emptyset) = 0$ but he does not study M_f in detail.)

Therefore, we prove some properties of M_f, for f not necessarily satisfying $f(\emptyset) = 0$, which we need in Section 5.

Given a matroid M, define $x \sim y$ for $x, y \in E$ if $x = y$ or there is an M-circuit containing both x and y. One can check that $\sim$ is an equivalence relation [13, Proposition 4.1.2]. The equivalence classes of $\sim$ are called the *components* of the matroid and if M consists of only one component then we say M is *2-connected* or, simply, *connected*. Notice that every loop creates one-element component.

Lemma 4 (Decomposition Lemma)**.** *Let $f : 2^E \to \mathbb{Z}$ be increasing and submodular. If a component A of the matroid M_f has at least two elements then* $\text{rank}(A) = f(A)$.

Proof. Let $M = (E, \mathcal{I}) = M_f$. For every component A of M, the function f truncated to A defines the matroid M' such that $\text{rank}_{M'}$ is the truncation of rank_M, so it is enough to prove the lemma for connected matroids only. Thus we assume further that M is connected and $|E| > 1$.

Let I be a base of M, that is, $|I| = \text{rank}(E)$. Let $D = E \setminus I$. Observe that M contains a circuit so $D \neq \emptyset$. By the maximality of I, for every $x \in D$ there is the (unique) $J_x \subseteq I$ such that $J_x \cup \{x\}$ is a circuit in M. By independence of J_x and monotonicity of f, the following chain of inequalities is valid for any $x \in D$:

$$f(J_x) \geq |J_x| = |J_x \cup \{x\}| - 1 \geq f(J_x \cup \{x\}) \geq f(J_x).$$

Hence,

$$f(J_x \cup \{x\}) = f(J_x) = |J_x|, \quad \text{for every } x \in D. \tag{9}$$

Let $t \geq 1$ and suppose that t distinct elements of D form a sequence $x_1, \dots, x_t$ with the property

$$J_{x_{i+1}} \cap \bigcup_{j=1}^{i} J_j \neq \emptyset, \quad \text{for every } i \in [t-1]. \tag{10}$$

For simplicity we denote $\cup_{j=1}^{i} J_j$ by $J_{[i]}$ and put $x_{[i]} = \{x_1, x_2, \dots, x_i\}$. We prove by induction on m that for every $m \leq t$

$$f(J_{[m]} \cup x_{[m]}) = f(J_{[m]}) = |J_{[m]}|. \tag{11}$$

The claim for $m = 1$ follows from (9). For $m \geq 2$, we have

$$\begin{aligned} f(J_{[m]} \cup x_{[m]}) &\leq f(J_{[m-1]} \cup x_{[m-1]}) + f(J_m \cup \{x_m\}) - f(J_{[m-1]} \cap J_m) \\ &\leq |J_{[m-1]}| + |J_m| - |J_{[m-1]} \cap J_m| \\ &= |J_{[m]}| \ \leq \ f(J_{[m]}) \ \leq \ f(J_{[m]} \cup x_{[m]}), \end{aligned}$$

which implies (11). We used the submodularity of f, induction argument, the fact that $J_{[m-1]} \cap J_m$ is independent and non-empty, and the monotonicity of f.

Pulling maximal sequences satisfying (10) out of D, we obtain partitions $\{D_1, D_2 \dots, D_k\}$ of the set D and $\{I_0, I_1, \dots, I_k\}$ of I for some $k \geq 1$, such that $I_0 = I \setminus \bigcup_{x \in D} J_x$, $J_x \subseteq I_i$ for every $x \in D_i$ and, by (11),

$$f(I_i \cup D_i) = |I_i| = \operatorname{rank}(I_i \cup D_i), \quad \text{for every } i \leq k.$$

Clearly, for every base B

$$|B \cap I_0| \geq \operatorname{rank}(E) - \sum_{i=1}^{k} \operatorname{rank}(I_i \cup D_i) = |I| - \sum_{i=1}^{k} |I_i| = |I_0|,$$

that is, $B \supset I_0$. This means that I_0 consists of *isthmuses* (M^*-loops). For any matroid the isthmuses form one-element components, so in our case (M is connected) $I_0 = \emptyset$. In view of that, putting $A_i = I_i \cup D_i$ for $i \leq k$, we have $E = \bigcup_{i=1}^{k} A_i$.

