On linear functorial operators extending pseudometrics

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Abstract. For a functor $F \supset Id$ on the category of metrizable compacta, we introduce a conception of a linear functorial operator $T = \{T_X : Pc(X) \to Pc(FX)\}$ extending (for each X) pseudometrics from X onto $FX \supset X$ (briefly LFOEP for F). The main result states that the functor SP^n_G of G-symmetric power admits a LFOEP if and only if the action of G on $\{1, \dots, n\}$ has a one-point orbit. Since both the hyperspace functor exp and the probability measure functor P contain SP^2 as a subfunctor, this implies that both exp and P do not admit LFOEP.

Keywords: linear functorial operator extending (pseudo)metrics, the functor of G-symmetric power

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The results of this note are related to recent authors' results [Ba] and [Pi] stating that every metrizable compact pair $X \subset Y$ admits a linear operator $T : Pc(X) \to Pc(Y)$ extending continuous pseudometrics from X onto Y. In the light of this result the question arises naturally: given a functor F putting in correspondence to each metrizable compactum X a space $FX \supset X$ is it possible for every X to define in some natural way a linear operator $T_X : Pc(X) \to Pc(FX)$ extending pseudometrics from X onto FX? This question is of interest because for many classical constructions such as the hyperspace functor exp or the functor P of probability measures all known operators extending (pseudo)metrics (e.g. the Hausdorff extension of metrics onto $\exp X$ or Kantorovich extension of metrics onto PX) are not linear. In this note we show that it is not occasionally and these functors *do not admit* any natural (or functorial) linear operator extending pseudometrics from X onto FX. This will be shown by proving that for $n > 1$ the symmetric power functor SP^n does not admit such a linear functorial extension operator, and noticing that both exp and P contain SP^2 as a subfunctor.

Now let us give precise definitions. For a topological space X by $Pc(X)$ the set of all continuous pseudometrics on X is denoted. The set $Pc(X)$ has the cone structure, i.e. given $t \in [0, \infty)$ and $p, p' \in Pc(X)$ we have $tp \in Pc(X)$ and $p + p' \in Pc(X)$.

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Let X, Y be two topological spaces. We say that a map $T : Pc(X) \to Pc(Y)$ is a *linear operator* if for every $t \geq 0$ and $p, p' \in Pc(X)$ we have $T(tp) = tT(p)$ and $T(p+p') = T(p)+T(p')$. In case $X \subset Y$ we call $T : Pc(X) \to Pc(Y)$ an *extension operator* if for every $p \in Pc(X)$ the pseudometric Tp extends p. Notice that any continuous map $f : X \to Y$ induces a linear operator $f^* : Pc(Y) \to Pc(X)$ acting by $f^*(p) = p(f \times f)$ for $p \in Pc(Y)$.

By $\mathcal{T}op$ we denote the category of all topological spaces and their continuous maps and by $\mathcal{MC}omp$ its full subcategory consisting of all metrizable compacta. *A natural transformation* $\eta : F \to G$ between two functors $F, G : \mathcal{MC}omp \to \mathcal{T}op$ is a family of morphisms (= continuous maps) $\eta = \{\eta_X : FX \to GX\}$ such that for every morphism $f : X \to Y$ in $\mathcal{MC}omp$ we get $Gf \circ \eta_X = \eta_Y \circ Ff$. A natural transformation $\eta = \{\eta_X\} : F \to G$ with all components η_X being embeddings is called an embedding of functors. This is denoted by $F \subset G$ and F is called a subfunctor of G. In this note we consider only functors F containing the identity functor Id as a subfunctor. Note that if F preserves one-point spaces then F admits at most one natural transformation $\eta : Id \to F$, see [Fe$_1$] or [FF].

Now we introduce the conception of a functorial operator extending pseudometrics, the central conception in this paper. Let $F : \mathcal{MC}omp \to \mathcal{T}op$ be a functor with $Id \subset F$. A collection $T = \{T_X : Pc(X) \to Pc(FX)\}$ of extension operators is called a *functorial operator extending pseudometrics* (briefly FOEP) for the functor F if for every morphism $f : X \to Y$ in $\mathcal{MC}omp$ the following diagram is commutative

$$\begin{array}{ccc} Pc(Y) & \xrightarrow{T_Y} & Pc(FY) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle (Ff)^*} \\ Pc(X) & \xrightarrow{T_X} & Pc(FX). \end{array}$$

If, moreover, all T_X's are linear operators, then $T = \{T_X\}$ is called a *linear functorial operator extending pseudometrics* (briefly LFOEP) for F.

