RECONSTRUCTING GENERAL PLANE QUARTICS FROM THEIR INFLECTION LINES

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Abstract. Let $C$ be a general plane quartic and let $\text{Fl}(C)$ denote the configuration of inflection lines of $C$. We show that if $D$ is any plane quartic with the same configuration of inflection lines $\text{Fl}(C)$, then the quartics $C$ and $D$ coincide. To formalize this result, we extend the notion of inflection lines to singular quartics, using a degeneration argument. We then perform a detailed analysis to show that the configurations associated to singular quartics do not arise as configurations of general quartics.

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Introduction

Let $k$ be an algebraically closed field and let $C \subset \mathbb{P}^2_k$ be a smooth plane quartic curve over $k$. The Plücker formulas show that if $C$ is general, then $C$ admits exactly 24 inflection lines. By inflection line for the smooth quartic $C$ we mean a line $L$ in $\mathbb{P}^2_k$ such that the intersection $L \cap C$ contains of a point $p$ of multiplicity at least 3; if the multiplicity of $p$ is 3, then we say that the line $L$ is a simple inflection line, if it is 4, then we say that $L$ is a hyperinflection line. In the Plücker formulas, inflection lines count for one, hyperinflection lines count for two.

A simple argument shows that, for a fixed line $L$, the space of all quartics admitting $L$ as an inflection line has codimension 2 in the space of all quartics. Thus, knowing the set of all 24 inflection lines of a general quartic imposes $2 \cdot 24 = 48$ conditions on the 14-dimensional space of plane quartic curves. It seems likely therefore that a general plane quartic be uniquely determined by the knowledge of its inflection lines. In this paper we answer affirmatively this question:

is a general plane quartic curve uniquely determined by the set of its inflection lines?

Our result shows that we can determine a general plane quartic curve from the finite set of cusps of its dual. As observed in [CS03a], the reconstructibility of a curve from its dual is an important step in Andreotti’s proof of the Torelli Theorem (see [ACGH85] for details).

The answer to our motivating question may depend on the characteristic of the ground field $k$. We work as much as we can with fields of arbitrary characteristic, however, we were only able to prove reconstructibility over fields of characteristic coprime to 6.

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**Previous work.** This question was inspired by a similar question on bitangent lines to plane quartics, introduced and addressed in a paper by Caporaso-Sernesi [CS03a]. A subsequent paper of Lehavi [Leh05] extended the result to all smooth quartic curves and again work of Caporaso-Sernesi [CS03b] generalized the question and resolved it for general canonical curves.

For the specific question on inflection lines, we had already obtained weaker results. In [PT13], we solved the analogous question for plane cubics; we were also able to reconstruct a general plane quartic from the knowledge of its inflection lines and a simple inflection point. In [PT14], we examined the families of plane quartics found by Vermeulen [Ver83]. From these families, we defined a list of quartics (that we called Vermeulen’s list) that, over fields of characteristic 0, consist of all the smooth plane quartics with at least 8 hyperinflection lines. We were able to show that different quartics in Vermeulen’s list have different configurations of inflection lines (with only two exceptions). We were not able though to argue that the configurations of the quartics in Vermeulen’s list could not also arise from singular quartics.

**Methods.** The key ingredients to follow through our proofs come from the classical invariant theory of binary and ternary quartic forms. In fact, our main tools are the two basic invariants of binary quartic forms of degrees 2 and 3 and their extension to contravariants of degrees 4 and 6 on ternary quartic forms (see Section 1). In particular, the vanishing of the quartic contravariant defines totally harmonic quartics (Definition 2.1). In our approach, the locus of totally harmonic quartics turns out to be the most degenerate.

In Section 2, we classify totally harmonic quartics in all characteristics. As a byproduct, we deduce that the only smooth plane quartics not having 24 inflection lines, counted with multiplicity, are projectively equivalent to the Fermat quartic curve over fields of characteristic 3 (Corollary 2.6).