Now we show that $k = 1$. Recall that every two distinct elements of E lay on a circuit. Hence, we are done if we prove that no circuit C can intersect more than one A_i. Suppose on the contrary that such C exists.

Choose $x \in C$, say $x \in A_k$. As $I_{[k-1]}$ spans $D_{[k-1]}$, the rank of $V = (C \cap A_k) \cup I_{[k-1]}$ does not decrease if we remove x. Hence, there is a circuit $C' \subseteq V \cup \{x\}$. As $C \cap A_k$ and $I_{[k-1]}$ are independent, C' intersects both these sets. What we have achieved is that $C' \cap D_{[k-1]} = \emptyset$. Similarly, choosing an element x' in $C' \cap I_{[k-1]}$, we can find a circuit C'' in $V' \cup \{x'\}$, where $V' = (C' \cap I_{[k-1]}) \cup I_k$, which means that C''

lies inside of (independent) I. This contradiction completes the proof. $\square$

Note that M_f is loopless if and only if

$$f(X) \geq 1, \quad \text{for every non-empty } X \subseteq E. \tag{12}$$

Lemma 5. *Let $f : 2^E \to \mathbb{Z}$ be increasing, submodular and satisfy (12). Then for any dependent $X \subseteq E(M_f)$ there is a non-empty $Y \subseteq X$ such that*

$$\frac{|X|}{\operatorname{rank}(X)} \leq \frac{|Y|}{f(Y)}. \tag{13}$$

Also, for every integer $d \geq 0$ and any dependent $X \subseteq E$ there are non-empty, pairwise disjoint $Y_1, Y_2, \dots, Y_k \subseteq X$ such that

$$\frac{|X| + d}{\operatorname{rank}(X) + 1} \leq \frac{\sum_{i=1}^k |Y_i| + d}{\sum_{i=1}^k f(Y_i) + 1}.$$

Proof. Observe that it is enough to prove the first part of the lemma for $X \neq \emptyset$ such that

$$\frac{|X|}{\operatorname{rank}(X)} > \frac{|Y|}{\operatorname{rank}(Y)}, \quad \text{for every } \emptyset \subsetneq Y \subsetneq X. \tag{14}$$

We also assume that X is dependent, so $|X| > \operatorname{rank}(X)$.

Let $M = (X, \mathcal{I})$ be the matroid defined by the function f truncated to the set 2^X, and let $\{A_1, \dots, A_k\}$ be the set of all components of M. Our assumption (12) implies that $\operatorname{rank}(A_i) \geq 1$ for every component A_i. Then straightforward calculations show that

$$\frac{|X|}{\operatorname{rank}(X)} = \frac{\sum_{i=1}^k |A_i|}{\sum_{i=1}^k \operatorname{rank}(A_i)} \leq \max\left\{\frac{|A_i|}{\operatorname{rank}(A_i)} \ : \ i \in [k]\right\},$$

which is a contradiction to (14) unless $k = 1$. Thus $X = A_1$. Since $|X| > \operatorname{rank}(X)$, we have $|A_1| \geq 2$ and from Lemma 4 we obtain

$$f(X) = f(A_1) = \operatorname{rank}(A_1) = \operatorname{rank}(X),$$

so $Y = X$ satisfies (13), as required.

In order to prove the second claim of the lemma, we consider all components $Y_1, \dots, Y_k$ of X, which have size at least 2 and put $Y_0 = X \setminus \bigcup_{i=1}^k Y_i$. Then Y_0 is independent and $|Y_i| \geq \operatorname{rank}(Y_i) + 1$ for every $i \geq 1$, which together with Lemma 4 give

$$\frac{|X| + d}{\operatorname{rank}(X) + 1} = \frac{|Y_0| + \sum_{i=1}^k |Y_i| + d}{|Y_0| + \sum_{i=1}^k \operatorname{rank}(Y_i) + 1} \leq \frac{\sum_{i=1}^k |Y_i| + d}{\sum_{i=1}^k f(Y_i) + 1}.$$

$\square$

As the inequality $|X|/\mathrm{rank}(X) \geq |X|/f(X)$ is obvious for every non-empty X, we obtain the following formula for $\mathrm{ind}(M_f)$.

