MA 1250

THE UNIVERSITY OF WARWICK

FIRST YEAR EXAMINATION: June 2014

INTRODUCTION TO GEOMETRY

Time Allowed: **1 hour**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 2 QUESTIONS.
If you have answered more than the required 2 questions in this examination, you will only be given credit for your 2 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) State three criteria for triangles to be congruent. **[6]**

b) Let ΔABC be a triangle. Show that the following are equivalent:

- the edges AB and AC have the same length;
- the angles $A\widehat{B}C$ and $B\widehat{C}A$ are congruent. [4]

c) Let ΔABC and $\Delta A'B'C'$ be triangles in the plane such that the internal angles in A, B and C in ΔABC coincide with the internal angles in A', B' and C' in $\Delta A'B'C'$ respectively. Suppose that the length of one edge of ΔABC coincides with the length of one edge of $\Delta A'B'C'$. Must the triangles ΔABC and $\Delta A'B'C'$ be congruent? Justify your answer. **[4]**

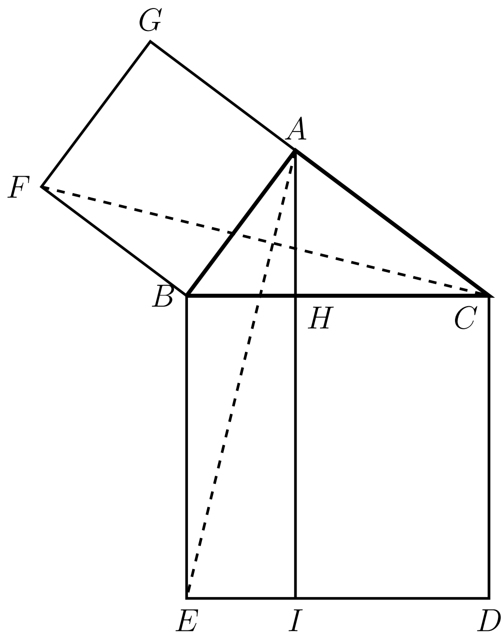
d) Let A be a point in the plane and let ℓ, m be distinct half lines originating from A. Show that the bisector of the angle in A formed by ℓ, m is the locus of points equidistant from ℓ and m. **[6]**

e) Let ΔABC be a triangle in the plane. Show that the bisectors of the three internal angles of the triangle meet in a point. Deduce that there is a circle admitting the three edges of the triangle ΔABC as tangent lines. **[5]**

 Question 2 continued overleaf

2. a) State Pythagoras' Theorem. **[4]**

Let ΔABC be a triangle with $C\widehat{A}B$ a right angle. Let $BCDE$ be the square with side BC and let $ABFG$ be the square with side AB. Draw the height of ΔABC with respect to the side BC and denote by H the intersection of this height with the edge BC. Extend the height AH until it meets the segment DE in a point I.

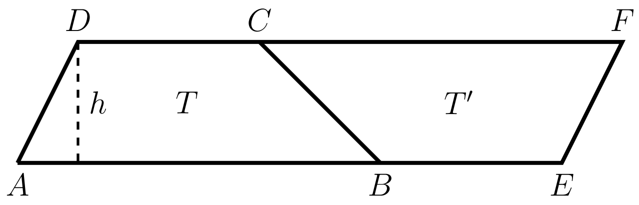


b) Show that the triangles ΔFBC and ΔABE are congruent. **[9]**

c) Prove that the area of triangle ΔFBC is half the area of square $ABFG$ and that the area of triangle ΔABE is half the area of rectangle $BHIE$. Draw a similar conclusion (without proof) for the square on AC. **[8]**

d) Conclude that the area of the square with edge BC is the sum of the areas of the squares with edge AB and AC. **[4]**

3. a) Let T be a trapezium in the plane, that is, a quadrilateral with two parallel edges. Label the vertices of T by $ABCD$ in such a way that the edges AB and DC are parallel. Rotate the trapezium T around the midpoint of the edge BC by $180°$ to obtain a second trapezium $T' = BEFC$ congruent to T.



(i) Show that the angles $A\widehat{B}E$ and $F\widehat{C}D$ are straight angles. **[6]**

(ii) Show that the edges AD and EF are parallel. **[6]**

(iii) Deduce that the area of T is one half the area of the parallelogram $AEFD$, and compute this area in terms of the lengths of the edges AB, CD and of the distance h between the line AB and the vertex D. **[5]**

b) Let S be a sphere in space and let O denote the centre of S. Determine the set of points of S that are fixed by the following isometries (proofs are not required).

