# BREAKING SYMMETRY ON COMPLETE BIPARTITE GRAPHS OF ODD SIZE

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## Abstract

Players $\mathcal{A}$ and $\mathcal{B}$ alternatively colour edges of a graph $G$, red and blue respectively. Let $L_{\mathrm{sym}}(G)$ be the largest number of moves during which $\mathcal{B}$ can keep the red and blue subgraphs isomorphic, no matter how $\mathcal{A}$ plays.

This function was introduced by Harary, Slany and Verbitsky who in particular showed that for complete bipartite graphs we have $L_{\mathrm{sym}}(K_{m,n}) = \frac{mn}{2}$ if $mn$ is even and that $L_{\mathrm{sym}}(K_{2m+1,2n+1}) \geq \max(m,n)$. Here we prove that

$$L_{\mathrm{sym}}(K_{2m+1,2n+1}) = O(n), \quad \text{if } m \leq n \leq m^{O(1)},$$

answering a question posed by Harary, Slany and Verbitsky.

## 1. Introduction

Let $G$ be a graph. The following *symmetry breaking-preserving game* on $G$ was introduced by Harary, Slany and Verbitsky [1, 2]. We have two players, $\mathcal{A}$ and $\mathcal{B}$, who alternatively select a previously uncoloured edge of $G$ and colour it red and blue respectively. Player $\mathcal{A}$ starts the game. A move of $\mathcal{A}$ followed by a move of $\mathcal{B}$ is called a *round*. Clearly, we have the same number of red and blue edges after every round. The aim of Player $\mathcal{B}$ is to keep the red and blue subgraphs isomorphic after every round of the game; as soon as $\mathcal{B}$ fails to do so, he loses.

Let $L_{\mathrm{sym}}(G)$ be the maximum number of moves during which $\mathcal{B}$ can keep the red and blue subgraphs isomorphic, no matter how $\mathcal{A}$ plays. Equivalently (see [2, Proposition 2.1])

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$L_{\rm sym}(G)$ is the smallest $k$ such that $\mathcal{A}$ can guarantee his win in at most $k+1$ moves. It is not quite clear how to define $L_{\rm sym}(G)$ in the cases when $\mathcal{B}$ can preserve the symmetry until the players run out of edges; following [2] we define $L_{\rm sym}(G) = \lfloor e(G)/2 \rfloor$ then.

One of the motivations of Harary, Slany and Verbitsky for introducing these notions was that $L_{\rm sym}(G)$ is clearly a lower bound on how long the second player can survive in any *graph avoidance game* on $G$. (The rules of the avoidance game are the same except that the player who first constructs a monochromatic copy of a certain forbidden subgraph loses.)

As it is observed in [1], if $G$ has an involutary automorphism $\psi$ without fixed edges, then $L_{\rm sym}(G) = e(G)/2$. Indeed, every orbit of the induced action $\psi^*$ on $E(G)$ consists of two edges, so $\mathcal{B}$ can use the *copycat strategy* of choosing $\psi^*(e) \in E(G)$ where $e$ is the edge previously coloured by $\mathcal{A}$.

The determination of $L_{\rm sym}(G)$ is suddenly getting rather complicated and deep when we consider graphs which do not admit a copycat strategy but for which this cannot be derived by looking at a part of the graph. For example, when one considers the path $P_n$ with $n$ vertices, one has to know (the parity of) its order $n$ in order to ascertain the existence of an appropriate automorphism $\psi$. So, if $\mathcal{A}$ follows some 'local' strategy, $\mathcal{B}$ might put up a strong resistance by playing copycat on a few separate parts of the graph. The surprising (at least to me) result of Harary, Slany and Verbitsky [2, Corollary 3.7 & Proposition 3.8] states that

$$(0.5 + o(1)) \log_2 n \le L_{\rm sym}(P_{2n}) \le (3.5 + o(1)) (\log_2 n)^2.$$

Its proof exploits some beautiful connections to the so-called *Ehrenfeucht-Fraïssé game.*

Complete bipartite graphs of even size clearly admit a copycat strategy. The following argument from [2] shows that

$$L_{\rm sym}(K_{m,n}) \ge \max(\tfrac{m-1}{2}, \tfrac{n-1}{2}), \quad \text{for odd } mn. \tag{1}$$

Indeed, let $X \subset V(K_{m,n})$ be the bigger part of $K_{m,n}$. Starting with $\psi$ being the identity automorphism, $\mathcal{B}$ uses the $\psi$-copycat strategy, that is, responds with $\psi^*(e)$ to the previous move $e$. This works unless $\psi^*(e) = e$ in which case $\mathcal{B}$ locally modifies $\psi$ so that it exchanges $x$ and $y$ now, where $\{x\} = X \cap e$ and $y \in X$ is a vertex not incident to any coloured edge. Such $y$ always exists during the first $(|X|-1)/2$ rounds during which $\psi$ remains an involutary automorphism of $K_{m,n}$ swapping the blue and red subgraphs.