Corollary 6. *Let $E \neq \emptyset$ and $f : 2^E \to \mathbb{Z}$ be increasing and submodular. If there is $x \in E$ with $f(\{x\}) \leq 0$, then $\mathrm{ind}(M_f) = \infty$. Otherwise,*

$$\mathrm{ind}(M_f) = \max_{\emptyset \subsetneq X \subseteq E} \left\lceil \frac{|X|}{\mathrm{rank}(X)} \right\rceil = \max_{\emptyset \subsetneq X \subseteq E} \left\lceil \frac{|X|}{f(X)} \right\rceil.$$

□

The following theorem is of independent interest. It is another illustration to the usefulness of M_f: the formula giving the rank function of $\vee^k M$ is more complicated, cf. (3).

Theorem 7. *Let $E \neq \emptyset$ and $f : 2^E \to \mathbb{Z}$ be increasing and submodular. Then, for any integer $k \geq 1$,*

$$\vee^k M_f = M_{kf}.$$

Proof. Clearly, $\vee^k M_f$ and M_{kf} have the same set of loops. Hence, it is enough to consider loopless M_f only.

Let X be independent in $\vee^k M_f$. Then any non-empty $I \subseteq X$ can be represented as $I_1 \cup \ldots \cup I_k$ with $I_i \in \mathcal{I}(M_f)$; we can additionally request that each $I_i \neq \emptyset$. Hence,

$$kf(I) \geq \sum_{i=1}^{k} f(I_i) \geq \sum_{i=1}^{k} |I_i| \geq |I|,$$

and we conclude that X is independent in M_{kf}.

On the other hand, suppose that X is not independent in $\vee^k M_f$. By Lemma 1 there is $Y \subseteq X$ such that $|Y|/\mathrm{rank}(Y) > k$. By Lemma 5 we can find non-empty $Z \subseteq Y$ such that $|Z|/f(Z) > k$. Thus, X cannot be independent in M_{kf}, as required. □

5. Applications of Theorem 2

In this section we present some consequences of Theorem 2 combined with the results of Section 4.

Theorem 8. *Let $E \neq \emptyset$, $f : 2^E \to \mathbb{Z}$ be increasing and submodular and let L be the set of loops of M_f, i.e. $L = \{x \in E : f(\{x\}) \leq 0\}$.*

i) If $L = \emptyset$ then the threshold for the circuit game $\mathfrak{C}(M_f, 1, q)$ is

$$q_0 = \max_{\emptyset \subsetneq X \subseteq E} \left\lceil \frac{|X|}{f(X)} \right\rceil - 1.$$

(In case $L \neq \emptyset$ Maker wins in his first move.)

ii) The threshold for the game $\mathfrak{C}^*(M_f, 1, q)$ *is*

$$q_0 = \max\Big\{|L|, \max_{\mathcal{Y}\in\Pi}\lfloor g(\mathcal{Y})\rfloor\Big\},$$

where Π *is the set of all families* $\mathcal{Y}$ *of nonempty, pairwise disjoint subsets of* $E \setminus L$ *and* $g(\mathcal{Y}) = \frac{\sum_{X\in\mathcal{Y}}|X|+|L|}{\sum_{X\in\mathcal{Y}}f(X)+1}$.

iii) If $f(\emptyset) \geq 0$ *then the threshold for the game* $\mathfrak{C}^*(M_f, 1, q)$ *is*

$$q_0 = \max_{X\subseteq E}\Big\lfloor\frac{|X|}{f(X)+1}\Big\rfloor.$$

Proof. The claim about $\mathfrak{C}(M_f, 1, q)$ follows from Theorem 2 and Corollary 6.

Let l be the number of loops of M_f and let us consider $\mathfrak{C}^*(M_f, 1, q)$, in the non-trivial case of $X \neq L$. Then, by Theorem 2

$$q_0 = \max_{X\subseteq E}\Big\lfloor\frac{|X|}{\mathrm{rank}(X)+1}\Big\rfloor = \max\Big\{l, \max_{\mathcal{I}\not\ni X\subseteq E\setminus L}\Big\lfloor\frac{|X|+l}{\mathrm{rank}(X)+1}\Big\rfloor\Big\}. \quad (15)$$

Since f truncated to $2^{E\setminus L}$ satisfies (12), we can apply here the second part of Lemma 5 and obtain

$$q_0 \leq \max\Big\{l, \max_{\mathcal{Y}\in\Pi}\lfloor g(\mathcal{Y})\rfloor\Big\}$$

with g and Π defined in the thesis of Part ii). The opposite inequality is obvious in view of (15) and the fact that $\mathrm{rank}(X) \leq f(X)$ for every non-empty $X \subseteq E \setminus L$. Thus we get the desired formula on q_0.

