MA133 Differential Equations

Week 7 Tutorial: Second Order Linear ODEs

What we've been doing.

- **Week 5** we finished off second order ODEs, so students should now be in a position to tackle the question below;
- **Week 6** we introduced difference equations and solved first and second order linear DEs with constant coefficients (homogenous and inhomogeneous);
- **Week 7** we'll be looking at first order nonlinear difference equations by way of the Logistic Equation.. would appreciate it if tutors who like these things don't get carried away and steal my thunder!

Exam question, example of modelling Something students seem to struggle with is turning a given situation into a set of equations and then solving them (plus interpreting the solution back into the context of the question). It would be useful to work through the following exam question. Part a) is simple bookwork, make sure that they can answer this, if they can't they'll need to go back to their notes and go over it again! Part b) is the crux of the question, and the bit they'll need most time to mull over and discuss. Get them to draw a diagram. In recent assignment many were also confused by frequency/period of the solution, so some discussion here may help. Rest should be straightforward, if running out of time leave as an exercise. [Note: I include solutions for you at the end] General comments about exam questions would also be useful, e.g. which bits might be worth more marks than others.

In particular they seem to get put off by all the words, so get them to break it up and discuss what information each bit is giving them and where it goes in the equation. I often find that the more words, the fewer students attempt a particular question, even though these questions tend to be the easier ones to actually answer.

1. For the mass-spring system

$$a\ddot{x} + b\dot{x} + cx = 0$$

   (a) Which term of this equation corresponds to the coefficient of the *damping* in the system?

   (b) In terms of the parameters a, b and c, under what conditions is the system modelled by the above equation *critically damped*?

   (c) In terms of the parameters a, b and c, under what conditions is the system modelled by the above equation *over-damped*?

2. A cubic block of side L and density $0 < \rho < 1$ per unit volume is floating in water of density 1 per unit volume.

   If the block is slightly depressed and then released, it oscillates in the vertical direction due to a buoyant force acting upwards and gravity pulling it downwards. i.e. the distance u that the bottom of the cube is from the surface varies periodically with time.

   The downwards force is equal to mass times gravitational acceleration, g, where mass is volume multiplied by density.

   Assuming there is no damping from the air or water, derive the differential equation of the motion, solve it for general initial conditions, and determine the period of the oscillation.

   Hint: Use Archimedes' principle that an object that is completely or partly submerged in a fluid is acted on by a vertical (buoyant) force equal to the weight of the displaced fluid. Weight is equal to the mass multiplied by g the gravitational acceleration. Pay particular attention to the signs (directions) of distances and forces.

3. If the block is floating at rest (i.e. there is no vertical motion), what fraction of the block is under water?

4. If we now assume that damping occurs due to the velocity that is proportional to the surface area, S of the vertical sides of the cube (damping is from both air and water and is assumed to be the same for both) then the damping term is given by $kS\dot{u}$. Show that critical damping occurs when

$$k = \frac{\sqrt{\rho g L}}{2}.$$

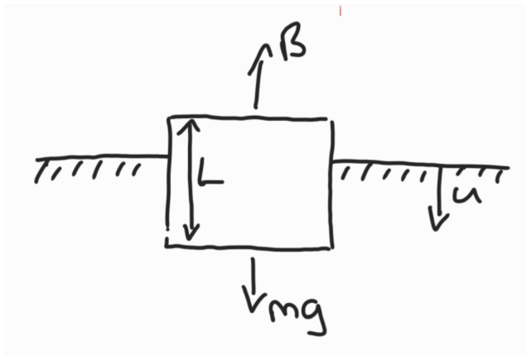
Additional if time: Matrices If you have time at the end, in week 8 we will start discussing systems of ODEs, so some discussion on 2×2 matrices would be beneficial. Making sure they're happy with determinants, inverses, eigenvalues and eigenvectors (distinct real eigenvalues). Giving some examples where the entries in the matrix are functions rather than constants, and then evaluating at specific values, will start to get them used to the idea of the Jacobian (but only use functions of one variable). They have had sight of this earlier in the term, but from experience many will still feel disconcerted by it.

I will be putting together a video for them to cover this anyway, so don't worry if you don't get around to this.

Solution to exam question

1. For the mass-spring system
   (a) damping term is $b\dot{x}$
   (b) critically damped if $b^2 - 4ac = 0$
   (c) overdamped if $b^2 > 4ac$

2. Diagram should look something like:



We have $m\ddot{u} = F$ so

$$\rho L^3 \ddot{u} = -L^2 ug + \rho g L^3$$

or

$$\ddot{u} + \frac{g}{\rho L} u = g.$$

This gives solution

$$u(t) = A \cos t\sqrt{\frac{g}{\rho L}} + B \sin t\sqrt{\frac{g}{\rho L}} + \rho L$$

Period is $2\pi\sqrt{\frac{\rho L}{g}}$.

3. If the block is floating at rest then this means $\ddot{u} = 0$ so that $\frac{g}{\rho L}u = g$ or $u = \rho L$, so fraction is ρ.

4. If we now assume that damping occurs due to the velocity that is proportional to the surface area then damping is $kS\dot{u}$ where $S = 4L^2$. So now

$$\rho L^3 \ddot{u} + 4kL^2\dot{u} + L^2 ug = \rho g L^3$$

$$\ddot{u} + 4kL^2\dot{u}\frac{g}{\rho L}u = g.$$

Homogeneous case (zero on RHS) gives auxiliary equation $m^2 + \frac{4k}{\rho L}m + \frac{g}{\rho L} =)$ so critical damping when $\frac{16k^2}{\rho^2 L^2} - \frac{4g}{\rho L} = 0$ giving $k^2 = \frac{\rho L g}{4}$ and so

$$k = \frac{\sqrt{\rho g L}}{2}$$