Harary, Slany and Verbitsky [2, Question 4.3] asked for the rate of growth of the function $L_{\rm sym}(K_{n,n})$ for odd $n$. The following more general result implies that $L_{\rm sym}(K_{m,n})$ grows linearly in $n$ if $m \le n \le m^{O(1)}$ and $mn$ is odd.

**Theorem 1** *Let odd integers $m$ and $n$ and an integer $k \ge 8$ satisfy*

$$51 \le m \le n \le \frac{(m-2k-3)^k}{(k+1)!\, m}. \tag{2}$$

*Then we have*

$$L_{\mathrm{sym}}(K_{m,n}) \le k(n-m+3)+2m+14. \tag{3}$$

*In particular, if $m=n$, then letting $k=8$ we obtain*

$$L_{\mathrm{sym}}(K_{n,n}) \le 2n+38, \quad \textit{for odd } n \ge 51. \tag{4}$$

In Section 3 we present the bound (3) as a more digestible, explicit function of $m$ and $n$.

Let us describe the main idea behind the proof of Theorem 1. In outline, $\mathcal{A}$ builds a red graph which has no non-trivial automorphism. If $\mathcal{B}$ has not lost yet, the isomorphism $\psi$ between the red and blue graphs is unique and $\mathcal{B}$ is *forced* to play the $\psi^*$-copycat strategy now. As the total number of edges is odd, $\psi^*$ has at least one odd orbit $D \subset E(G)$. No matter how $D$ has been coloured, Player $\mathcal{A}$ can beat the $\psi^*$-copycat strategy on $D$ with at most two moves. So, if $\psi$ remains the unique isomorphism during these two moves of $\mathcal{A}$, then $\mathcal{B}$ loses.

This method might be applicable to many graphs with odd size. Here is its concrete realisation for complete bipartite graphs.

## 2. Proof of Theorem 1

We will describe the appropriate strategy of $\mathcal{A}$ which consists of a few phases. Assume that $\mathcal{B}$ keeps the red and blue subgraphs isomorphic throughout our strategy.

Let $V \cup V' = V(K_{m,n})$ be the vertex classes, where $|V| = m$.

PHASE 1. $\mathcal{A}$ builds a red cycle of length $2l \in [2m-6, 2m-2]$, say visiting vertices $x_1, x'_1, \dots, x_l, x'_l$ in this order, plus one edge from some vertex $y_0 \in Y$ to $X'$, where we denote $X = \{x_1, \dots, x_l\} \subset V$, $X' = \{x'_1, \dots, x'_l\} \subset V'$, $Y = V \setminus X$, and $Y' = V' \setminus X'$.

The red/blue subgraph will have maximum degree at most 2 at any moment before the end of Phase 1, so in particular $\mathcal{A}$ can easily create a red path of length $2m-7$ by choosing vertices $x_1, x'_1, \dots, x_{m-3}, x'_{m-3}$ one by one in this order. If the edge $\{x_1, x'_{m-3}\}$ is available, then $\mathcal{A}$ colours it and we are done because the addition of an edge between $X'$ and $Y$ is always possible as $m-3>2$. Otherwise, $\mathcal{A}$ extends the path by two more edges. Now, if $\{x_1, x'_{m-2}\}$ is available, then $\mathcal{A}$ selects it and we are done again. Otherwise, $\mathcal{A}$ can add $x_{m-1}$ to the path: indeed, the vertex $x'_{m-2}$ sends a blue edge to $x_1$, so it sends at most one blue edge to $V \setminus X$. If $\{x_{m-1}, x'_1\}$ is available, then $\mathcal{A}$ selects it, obtaining the desired configuration (up to relabelling). Otherwise, the edge $\{x_{m-1}, x'_1\}$ is blue and $\mathcal{A}$ can extend the path to $x'_{m-1}$. In the next move $\mathcal{A}$ colours the edge $\{x_1, x'_{m-1}\}$ which cannot be blue because we have already encountered two blue edges incident to

$x_1$. Finally, $\mathcal{A}$ connects some $y_0 \in Y$ to $X'$, obtaining the desired configuration in all cases.