The third part of our thesis follows from the Part ii), since if additionaly $f(\emptyset) \geq 0$ then for any $\mathcal{Y} \in \Pi$

$$\sum_{X\in\mathcal{Y}} f(Y) \geq f\big(\bigcup_{X\in\mathcal{Y}} X\big)$$

and $f(X \cup L) = f(X)$ for every $X \subseteq E$, by submodularity and monotonity of f.

□

The above theorem is a useful tool for calculating the threshold for games played on graphs, in which Maker tries to build a subgraph F with the property $e(F) > av(F)+b$, for given integers a, b. By $e(F)$ we denote the number of edges of F and by $v(F)$ the number of vertices covered by the edges of F. Such games are equivalent to the games played on count matroids, constructed in the following way.

With a given graph $G = (V(G), E(G))$ and integers

$$a \geq 1 \text{ and } b \geq 1 - 2a$$

we can associate the *count matroid* $\mathcal{N}_{a,b}(G)$ which is the matroid M_f on $E(G)$ defined by the submodular and increasing function

$$f(F) = av(F) + b, \quad F \subseteq G,$$

From now on, for the brevity of notation, we do not distinguish a graph F from its edge set $E(F)$ and write $|F|$ instead of $e(F)$. Note the restriction $b \geq 1 - 2a$: otherwise $\mathcal{N}_{a,b}$ consists of loops only. This construction was introduced by White and Whiteley [18] (see also Whiteley [19]).

For example, the matroid $\mathcal{N}_{1,-1}$ is the *cycle matroid* we met; its circuits are formed by cycles of G. In the matroid $\mathcal{N}_{1,0}$ the independent sets are vertex-disjoint unions of trees and unicyclic graphs.

Theorem 8 gives formulae for computing the threshold in the circuit games on count matroids. Sometimes the corresponding maximum is easy to compute. Here are just a few examples; we compute q_0 for games played on the complete graph K_n and on the complete bipartite graph $K_{n,n}$.

Theorem 9. *For integers $n \geq 2$ and $2a + b \geq 1$, let $M = \mathcal{N}_{a,b}(K_n)$. Then, the threshold for the game $\mathfrak{C}(M, 1, q)$ is*

$$q_0 = \left\lceil \frac{n(n-1)}{2(an+b)} \right\rceil - 1, \tag{16}$$

and the threshold for $\mathfrak{C}^*(M, q, 1)$ is

$$q_0 = \left\lfloor \frac{n(n-1)}{2(an+b+1)} \right\rfloor. \tag{17}$$

Proof. To solve $\mathfrak{C}(M, 1, q)$ we apply Theorem 8. Given $f(F)$, the maximum of $|F|$ is attained when F is a clique. Hence

$$q_0 + 1 = \max_{2 \leq i \leq n} \lceil g(i) \rceil, \text{ where } g(i) = \frac{\binom{i}{2}}{ai + b}.$$

It is routine to calculate that $\lceil g(i) \rceil$ is maximal for $i = n$, which gives the threshold (16).

For $\mathfrak{C}^*(M, q, 1)$ it is enough to compute the maximum of the function

$$g(F_1, \ldots, F_k) = \frac{\sum_{i=1}^{k} |F_i|}{\sum_{i=1}^{k} (av(F_i) + b) + 1}$$

over all $k \leq n$ and sequences $(F_1, \ldots, F_k)$ of pairwise edge-disjoint subgraphs $F_i \subseteq K_n$ with $|F_i| \geq 1$.

Suppose that the $g_{\max}$ is the maximum value of g and is obtain by a sequence $G_1, \ldots, G_k \subseteq K_n$. Then

$$\frac{\binom{n}{2}}{an+b+1} = g(K_n) \leq g_{\max} = g(G_1, \ldots, G_k) = \frac{\sum_{i=1}^k |G_i|}{\sum_{i=1}^k (av(G_i)+b)+1}$$
$$< \max_{1\leq i\leq k} \frac{|G_i|}{av(G_i)+b} \leq \frac{\binom{m}{2}}{am+b} \tag{18}$$

for some $m \leq n$.