In Section 3, we introduce and analyze the natural map $\mathcal{F}$ assigning to a general plane quartic its configuration of inflection lines in the Hilbert scheme of points in $\mathbb{P}^2_k$. Our goal is to show that $\mathcal{F}$ is birational onto its image. The locus of totally harmonic quartics is a subset of the indeterminacy locus of $\mathcal{F}$.

We argue that the configurations of lines arising from curves in the indeterminacy locus of $\mathcal{F}$ are disjoint from the configurations of inflection lines of general plane quartics (Theorem 4.7). These arguments are contained in Section 4 and are reminiscent of techniques used in tropical geometry (see for instance the proofs of Propositions 4.5 and 4.6). We have not been able to use the stable reduction technique employed in [CS03a, PT13]. Instead, we replace this step of the reconstruction process by an analysis of singular quartics—the ones containing a line with multiplicity at least three prove the hardest to handle.

In summary, let $F_l \subset \mathbb{P}^2_k$ be a 0-dimensional scheme of length 24, corresponding to a configuration of 24 lines in $\mathbb{P}^2_k$. We show that if $F_l$ arises as a configuration of inflection lines associated to a singular quartic curve, then it satisfies one of the following properties:

- one point of $F_l$ has multiplicity at least 3;
- the reduction of $F_l$ is contained in a line;
- there is a singular quartic containing $F_l$.

None of these properties is satisfied for the configuration of inflection lines of a general quartic.

Next, we compute the generic degree of the map $\mathcal{F}$. First, we identify a non-empty locus of smooth plane quartic curves in which we can explicitly reconstruct the curves from their configuration of inflection lines (Corollary 5.5). It is important for our reasoning that these quartics have hyperinflection lines, since we exploit their presence to perform explicitly the reconstruction.

To transport the reconstructibility from our specific examples to a general plane quartic we use a computation of the tangent space to the locus of plane quartics with preassigned configuration of inflection lines (Lemma 5.8). Hyperinflection lines cause problems to the tangent space computation, but simplify the explicit reconstruction step. We need to strike a balance between the number of hyperinflection lines and simple inflection lines: at least 5 and at most 10 hyperinflection lines are our bounds!

We chose to work with Hilbert scheme as target of the rational map $\mathcal{F}$. Another natural choice would have been to work with the symmetric product $\text{Sym}^{24}(\mathbb{P}^2_k)$. Of course, these two varieties are birational and for a general curve, there is no difference in using one or the other. Nevertheless, we were only able to perform the reconstruction of a plane quartic from its inflection lines explicitly, in the presence of at least 5 hyperinflection lines. From the point of view of the Hilbert scheme, this means that we examined points corresponding to zero-dimensional subschemes with at least 5 non-reduced points. From the point of view of the symmetric product, these are 24-tuple of points with at least 5 repetitions. On these loci, the Hilbert
scheme and the symmetric product differ: the former is smooth, while the latter is singular. This difference is illustrated by Remark 5.7.

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1. Harmonic forms

In this section, we identify the locus of binary quartic forms with a triple root. Classically, this ties in with the invariant theory of binary quartic forms and this is our starting point. We work over the integers, since we do not want to impose restrictions on the characteristic of the ground-field.

Let \( n \) be a non-negative integer and let \( R \) be a commutative ring with identity. Denote by \( \mathbb{P}_n^R \) the projective space of dimension \( n \) over the spectrum \( \text{Spec} \ R \) of \( R \).

Let \( f_0, \ldots, f_4 \) be homogeneous coordinates on \( \mathbb{P}_2^4 \). We identify \( \mathbb{P}_2^4 \) with the projectivization of the space of binary quartic forms associating to the point \([f_0, \ldots, f_4]\) in \( \mathbb{P}_2^4 \) the binary quartic form

\[
f = f_0 y^4 + f_1 y^3 z + f_2 y^2 z^2 + f_3 y z^3 + f_4 z^4.
\]

The group-scheme \( \text{GL}_2 \) acts on forms of \( \mathbb{P}_2^4 \) by the rule

\[
\begin{pmatrix}
a & b \\
c & d 
\end{pmatrix}
\begin{pmatrix} y \\ z 
\end{pmatrix} = \begin{pmatrix} ay + bz \\ cy + dz 
\end{pmatrix}.
\]