PHASE 2. $\mathcal{A}$ connects $Y$ to $X'$.

The vertex $y_0 \in Y$ will play a special role. Assume that $x'_1$ is the vertex in $X'$ connected to $y_0$ by a red edge.

Consider first the case when $y_0$ is incident to at least one blue edge. By reversing, if necessary, the direction of the red $2l$-cycle, we can assume that $y_0$ has at least one blue neighbour *outside* $\{x'_1, \ldots, x'_{10}\}$. In the next five moves $\mathcal{A}$ colours $\{y_0, x'_i\}$ for the smallest possible index $i \geq 2$ each time. Suppose that the last such edge was $\{y_0, x'_s\}$. When $\mathcal{A}$ was colouring it, at most 5 edges incident to $y_0$ were blue of which at most 4 lie in $X'_0 = \{x'_1, \ldots, x'_s\}$. Hence $s \leq 10$.

In the next three moves $\mathcal{A}$ colours $\{y_0, x'_i\}$, where $i$ is the smallest available index with $i \geq 2s$ except, when colouring the third edge in the case $s = 6$, the additional condition on $i$ is that the indexes of the last three red neighbours of $y_0$ do not form an arithmetic progression. (Note that for $s \geq 7$ the indexes of the *first* six neighbours of $y_0$ cannot form a six-term arithmetic progression as $s \leq 10$.)

Let $t > 2s$ be the largest index of a red neighbour of $y_0$ and let $X'_1 = \{x'_{2s}, \ldots, x'_t\}$. We claim that $t \leq 26$. If $s \geq 7$, then $y_0$ sends at most $8 - (s - 6) = 14 - s$ blue edges to $X'_1$ because $y_0$ sends $s - 6$ blue edges to $X'_0$; thus

$$t - 2s + 1 = |X'_1| \leq 3 + (14 - s), \tag{5}$$

which gives the required in view of $s \leq 10$. If $s = 6$, then we have to add 1 to the bound (5), still obtaining the claimed inequality $t \leq 26$.

If $y_0$ is not incident to a blue edge at the end of Phase 1, then this stays so, in whichever manner we add red edges incident to $y_0$. In particular, $\mathcal{A}$ can connect $y_0$ to

$$\{x'_1, x'_2, \ldots, x'_6, x'_{12}, x'_{13}, x'_{15}\}$$

and we have $s = 6$ and $t = 15$.

Next, $\mathcal{A}$ connects the vertices of $Y \setminus \{y_0\}$ to $X'$, by five edges each, so that distinct vertices of $Y$ have disjoint neighbourhoods in $X'$, which is possible as $l > 2 \cdot 9 + 5 \cdot (|Y| - 1)$.

The first two phases last for $r_2 = 2l + 9 + 5(m - l - 1)$ rounds.

PHASE 3. If $m = n = l + 1$, then $\mathcal{A}$ connects the (unique) vertex of $Y'$ to arbitrary 12 vertices of $X$. Otherwise, $\mathcal{A}$ picks vertices of $Y'$ one by one and connects each selected vertex by $k$ edges to $X$. Suppose that $\mathcal{A}$ has already dealt with $y'_1, \ldots, y'_i \in Y'$ and connected the next vertex $y' \in Y'$ to a set $H \subset X$ of size $h \in [0, k-1]$. Let $\Gamma_{\mathrm{red}}(y)$ (resp. $\Gamma_{\mathrm{blue}}(y)$) denotes the red (resp. blue) neighbourhood of a vertex $y$.

We claim that $f$, the number of vertices in $F = \Gamma_{\text{blue}}(y') \cap X$, will be at most $k+1$ when we deal with the vertex $y'$. This is true for $i \leq 3$ when the maximum degree of the blue graph is at most $\max(k, 9)$. If $i \geq 4$, then the isomorphism between the red and blue graphs must respect the classes $V$ and $V'$ because the non-trivial red component has at least $l + i > |V|$ vertices in $V'$. So the blue degree of $y'$ is at most the maximum red degree of a vertex in $V'$ which is $k$, giving the desired bound on $f$.