If $m = 2$ then the above implies that $g_{\max} < 1/(2a+b) \leq 1$ and hence $\lfloor g_{\max} \rfloor \leq \lfloor g(K_n) \rfloor$.

In case of $3 \leq m \leq n$, standard calculations show that if $2a+b \geq 1$ then $\binom{n}{2}/(an+b+1) < \binom{m}{2}/(am+b)$ only for $m = n$. Therefore $|G_i| = \binom{n}{2}$ for some $i \leq k$ and so $g_{\max} = g(K_n)$.

Thus, in any case, $\lfloor g(K_n) \rfloor$ is the maximum of $\lfloor g \rfloor$, which implies the formula (17). □

Corollary 10. *Suppose that Maker and Breaker select respectively* 1 *and q edges of K_n ($n \geq 2$) and Maker wants to build a cycle. Then Maker wins the game (no matter who starts) if and only if $q < \lceil n/2 \rceil - 1$.* □

Let us add that the above result, with assumption that Maker is the first player, can be obtain directly from Theorem 2: K_n is a union of $\lceil n/2 \rceil$ trees, so for the corresponding matroid $M = \mathcal{N}_{1,-1}(K_n)$ we have $\mathrm{ind}(M) = \lceil n/2 \rceil$.

Theorem 11. *For integers $n \geq 1$ and $2a+b \geq 1$, let $M = \mathcal{N}_{a,b}(K_{n,n})$. Then, the threshold for the game $\mathfrak{C}(M,1,q)$ is*

$$q_0 = \left\lceil \frac{n^2}{2an+b} \right\rceil - 1,$$

and the threshold for $\mathfrak{C}^*(M,q,1)$ is

$$q_0 = \left\lfloor \frac{n^2}{2an+b+1} \right\rfloor.$$

Proof. Let us solve the game $\mathfrak{C}^*(M,1,q)$ only, since the calculating of the corresponding maximum for $\mathfrak{C}(M,q,1)$ is much simpler.

The analysis of $\mathfrak{C}^*(M,1,q)$ is similar to that presented in the proof of the second part of Theorem 11. We define the function g analogously and compute the maximum of g by the following modification of the formula (18):

$$\frac{n^2}{2an+b+1} = g(K_{n,n}) \le g_{\max} = g(G_1,\ldots,G_k) = \frac{\sum_{i=1}^{k}|G_i|}{\sum_{i=1}^{k}(av(G_i)+b)+1}$$
$$< \max_{1\le i\le k} \frac{|G_i|}{av(G_i)+b} \le \frac{\lfloor m/2\rfloor\lceil m/2\rceil}{am+b}$$

for some $m \le 2n$. Then we conclude that either $m=2$ and $\lfloor g_{\max}\rfloor = 0 \le g(K_{n,n})$ or $3 \le m \le 2n$ and the above holds only if $m = 2n$. Thus $\lfloor g(K_{n,n})\rfloor$ maximises $\lfloor g\rfloor$ and by Theorem 8 we obtain q_0 as desired. $\square$

Corollary 12. *Suppose that Maker and Breaker select respectively* 1 *and q edges of $K_{n,n}$ ($n \ge 1$) and Maker wants to build a cycle. Then Maker wins the game (no matter who starts) if and only if $q < \lfloor n/2\rfloor$.*

Pikhurko [15] generalised count matroids to r*-graphs*. By an r-graph we mean a subset of $\binom{[n]}{r} = \{X \subseteq [n] \,:\, |X| = r\}$, where $[n] = \{1,\ldots,n\}$. In order not to mess with details, we state a special case which admits a nice solution.

For non-negative integers $a_0,\ldots,a_{r-1}$ define

$$f(H) = a_0 + \sum_{i=1}^{r-1} a_i p_i(H), \quad H \subseteq \binom{[n]}{r}, \tag{19}$$

where $p_i(H)$ is the number of i-sets covered by at least one edge of H. For example, $p_1(H) = |\cup_{e\in H} e|$ and $p_r(H) = |H|$.