There are two basic invariants under the action of \( \text{SL}_2 \subset \text{GL}_2 \) on binary quartic forms:

\[
\begin{align*}
I(f) &= 12f_0f_4 - 3f_1f_3 + f_2^2, \\
J(f) &= 72f_0f_2f_4 - 27f_0f_3^2 - 27f_2^2f_4 + 9f_1f_2f_3 - 2f_2^3.
\end{align*}
\]

Denote by \( V_3 \subset \mathbb{P}_2^4 \) the scheme defined by the vanishing of \( I \) and \( J \) and by \( U_3 \subset \text{Spec} \ (\mathbb{Z}) \) the open subset \( U_3 = \text{Spec} \ (\mathbb{Z}[\frac{1}{3}]) \) of \( \text{Spec} \ (\mathbb{Z}) \). Let \( V'_3 \subset \mathbb{P}_2^4 \) denote the image of the morphism

\[
\mathbb{P}_2^4 \times \mathbb{P}_2^4 \to \mathbb{P}_2^4
\]

\[
([\alpha, \beta], [\gamma, \delta]) \mapsto (ay - \beta z)^3(\gamma y - \delta z)
\]

with the reduced induced subscheme structure. We refer informally to the scheme \( V'_3 \) as the locus of binary quartic forms with a triple root.

**Lemma 1.1.** The morphism \( V_3 \to \text{Spec} \ (\mathbb{Z}) \) is flat over the open set \( U_3 \subset \text{Spec} \ (\mathbb{Z}) \). Moreover, the schemes \( V_3 \) and \( V'_3 \) coincide over the open set \( U_3 \).

**Proof.** To prove flatness, we will show that the scheme \( V_3 \) is the complete intersection of \( I \) and \( J \) of codimension 2 above all primes of \( U_3 \). This suffices by [SPA17, Tag 0123], since \( \text{Spec} \ (\mathbb{Z}) \) is regular and \( V_3 \) is Cohen-Macaulay as a consequence of being a complete intersection on \( U_3 \).

The quadric \( V(I) \) defined by the vanishing of \( I \) has rank 3 at the prime \( (2) \), rank 1 at the prime \( (3) \) and rank 5 at all remaining primes. Therefore \( V(I) \) is irreducible at every prime (and it is non-reduced at \( (3) \)). Thus, prove flatness at a prime \( (p) \), it suffices to show that there are binary quartic forms contained in \( V(I) \) on which \( J \) does not vanish. The form \( I(y, z) = y^4 - yz^3 \) is one such example at all primes \( p \neq 3 \), since \( I(h) = 0 \) and \( J(h) = -27 \). This proves the first part of the statement.

We now prove the second part of the statement. First, it is an immediate check that the forms \( I \) and \( J \) vanish identically on \( V'_3 \), so that the locus \( V'_3 \) is contained in \( V_3 \). Next, observe that \( V_3 \) has pure codimension 2, just like \( V'_3 \), so that the intersection of \( V_3 \) with any 2-dimensional subvariety of \( \mathbb{P}_2^4 \) contains at least one point from each irreducible component of \( V_3 \). We are going to show that the plane \( \pi \) with equations \( f_0 = f_4 = 0 \) intersects \( V_3 \) at smooth points corresponding to quartic forms in \( V'_3 \). It will then follow that every component of \( V_3 \) is also a component of \( V'_3 \) and we will be done.

The intersection of \( V_3 \) and \( \pi \) consists of the forms with coefficients satisfying the system

\[
\begin{align*}
f_0 &= 0, & -3f_1f_3 + f_2^2 &= 0, \\
f_4 &= 0, & f_2(9f_1f_3 - 2f_2^2) &= 0.
\end{align*}
\]
Thus, away from the prime (3), the intersection $V_3 \cap \pi$ consists of the closed points corresponding to the forms $y^3 z$ and $yz^3$ that are clearly in $V_3'$. The Zariski tangent space to $V_3$ at the point $[0, 1, 0, 0, 0]$ has equations $-3f_3 = -27f_4 = 0$, so that $[0, 1, 0, 0, 0]$ is smooth on $V_3$ at every prime of $U_3$. Exchanging the roles of $y$ and $z$, the same is true for the point $[0, 0, 1, 0]$.