Here is the strategy: $\mathcal{A}$ connects $y'$ to a vertex $x \in X \setminus (F \cup H)$ such that $\mu(H \cup \{x\})$ is minimum, where

$$\mu(Z) = \sum_{j=1}^{i} c_{|Z \setminus \Gamma_{\text{red}}(y'_j)|}, \quad Z \subset X,$$

where $c_0 = l$, $c_1 = 1$, and all other $c$'s are zero. In other words,

$$\mu(Z) = l \left| \left\{ j \in [i] \mid Z \subset \Gamma_{\text{red}}(y'_j) \right\} \right| + \left| \left\{ j \in [i] \mid |Z \setminus \Gamma_{\text{red}}(y'_j)| = 1 \right\} \right|.$$

It is easy to see that

$$\begin{aligned} \sum_{x \in X \setminus (F \cup H)} \mu(H \cup \{x\}) \;&\leq\; \sum_{\substack{j \in [i] \\ H \subset \Gamma_{\text{red}}(y'_j)}} (c_0(k-h) + c_1(l-k)) \\ &+ \sum_{\substack{j \in [i] \\ |H \setminus \Gamma_{\text{red}}(y'_j)| = 1}} c_1(k-h+1) \;\leq\; (k-h+1)\mu(H), \end{aligned}$$

by straightforwardly comparing the corresponding terms. Hence, $\mathcal{A}$ can choose an $x \in X \setminus (F \cup H)$ such that

$$\mu(H \cup \{x\}) \leq \frac{k-h+1}{l-f-h}\,\mu(H) \leq \frac{k-h+1}{m-2k-3}\,\mu(H).$$

As $\mu(\emptyset) = li < mn$, the inequality

$$\mu(H) < \frac{(k+1)!}{(m-2k-3)^k}\,mn \leq 1$$

holds when we reach the case $|H| = k$. As $\mu$ is integer-valued, it must be the case that $\mu(H) = 0$, which means by the definition that $|H \setminus \Gamma_{\text{red}}(y'_j)| \geq 2$ for any $j \in [i]$. Thus $\mathcal{A}$ can ensure that the hamming distance between any two sets in $\{\Gamma_{\text{red}}(y') \mid y' \in Y'\}$ is at least 4 at the end of Phase 3.

The first three phases took $r_3 = r_2 + 12$ rounds if $n = m = l+1$ and $r_3 = r_2 + k(n-l)$ rounds otherwise.

Phase 4. $\mathcal{A}$ adds at most two more edges and wins.

This phase needs some analysis before we can specify the moves of $\mathcal{A}$. Let $A_i$ (resp. $B_i$) consist of red (resp. blue) edges after $i$ rounds, viewed as a subset of $E(K_{m,n})$. Thus

$A = A_{r_3}$ is the red graph at the end of Phase 3. Note that $X \cup X'$ spans in $A$ an induced $2l$-cycle which we denote by $C \subset A$.

**Claim 1.** Let $A'$ be obtained by adding to $A$ at most two edges of the encompassing graph $K_{m,n}$. Let $\phi : V(K_{m,n}) \to V(K_{m,n})$ be a bijection such that $\phi^*(A) \subset A'$, where $\phi^*$ denotes the induced action on 2-point sets. Then $\phi(y_0) = y_0$, $\phi(Y) = Y$, $\phi(X) = X$, $\phi(Y') = Y'$ and $\phi(X') = X'$.

If, furthermore, $\phi^*(C) = C$, then $\phi$ is the identity map.

*Proof.* As $A, A'$ are connected bipartite graphs, we have $\phi(V) = V$ or $\phi(V) = V'$.

Let us show first that $\phi(V) = V$, which needs justification when $|V| = |V'|$. If $|Y'| = 1$, then the (unique) vertex of $Y'$ of degree at least 12 must be preserved by $\phi$ as any other vertex has $A'$-degree at most 11; thus $\phi(V) = V$, as required. Suppose that $|Y'| = |Y| \geq 2$. The vertices in $X$ are incident to at most $4 + |Y'| \leq 7 < k$ $A'$-edges each. Thus $V$ contains at most one vertex of $A'$-degree at least $k$ and $\phi$ must map $Y'$ into $V'$. Now it follows that $\phi(V) = V$ and $\phi(V') = V'$.