It is easy to prove that f satisfies (7) and (8) and hence defines a matroid M_f on $\binom{[n]}{r}$. (In general, a_0 could be negative, see [15, Section 3].)

Corollary 13. *Let $a_0,\ldots,a_{r-1}$ be non-negative integers, $n \ge 1$, and f be defined by (19). The threshold for the game $\mathfrak{C}(M_f,1,q)$ is*

$$q_0 = \left\lceil \frac{\binom{n}{r}}{\sum_{i=0}^{r-1} a_i\binom{n}{i}} \right\rceil - 1.$$

Proof. By Theorem 8 it is enough to show that

$$\frac{|H|}{\sum_{i=0}^{r-1} a_i p_i(H)} \le \frac{\binom{n}{r}}{\sum_{i=0}^{r-1} a_i\binom{n}{i}}, \quad \text{for every } H \subseteq \binom{[n]}{r}.$$

This can be rewritten as

$$\sum_{i=0}^{r-1} a_i \left(p_i(H)\binom{n}{r} - |H|\binom{n}{i} \right) \ge 0.$$

One can verify that

$$p_i(H)\binom{n}{r} = p_i(H)\binom{n}{i}\binom{n-i}{r-i}/\binom{r}{i} \geq |H|\binom{n}{i},$$

where the last inequality follows from double-counting of all pairs (A, B) such that $A \in \binom{[n]}{i}$, $B \in H$ and $A \subseteq B$. Thus each term in the previous sum is non-negative, giving the required. □

Finally, let us propose a *density game* $\mathfrak{D}_a(K_n, 1, q)$, played on K_n, in which Maker's aim is to claim a graph H of density $|H|/v(H) \geq a$, for given $a > 0$. If a is integer then by Theorem 9 the threshold for that game is

$$q_0 = (1 + o(1))\,\frac{n}{2a}. \tag{20}$$

However, the matroid approach breaks down if a is not an integer. The lower bound $(1 + o(1))\frac{n}{2a}$ on q_0 still holds: if $q + 1 \leq \frac{n-1}{2a}$ then the game lasts at least *an* turns, so Maker wins playing arbitrarily. Theorem 9 implies only that $q_0 \leq (1 + o(1))\,\frac{n}{2\lfloor a\rfloor}$. Also, note that (20) is not generally true for $a < 1$; e.g. for $a \leq 2/3$ Maker wins in two moves unless $q \geq 2n - 4$. Nevertheless, we conjecture that (20) holds for every $a \geq 1$.

Conjecture 14. *For any real $a \geq 1$, the threshold of the density game $\mathfrak{D}_a(K_n, 1, q)$ is $q_0 = (1 + o(1))\,\frac{n}{2a}$.*

6. Computational Considerations

An important question is whether there are good algorithms realising the strategies given by the proof of Theorems 2 and 3. A moment's thought reveals that we have to find an algorithm *demonstrating* whether

$$\operatorname{rank}(\vee^k M) \geq r \tag{21}$$

for a given matroid M and integers k and r, that is, producing either k independent sets whose union has at least r elements or a set Y which disproves (21) via (3).

There are quite a few such algorithms (e.g. the proof of Edmonds [4] gives one). The algorithm of Kelmans and Polesskii [9] runs in polynomial in $|E|$ time and makes $O(|E|^2)$ calls to the *independence oracle*, that is, the subroutine which tests whether a set $Y \subseteq E$ is independent or not. Note that the cases $k > |E|$ or $r > |E|$ are trivial, so we choose $|E|$ as the sole parameter for measuring efficiency. Thus, if the independence oracle runs in polynomial time, then Breaker's/Maker's winning strategy in Theorems 2 and 3 can be computed in polynomial time.

For the matroid M_f defined by a submodular function f, which we considered in the previous sections, one can always devise an independence oracle which runs in polynomial time. Indeed, a non-empty set X is independent if and only if for any $x \in X$ we have

$$\min\{g(Y) \,:\, Y \subseteq X \setminus \{x\}\} \geq 0, \tag{22}$$

where $g(Y) = f(Y \cup \{x\}) - |Y \cup \{x\}|$. The function g is submodular so its minimum can be computed in polynomial time by the ellipsoid method as it was shown by Grötschel, Lovász and Schrijver [6, 7]. (Schrijver [16] presents another, more combinatorial, minimisation algorithm.)

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