Notice that the introduced conceptions are near to the notion of a metrizable functor [Fe$_2$].

Classical examples of FOEP are the Hausdorff extension of (pseudo)metrics from a compactum X onto the hyperspace $\exp X$ of all non-empty compact sets in X and Kantorovich extension of (pseudo)metrics from X onto the space PX of probability measures on X, see [FF] or [Fe$_2$]. These operators are not linear (and as we will see later they cannot be linear). An important example of a functor admitting a linear FOEP is the functor M putting in corresponding to a compactum X the space $M(X)$ of all Borel-measurable functions $[0,1] \to X$ [BP]. A linear FOEP for the functor M can be defined by the formula

$$T_X(d)(f,g) = \int_0^1 d(f(t), g(t))\, dt, \quad \text{where } f, g \in M(X) \text{ and } d \in Pc(X).$$

The functor $M(X)$ and defined above LFOEP play a crucial role in the construction of linear extension operators in [Za].

Therefore, the question is: which functors admit and which do not admit linear FOEP's? It turns out that depends much on relationships between F and the functors SP^n_G of G-symmetric power which definitions we are going to recall now.

Let $G \subset S_n$ be a subgroup of the symmetric group S_n (i.e. the group of all bijections of the set $\mathbf{n} = \{1, \dots, n\}$). For a compactum X let $SP^n_G(X)$ be the quotient space of X^n with respect to the equivalence relation $\sim$: $(x_1, \dots, x_n) \sim (y_1, \dots, y_n)$ iff $(x_1, \dots, x_n) = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$ for some $\sigma \in G$. Further by $[x_1, \dots, x_n] \in SP^n_G(X)$ the equivalence class of an element $(x_1, \dots, x_n) \in X^n$ is denoted. It is easily seen that the construction of SP^n_G determines a functor on the category $\mathcal{MC}omp$.

The principal result of this note is the following

Theorem. *The functor SP^n_G admits a linear functorial operator extending pseudometrics if and only if the action of G on $\{1, \dots, n\}$ has a one-element orbit (i.e. $G \cdot k = \{\sigma(k) \mid \sigma \in G\} = \{k\}$ for some $k \in \{1, \dots, n\}$).*

Applications of this theorem rely on the following simple

Proposition. *Let $F_1, F_2 : \mathcal{MC}omp \to \mathcal{T}op$ be two functors such that each F_i, $i = 1, 2$, preserves point and contains the identity functor Id. If there is a natural transformation $\varphi = \{\varphi_X\} : F_1 \to F_2$ and the functor F_2 admits LFOEP then F_1 admits LFOEP either.*

Proof: For $i = 1, 2$ denote by $\eta_i : Id \to F_i$ the functorial embedding. Since F_i preserves point, the transformation η_i is unique. Hence $\varphi \circ \eta_1 = \eta_2$.

If $T_2 = \{T_{2,X} : Pc(X) \to Pc(F_2 X)\}$ is a LFOEP for F_2 then letting $T_{1,X}(d) = T_{2,X}(d)(\varphi_X \times \varphi_X)$ for $X \in \mathcal{MC}omp$ and $d \in Pc(X)$, we obtain a LFOEP $T_1 = \{T_{1,X}\}$ for F_1. $\square$

Since both functors exp and P contain the symmetric square functor $SP^2 = SP^2_{S_2}$ as a subfunctor, Theorem and Proposition imply

Corollary. *The functors* exp *and P on $\mathcal{MC}omp$ do not admit any linear functorial operator extending pseudometrics.*

Proof of Theorem

To prove the theorem we will need two simple lemmas first.