As each vertex of $Y'$ has degree at least 8 while the $A'$-degrees in $X'$ are all at most 5, we conclude that $\phi(Y') = Y'$. As each vertex of $\phi(Y)$ sends at least five edges to $\phi(X') = X'$, we have $\phi(Y) = Y$. Similarly, $\phi(y_0) = y_0$, which proves the first part of the claim.

Suppose furthermore that $\phi^*(C) = C$. This means that the restriction of $\phi$ to $X \cup X'$ is a cyclic rotation, possibly composed with the reflection $x_i \mapsto x'_{l-i+1}$, $x'_i \mapsto x_{l-i+1}$. We are going to show that $\phi|_{X \cup X'}$ is the identity by considering the neighbourhood of $y_0$.

Recall that the set $X'_0 = \{x'_1, \ldots, x'_s\}$ contains six $A$-neighbours of $y_0$ and $X'_1 = \{x'_{2s}, \ldots, x'_t\}$ the remaining three. The sets $X'_0$ and $X'_1$ cannot both intersect $\phi(X'_0)$ as they are separated by $s-1$ other vertices of $X'$. Hence, $\phi(X'_0) \cap X'_1 = \emptyset$ and $\phi(X'_0) \cap X'_0$ contains at least four $A$-neighbours of $y_0$. If $\phi(X'_1)$ is situated at the 'wrong' side of $\phi(X'_0)$, then (as $l \geq 2t - 4$) we have $\phi(X'_1) \cap X'_1 = \emptyset$ and at least three $A'$-neighbours of $y_0$ fall outside $\phi(X'_0 \cup X'_1)$, a contradiction. Thus $\phi|_{X \cup X'}$ is a cycle rotation (without any reflection). Moreover, it is the identity rotation for otherwise we obtain the contradiction $|\phi(\Gamma_{\mathrm{red}}(y_0)) \setminus \Gamma_{\mathrm{red}}(y_0)| \geq 3$. (The latter inequality holds because in Phase 2 we excluded the possibility that the indexes of the red neighbours of $y_0$ form two arithmetic progressions.)

Now, for any two vertices of $Y \cup Y'$ the hamming distance between their $A$-neighbourhoods is at least 4. This clearly implies that $\phi$ is the identity bijection. ∎

Claim 1 implies in particular that $A$ has no non-trivial automorphism. Thus $\psi = \psi_{r_3}$ is uniquely determined, where $\psi_r$ denotes the red-blue isomorphism after $r$ rounds. The bipartition $V \cup V'$ is clearly preserved (or reversed) by $\psi$ so we have the induced action $\psi^*$ on $E(K_{m,n})$.

**Claim 2.** For any $(r_3+1)$st move $e$ of $\mathcal{A}$, the player $\mathcal{B}$ is forced to reply with $\psi^*(e)$.

*Proof.* By Claim 1 any bijection $\phi$ with $\phi^*(A) \subset A_{r_3+1}$ preserves $X \cup X'$. As $X \cup X'$ induces in $A_{r_3+1}$ a cycle with at most one chord added, we have $\phi^*(C) = C$ and, again by Claim 1, $\phi$ is the identity map. This means that $A$ is the only subgraph of $A_{r_3+1}$ isomorphic to $A$ and the analogous claim holds for the blue edges. Thus $\psi^*_{r_3+1}(A) = B_{r_3}$, which implies that $\psi_{r_3+1} = \psi$. Now, $\psi^*$ must map $\{e\} = A_{r_3+1} \setminus A$ into $B_{r_3+1} \setminus B_{r_3}$, as required. ∎

As the total number of edges $mn$ is odd, some $\psi^*$-orbit $D \subset E(K_{m,n})$ has the odd number of elements.

If at least one element of $D$ is already coloured, then we can find via a parity argument an uncoloured $e \in D$ such that $\psi^*(e)$ is red. Now, $\mathcal{A}$ colours $e$ red and $\mathcal{B}$ loses by Claim 2.

Suppose that no edge of $D$ has been coloured. If $D = \{e\}$, then $\psi^*(e) = e$ and $\mathcal{A}$ wins by choosing $e$ by Claim 2. Suppose that $|D| > 1$ and let $e \in D$. Now $e' = \psi^*(e)$ and $e'' = (\psi^*)^{-1}(e)$ are two distinct edges belonging to $D$.