\[\Box\]

**Remark 1.2.** At the prime (3), the reductions of the invariants $I$ and $J$ are $f_2^3$ and $f_3^3$, so that they no longer define the locus of binary quartic forms with a triple root. It is an easy computation to show that the flat limit of $V_3$ at (3) is the scheme defined by the ideal $(f_2, f_0 f_4 - f_1 f_3)^2$. The scheme defined by the radical ideal $(3, f_2, f_0 f_4 - f_1 f_3)$ is the reduced subscheme of $V_3$ consisting of polynomials over a field of characteristic 3 with a root of multiplicity at least 3: it is a smooth quadric of dimension 2. We observe that adding to the classical invariants the following two expressions

\[
\begin{align*}
I_3(f) &= \frac{1}{3} (J(f) + 2f_2 I(f)) = 32f_0 f_2 f_4 - 9f_0 f_3^2 - 9f_1^2 f_4 + f_1 f_2 f_3 \\
I_4(f) &= \frac{1}{5} (I_3(f) f_2 + I(f) (f_0 f_4 - f_1 f_3)) = -128f_0^2 f_2^2 + 28f_0 f_1 f_3 f_4 - 3f_0 f_2 f_3^2 - 3f_1^2 f_2 f_4 + f_1^2 f_3^2
\end{align*}
\]

we obtain a subscheme of $V_3$ whose structure map to Spec ($\mathbb{Z}$) is flat also above (3). For our main result, we exclude the case of fields of characteristic 3.

The invariant $I$ is a quadratic form on $\mathbb{P}_k^4$ and we will use the associated bilinear form in our arguments. We denote this bilinear form by $\langle - , - \rangle_2$: if $f = \sum f_i y^{4-i}z^i$ and $g = \sum g_i y^{4-i}z^i$ are forms in $\mathbb{P}_k^2$, then we have

\[
\langle f, g \rangle_2 = 12f_0 g_4 - 3f_1 g_3 + 2f_2 g_2 - 3f_3 g_1 + 12f_4 g_0.
\]

We will also need the identity

\[
I(f + g) = I(f) + \langle f, g \rangle_2 + I(g).
\]

**Remark 1.3.** Let $k$ be an algebraically closed field. The vanishing set of $I$ in $\mathbb{P}_k^4$ decomposes into three $SL_2(k)$-orbits:

- one orbit of forms with no repeated roots;
- the orbit of forms with a root of multiplicity exactly 3;
- the orbit of forms with a root of multiplicity exactly 4.

Indeed, in each $SL_2(k)$-orbit of forms having at least three distinct roots, there is a representative proportional to $yz(y - z)(y - \lambda z)$, for some $\lambda \in k$. We then conclude easily evaluating the invariant $I$ on such a representative. If a form has at most two distinct roots and no root of multiplicity at least 3, then it is equivalent to $\lambda y^2 z^2$, for some non-zero $\lambda \in k$, and $I(\lambda y^2 z^2) = \lambda^2$ does not vanish. We already saw that $I$ vanishes on forms with roots of multiplicity at least 3. If the characteristic of $k$ is different from 3, the form $y^4 - y z^3$ has distinct roots and $I(y^4 - y z^3)$ vanishes. If the characteristic of $k$ is 3, then the form $y^4 + z^4$ has distinct roots and $I(y^4 + z^4)$ vanishes.