Lemma 1. *Suppose for a finite space $X = \{x_1, \dots, x_m\}$ and reals a_{ij}, $1 \le i < j \le m$, the equality*

$$\sum_{i<j} a_{ij} d(x_i, x_j) = 0, \tag{1}$$

holds for every metric d on X. Then all a_{ij} are equal to 0.

Proof: Choose two different metrics on X, d_1 and d_2: in the first metric all distances between different points are equal to 1, the second is the same, except the distance between x_i and x_j is equal to 2. Subtracting the corresponding equalities (1), we obtain $a_{ij} = 0$. □

Lemma 2. *Any pseudometric d on a finite $X = \{x_1, \dots, x_m\}$, $m > 2$, may be expressed as a linear combination of E_{ij} (E_{ij} is defined as a pseudometric on X gluing together points x_i and x_j, while all other non-zero distances are equal to 1), i.e. there exist real e_{ij} such that*

$$d = \sum_{i<j} e_{ij} E_{ij}. \tag{2}$$

PROOF: Evaluating both sides of (2) on the pair (x_k, x_l) we receive the following linear system of equations (in terms of e's):

$$d(x_k, x_l) = \sum_{i<j} e_{ij} E_{ij}(x_k, x_l) = -e_{kl} + \sum_{i<j} e_{ij}. \tag{3}$$

Summing the above equality over all pairs (x_k, x_l) we have $\sum_{i<j} d(x_i, x_j) = (\frac{m^2-m-2}{2}) \sum_{i<j} e_{ij}$ and finally (taking into the account (3)):

$$e_{kl} = \frac{2\sum_{i<j} d(x_i, x_j)}{m^2 - m - 2} - d(x_k, x_l). \tag{4}$$

□

PROOF OF THE THEOREM: Suppose that there is a one-element orbit: for some $k \, \forall g \in G \; g(k) = k$. We may define $T = (Pr_k)^*$, where $Pr_k : SP^n_G \to Id$ is natural transformation of functors, taking $[x_1, \dots, x_n]$ to x_k. The explicit formula looks as (here and further on we omit sometimes subscripts for the clarity of language):

$$T(d)([x_1, \dots, x_n], [y_1, \dots, y_n]) = d(x_k, y_k).$$

The routine verification will show that so defined T is a desired LFOEP.

Conversely, suppose that such operator T exists and there is no stationary elements in $\mathbf{n}$ with respect to G. Consider some finite X, $|X| \geq 2n$ and calculate $T(d)$ on elements $[x_1, \dots, x_n]$ and $[y_1, \dots, y_n]$ where all x_i and y_i are different. Taking into the account (2) and (4) and using the linearity of T, we have:

$$\begin{aligned} T(d)([x_1, \dots, x_n], [y_1, \dots, y_n]) &= \sum_{i<j} e_{ij} T(E_{ij})([x_1, \dots, x_n], [y_1, \dots, y_n]) \\ &= \sum_{i,j} a_{ij} d(x_i, y_j) + \sum_{i<j} b_{ij} d(x_i, x_j) + \sum_{i<j} c_{ij} d(y_i, y_j) \end{aligned} \tag{5}$$

for some real constant a_{ij}, b_{ij}, c_{ij}. Note, that is general all coefficients e_{ij} are not necessarily nonnegative, but formula (5) still holds. Really, if for pseudometrics

d_1 and d_2 the function $d_1 - d_2$ (pointwise subtraction) is a pseudometric, then $T(d_1) = T(d_2 + (d_1 - d_2)) = T(d_2) + T(d_1 - d_2)$, so $T(d_1 - d_2) = T(d_1) - T(d_2)$, for any linear T.

From functoriality of T we can read that formula (5) is true for all X, d and distinct $x_i, y_i \in X$: just consider embeddings of some fixed space with $2n$ points mapping it onto $\{x_1, \ldots, x_n, y_1, \ldots y_n\}$. It must be true for all (not necessarily distinct) x_i, y_i as $T(d)$ is continuous function on X^2: take appropriate connected metric space, and consider limits of both sides of (5) when some of x's and y's approach each other.