(i) The antipodal map, sending each point P of the sphere to the point P' at the opposite end of the diameter of the sphere containing P. **[2]**

(ii) A rotation of angle α, with $0 < \alpha < 2\pi$, around an axis ℓ through the centre O of the sphere. **[3]**

(iii) A reflection in a plane Π through the centre O of the sphere. **[3]**

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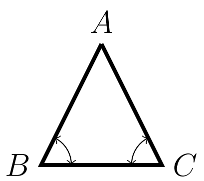
Model Solution No: 1

a)
- Two triangles ΔABC and $\Delta A'B'C'$ are congruent if the edges AB and $A'B'$ have the same length, the edges BC and $B'C'$ also have the same length and the angles $A\hat{B}C$ and $A'\hat{B}'C'$ are congruent. **[2]**
- Two triangles ΔABC and $\Delta A'B'C'$ are congruent if the angles $A\hat{B}C$ and $A'\hat{B}'C'$ are congruent, the angles $B\hat{C}A$ and $B'\hat{C}'A'$ are also congruent and the edges BC and $B'C'$ have the same length. **[2]**
- Two triangles ΔABC and $\Delta A'B'C'$ are congruent if the equalities $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|BC| = |B'C'|$ hold. **[2]**

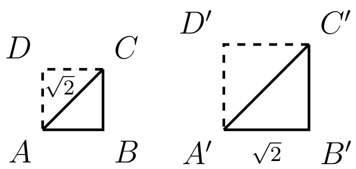
b)



Suppose that the edges AB and AC have the same length. The triangles ΔCAB and ΔBAC are congruent by the side-angle-side criterion, since the angles $C\hat{A}B$ and $B\hat{A}C$ coincide and the two adjacent edges have the same length by hypothesis. It follows that the internal angles $A\widehat{B}C$ and $B\widehat{C}A$ are congruent. **[2]**

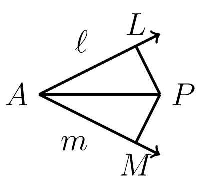
Conversely, suppose that the angles $A\widehat{B}C$ and $B\widehat{C}A$ are congruent. The triangles ΔABC and ΔBCA are congruent by the angle-side-angle criterion, since the sides BC and CB coincide and the two adjacent angles are congruent by hypothesis. It follows that the sides AB and AC are congruent. **[2]**

c)



The two triangles do not have to be congruent: suppose that $ABCD$ is a square of side-length 1 and $A'B'C'D'$ is a square of side-length $\sqrt{2}$. The two triangles ΔABC and $\Delta A'B'C'$ are not congruent, but they satisfy the hypothesis: their angles are congruent and the edges BC and $A'B'$ have the same length. **[4]**

d)



Suppose that P is a point on the bisector of the angle in A formed by ℓ, m. Let L be the point on ℓ closest to P and let M be the point on m closest to P. By a result in the lectures (Lemma 11) the internal angles $P\hat{L}A$ and $P\hat{M}A$ are right angles. Since the angles $P\hat{A}L$ and $M\hat{A}P$ are congruent by assumption, it follows that the angles $L\hat{P}A$ and $A\hat{P}M$ are also congruent. By the angle-side-angle criterion, the triangles ΔPLA and ΔPMA are congruent and it follows that the segments PL and PM have the same length, as required. **[3]**

Conversely, suppose that P is a point on the plane equidistant from ℓ and m. Let L be the point on ℓ with minimum distance from P and let M be the point on m with minimum distance from m, so that the segments PL and PM have the same length. The triangle ΔPLM is isosceles, and by part (b) it follows that the angles $M\hat{L}P$ and $P\hat{M}L$ are congruent. We deduce that the angles $L\hat{M}A$ and $A\hat{L}M$ are congruent, since they are complementary to congruent angles. We deduce that the triangle ΔALM is isosceles, so that the lengths of the edges AL and AM coincide. Again by part (b) we obtain that the angles $A\hat{M}L$ and $L\hat{M}A$ are congruent. By the side-side-side criterion, the triangles ΔPLA and ΔPMA are congruent, so that the angles $P\hat{A}L$ and $M\hat{A}P$ are congruent. **[3]**

e) By the standard convention, triangles are non-degenerate, so that the bisectors of the angles $A\hat{B}C$ and $B\hat{C}A$ are not parallel: denote by P be the common point of these bisectors. By part (d), the point P is equidistant from the lines AB and BC, as well as from the lines BC and CA. It follows that P is equidistant from the lines CA and AB, so that by part (d) again, the point P is also on the bisector of the angle $C\hat{A}B$. **[3]** It follows that the circle centred at P and with radius the distance between P and the line AB admits the three edges of ΔABC as tangent lines. **[2]**