We now move on to homogeneous polynomials of degree 4 in three variables $x, y, z$. Let $\mathbb{P}_k^2$ be the projective plane with homogeneous coordinates $x, y, z$. We denote by $\mathbb{P}_k^2^\vee$ the dual projective plane with coordinates $u, v, w$ dual to the coordinates $x, y, z$. Let $\mathbb{P}_k^3^\vee$ denote the projective space of dimension 14 that we think of as the space of quartic curves in $\mathbb{P}_k^2$. We extend the definition of the invariants $I$ and $J$ to contravariant forms $H$ and $K$ on the space $\mathbb{P}_k^3^\vee$.

\[
H(q)(u, v, w) = u^4 I \left( q \left( -\frac{v}{u} y - \frac{w}{u} z, y, z \right) \right),
\]

\[
K(q)(u, v, w) = u^6 I \left( q \left( -\frac{v}{u} y - \frac{w}{u} z, y, z \right) \right).
\]

The expressions above are clearly rational sections of $\mathcal{O}_{\mathbb{P}_k^3^\vee}(4)$ and of $\mathcal{O}_{\mathbb{P}_k^3^\vee}(6)$; an easy check using the invariance of $I$ and $J$ under $SL_2$ shows that they are in fact global sections of the corresponding sheaves: they are ternary forms of degrees 4 and 6 respectively. The coefficients of $H(q)$ and $K(q)$ are forms of degree 2 and 3 respectively in the coefficients of $q$.

**Definition 1.4.** Let $q(x, y, z)$ be a ternary quartic form; we call the ternary quartic form $H(q)(u, v, w)$ the harmonic quartic associated to $q$ and we call the ternary sextic form $H(q)(u, v, w)$ the harmonic sextic associated to $q$. 

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The coefficients of the form $H(q)$ are quadratic forms on the space $\mathbb{P}^{14}_3$ of quartic curves. We shall make repeated use of the associated bilinear map: if $q, r$ are ternary quartic forms, we define

$$
\langle q, r \rangle_3 = u^4 \left\langle q \left( -\frac{v}{u}y - \frac{w}{u}z, y, z \right), r \left( -\frac{v}{u}y - \frac{w}{u}z, y, z \right) \right\rangle_2
$$

and again observe that the equality

$$
H(q + r) = H(q) + \langle q, r \rangle_3 + H(r)
$$

holds.

Let $\ell \subset \mathbb{P}^2_3$ be the line with equation

$$
\ell: u_0x + v_0y + w_0z = 0
$$

and fix a parameterization of $\ell$. Let $q(x, y, z)$ and $r(x, y, z)$ be ternary quartic forms and denote by $q|_\ell$ and $r|_\ell$ the restrictions of $q$ and $r$ to the line $\ell$ using the chosen parameterization: these restrictions are therefore binary quartic forms. It follows at once from the definitions that there is a non-zero constant $\lambda \in k$ such that the identities

$$
\begin{align*}
H(q)(u_0, v_0, w_0) &= \lambda^4 I(q|_\ell) \\
\langle q, r \rangle_3 (u_0, v_0, w_0) &= \lambda^4 \langle q|_\ell, r|_\ell \rangle_2 \\
K(q)(u_0, v_0, w_0) &= \lambda^6 J(q|_\ell)
\end{align*}
$$

hold. In what follows, we shall be concerned mostly with the vanishing of these expressions, so that the constant $\lambda$ is harmless.

**Definition 1.5.** Let $k$ be a field, let $q$ be a non-zero ternary quartic form and let $C \subset \mathbb{P}^2_3$ be the plane quartic defined by the vanishing of $V(q)$. The inflection scheme of $C$ is the subscheme of $\mathbb{P}^2_3$ defined by the simultaneous vanishing of the forms $H(q)$ and $K(q)$; we denote the inflection scheme of $C$ by $\text{Fl}(C)$.

It follows from Lemma 1.1 that over a field of characteristic different from 3, the inflection scheme of a plane quartic $C$ consists exactly of the lines having a point of intersection multiplicity at least 3 with $C$. In particular, for smooth curves, the points of the inflection scheme correspond to some tangent lines to the curve. Using the dimension of the inflection scheme, we obtain a stratification of the space of plane quartics.

**Remark 1.6.** Over a field of characteristic 3, we really should extend the formulas in (2) to ternary forms and use these to define the inflection scheme. As it is, the scheme that we are calling the inflection scheme contains all inflection lines as a closed subscheme, but it usually has the wrong dimension: indeed, the dimension of the inflection scheme that we just defined is always at least 1. Since we will not use inflection schemes over fields of characteristic 3, we will not pursue this here. Besides, since flat limit of the scheme defined by the vanishing of the invariants at (3) is non-reduced, it might be better for our purposes to use the reduced induced scheme structure, rather than the limit structure.