Now, $T(d)$ as a pseudometric is symmetric. So, swap y and x in (5) and compare. We obtain:

$$\sum_{i<j} d(x_i, x_j)(b_{ij} - c_{ij}) + \sum_{i<j} d(y_i, y_j)(c_{ij} - b_{ij}) + \sum_{i,j} d(x_i, y_j)(a_{ij} - a_{ji}) = 0$$

and, according to Lemma 1,

$$b_{ij} = c_{ij} \text{ and } a_{ij} = a_{ji}. \tag{6}$$

Next, $T(d)([x_1, \ldots, x_n], [x_1, \ldots, x_n]) = 0$. After simple transformations we obtain: $\sum_{i<j} d(x_i, x_j)(a_{ij} + a_{ji} + b_{ij} + c_{ij}) = 0$. Therefore (applying (6)):

$$a_{ij} = a_{ji} = -b_{ij} = -c_{ij}. \tag{7}$$

Suppose that we have $g \in G$ which moves k to l. Then, the two elements $[x, \ldots, x, z, x, \ldots, x]$ with one z at k-th and l-th positions respectively are equivalent, and therefore, for every $[y_1, \ldots, y_n]$ formula (5) should yield the same values. After routine transformations we obtain: $\sum_i d(z, y_i)(a_{ki} - a_{li})$+(other terms) $= 0$. Therefore for all i $a_{ki} = a_{li}$. So, assuming (6) $a_{ij} = a_{kl}$, if i and k are G-related and j and l are G-related. The same is true for b's and c's.

If we have a 2-element orbit (let it be $\{1, 2\}$) then consider the following three points $[x, x, z, \ldots, z]$, $[y, y, z, \ldots, z]$ and $[x, y, z, \ldots, z]$ and use all that we know about the coefficients:

$$\begin{aligned} T(d)([x, x, z, \ldots, z], [y, y, z, \ldots, z]) &= 4a_{11} d(x, y), \\ T(d)([x, x, z, \ldots, z], [x, y, z, \ldots, z]) &= a_{11} d(x, y), \\ T(d)([x, y, z, \ldots, z], [y, y, z, \ldots, z]) &= a_{11} d(x, y). \end{aligned}$$

To satisfy the triangular inequality we must put $a_{11} = 0$.

If we have a k-element ($k > 2$) orbit (let it be $\{1, \ldots, k\}$) then consider the following two points in $SP^n_G(X)$: $[x_1, \ldots, x_k, z, \ldots, z]$ and $[y_1, \ldots, y_k, z, \ldots, z]$ with following original distances in X: all nonzero distances are 1 except $d(x_i, y_j) = 2$, all i, j. Calculate:

$$T(d)([x_1, \ldots, x_k, z, \ldots, z], [y_1, \ldots, y_k, z, \ldots, z]) = (2k - k^2) a_{11}.$$

Since, $2k - k^2 < 0$ when $k > 2$, $a_{11} \leq 0$.

So, if all orbits are non-degenerated then for all i $a_{ii} \leq 0$. Finally, let us for some x, y with $d(x,y) > 0$ find:

$$d(x,y) = T(d)([x,\dots,x],[y,\dots,y]) = \sum_i a_{ii} d(x,y) \leq 0.$$

Contradiction. □

References

[Ba] Banakh T., *AE(0)-spaces and regular operators extending (averaging) pseudometrics*, Bull. Polon. Acad. Sci. Ser. Sci. Math. **42** (1994), 197–206.

[BP] Bessaga C., Pełczyński A., *On the spaces of measurable functions*, Studia Math. **44** (1972), 597–615.

[Fe_1] Fedorchuk V.V., *On some geometric properties of covariant functors* (in Russian), Uspekhi Mat. Nauk **39** (1984), 169–208.

[Fe_2] Fedorchuk V.V., *Triples of infinite iterates of metrizable functors* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), 396–418.

[FF] Fedorchuk V.V., Filippov V.V., *General Topology. Principal Constructions* (in Russian), Moscow Univ. Press, Moscow, 1988.

[Pi] Pikhurko O., *Extending metrics in compact pairs*, Mat. Studiï **3** (1994), 103–106.

[Za] Zarichnyi M., *Regular linear operators extending metrics: a short proof*, Bull. Polish. Acad. Sci. **44** (1996), 267–269.

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