Suppose first that $e = \{x_i, x'_j\}$ and that both $C \cup \{e, e'\}$ and $C \cup \{e, e''\}$ have a $2l$-cycle passing through the edge $e$. Trivial considerations show that

$$\{e', e''\} = \Big\{ \{x_j, x'_{i-1}\}, \{x_{j+1}, x'_i\} \Big\}. \tag{6}$$

(Of course, we do all index calculations modulo $l$.) If $\psi(V) = V$, then (6) means that for some $\delta = \pm 1$ we have $\psi^\delta(x_i) = x_{j+1}$, $\psi^\delta(x'_j) = x'_i$, $\psi^{-\delta}(x_i) = x_j$ and $\psi^{-\delta}(x'_j) = x'_{i-1}$. But then $\psi^\delta$ maps the red edge $\{x_j, x'_j\}$ into the red edge $\{x_i, x'_i\}$, a contradiction. In the same way we obtain a contradiction in the case $\psi(V) = V'$.

Thus, by the first part of Claim 1, we can assume that we can recover $C$ from knowing, for example, $A \cup \{e, e'\}$. Now, $\mathcal{A}$ selects $e'$ to which, by Claim 2, $\mathcal{B}$ is forced to reply by colouring $\psi^*(e')$. Then $\mathcal{A}$ selects $e \neq \psi^*(e')$. Claim 1 implies that $A$ is the unique subgraph of $A \cup \{e, e'\}$ isomorphic to $A$ and the argument of Claim 2 shows that $\mathcal{B}$ loses.

Let us count the total number $r_4$ of rounds. If $n = m = l+1$, then

$$r_4 \le 2(m-1) + 9 + 12 + 2 \le 3(n-m+k) + 2m + 15,$$

as $k \ge 8$. Otherwise,

$$r_4 \le 2l + 9 + 5(m-l-1) + k(n-l) + 2 \le k(n-m+3) + 2m + 15,$$

where we used the inequality $l \ge m-3$. This finishes the proof of Theorem 1 as $L_{\mathrm{sym}}(K_{m,n}) \le r_4 - 1$.

## 3. Concluding Remarks

It is not hard to convert our strategy into an algorithm which has running time $O(mn)$ per one move of $\mathcal{A}$.

Unfortunately, not for every pair $(m, n)$ an integer $k$ satisfying (2) can be found. For example, if $n > 2^m$, then any subgraph of $K_{m,n}$ contains two vertices in $V'$ with the same neighbourhood in $V$, which ruins our strategy. The value of $n$ can range up to

$$\max_{k\ge 8} \frac{(m-2k-3)^k}{(k+1)!\, m} = (1.531... + o(1))^m.$$

The optimal choice is to take the smallest integer $k \ge 8$ satisfying (2). If $m$ is large, then the following formulae can be routinely verified. If $\log n/\log m < 7 - o(1)$, then $k = 8$. If $7 - o(1) \le \log n/\log m = O(1)$, then

$$\frac{\log n}{\log m} + 1 < k < \frac{\log n}{\log m} + 3.$$

If $O(\log m) < \log n = o(m)$, then

$$k = (1 + o(1))\, \frac{\log n}{\log m - \log\log n}.$$

If $(1 + o(1))^m < n < (1.531... + o(1))^m$, then $k = (x_0 + o(1))\, m$, where $x_0$ is the smallest positive root of equation

$$((1-2x)\mathrm{e}/x)^x = \mathrm{e}^{\log n/m}.$$

The present best known bounds on $L_{\rm sym}(K_{n,n})$ for odd $n$ are given by (1) and (4). Unfortunately, it seems that a minor modification of our method cannot give any considerable improvement on (4) as any graph $H$ with no non-trivial automorphism has at least $(1 - o(1))\, v(H)$ edges. Indeed, if we fix some large constant $C$, then $H$ has $O(1)$ components with less than $C$ vertices (as no two can be isomorphic), while the remaining components span at least $(v(H) - O(1))\, \frac{c-1}{c}$ edges. Also, one meets difficulties when trying to improve on (1): it is not hard to see that $\mathcal{A}$ can ensure that within first $\frac{n+1}{2}$ rounds there was a position when no blue-red isomorphism could be generated by an involutary automorphism of $K_{n,n}$. Hence, $\mathcal{B}$ must go beyond copycat if he wants to survive for more than $(\frac{1}{2} + o(1))\, n$ rounds.

In fact, we do not even have any solid conjecture what the value of

$$\lim_{n\to\infty} \frac{L_{\rm sym}(K_{2n+1,2n+1})}{n}$$

is (if the limit exists).

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