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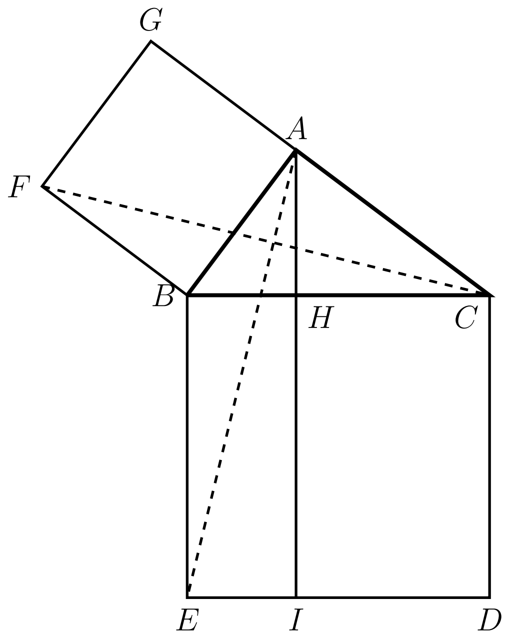
Model Solution No: 2

a) In a right triangle, the square of the hypothenuse is equal to the sum of the squares of the two remaining sides. **[4]**



b) The edges FB and AB have the same length, by construction; for the same reason, the edges BC and BE have the same length. **[3]** The angle $C\hat{B}F$ is the sum of the angle $C\hat{B}A$ and of a right angle; the same is true of the angle $E\hat{B}A$. **[3]** It follows from the side-angle-side criterion that the triangles ΔFBC and ΔABE are congruent. **[3]**

c) The angle $C\hat{A}G$ is a straight angle, since the angles $C\hat{A}B$ and $B\hat{A}G$ are right angles by assumption. **[2]** It follows that the distance between the point C and the line FB is equal to the length of the segment FG. It follows that the area of the triangle ΔFBC equals $\frac{1}{2}|FB| \cdot |FG| = \frac{1}{2}|FB|^2$. **[2]** The lines BE and AI are parallel, since they are both perpendicular to the line BC. **[2]** It follows that the distance between the line BE and the point A equals the length of the segment BH. We conclude that the area of the triangle ΔABE is $\frac{1}{2}|BE| \cdot |BH|$, as required. **[2]**

d) By part (c) the square of the length $|AB|$ is equal to the area of the rectangle $BEIH$. Applying the same argument to the edge AC we find that the square of the length $|AC|$ is equal to the area of the rectangle $HIDC$. **[2]** Since the two rectangles $BEIH$ and $HIDC$ have only one edge in common and their union is the square $BEDC$, the result follows. **[2]**

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Model Solution No: 3

a) (i) The line AB is parallel to the line CD by assumption and it is also parallel to the line CF by construction. It follows from Playfair's version of the Parallel Postulate that the lines CD and CF coincide, since they are both parallel to the line AB and share the point C. In particular, the angle $F\hat{C}D$ is straight. A similar argument shows that also the angle $A\hat{B}E$ is straight. **[3]**

(ii) By part (i), the lines AE and DF are parallel. By construction, the angles $E\hat{A}D$ and $D\hat{F}E$ are supplementary, so that, by the Parallel Postulate, the lines AD and EF are parallel, as required. **[3]** **[2]** **[2]**

(iii) The area of T is half the area of the sum of the areas of T and of T'. From parts (i) and (ii), we know that the quadrilateral $AEFD$, union of T and T', is a parallelogram and therefore the area of T is $\frac{1}{2}|AE| \cdot h = \frac{1}{2}(|AB| + |CD|) \cdot h$. **[2]**

[5]

b) (i) The distance between a point on S and its image under the antipodal map equals the diameter of the sphere: no point of S is fixed in this case. **[2]**

(ii) The only fixed points of a non-trivial rotation are the points on the axis of rotation: these are the two antipodal points on S at the intersection of ℓ and S. **[3]**

(iii) The only fixed points of a reflection are the points on the reflecting plane: these are the points of S on the great circle $\Pi \cap S$. **[3]**