**Example 1.7.** Let $k$ be a field and let $C$ denote the plane Klein quartic curve with equation $q(x, y, z) = x^3y + y^3z + z^3x = 0$. In this case, the contravariants $H$ and $K$ evaluate to

$$
\begin{align*}
H(q)(u, v, w) &= 3(u^3v + u^3w + w^3u), \\
K(q)(u, v, w) &= 27(u^5w + v^5u + w^5v - 5u^2v^2w^2).
\end{align*}
$$

If the characteristic of $k$ is different from 3, then it follows at once that the inflection scheme of $C$ is finite of degree 24. Thus, the same is true for a general plane quartic over fields of characteristic different from 3.

As a consequence of this example, the general plane quartic over a field of characteristic different from 3 has zero-dimensional inflection scheme. We will study the degeneracy of the inflection scheme in Section 2, focusing on the harmonic quartic.
forms
quartic
Definition 2.1. Let \( q(x, y, z) \) be a ternary quartic form; we say that \( q \) is totally harmonic if the harmonic quartic \( H(q) \) vanishes.

We list in Table 1 some computations involving the harmonic quartic \( H(\cdot) \) and the associated bilinear form \( \langle \cdot, \cdot \rangle_3 \); we will freely use these formulas that are simple consequences of the definitions. Recall that the identity (4) holds. Let \( q_4(y, z), q_3(y, z), q_2(y, z), q_1(y, z) \) be binary forms of degree 4, 3, 2, 1 respectively in \( y, z \).

\[
\begin{align*}
\langle x^aq_4-a(y, z), x^bq_4-b(y, z) \rangle_3 &= 0, & \text{if } a+b \geq 5 \\
\langle x^aq_4(y, z) \rangle_3 &= 12q_4(-w, v) \\
\langle x^aq_4(y, z) \rangle_3 &= -3q_1(-w, v)q_4(-w, v) \\
\langle x^aq_4(y, z) \rangle_3 &= -3u\partial_uq_4(-w, v) \\
H(x^2q_2(y, z)) &= q_2(-w, v)^2 \\
H(x^2y^2, q_4(y, z))_3 &= u^2\partial_u\partial_vq_4(-w, v)
\end{align*}
\]

Table 1. Some identities involving \( \langle \cdot, \cdot \rangle_3 \) and \( H(\cdot) \)

In the case of fields of characteristic 3, the harmonic quartic \( H(\cdot) \) is always a square. Indeed, suppose that the coefficients are in a field \( k \) of characteristic 3 and that \( q(x, y, z) \) is a quartic form in \( k[x, y, z] \). Write \( q(x, y, z) = q_1(x, y, z) + q_2(x, y, z) + q_3(x, y, z) \), where

- \( q_1(x, y, z) \) is the sum of all the terms of \( q \) corresponding to the monomials \( x^2yz, xy^2z, x^2y^2z; \)
- \( q_2(x, y, z) \) is the sum of all the terms of \( q \) corresponding to the monomials \( x^2y^2, x^2z^2, y^2z^2; \)
- \( q_3(x, y, z) = q(x, y, z) - q_1(x, y, z) - q_2(x, y, z). \)

In this case, the formula for \( H(q) \) becomes

\[
H(q) = \left( \frac{q_2(vw, uw, uw) - q_1(vw, uw, uw)}{(vw)^2} \right)^2.
\]

We now go back to the case of ternary quartic forms over a field \( k \) of arbitrary characteristic.

Lemma 2.2. Let \( c(x, y, z) \) be a quadratic form and let \( q = c^2 \) be its square. The harmonic quartic \( H(q) \) is the square of a quadratic form.

Proof. By the invariance of \( H \) under \( SL_2 \), it suffices to check the statement when \( c \) is one the following four forms \( xy-z^2, xy, x^2, 0 \), which is an easy computation. \( \Box \)

Lemma 2.3. Let \( q(x, y, z) \) be a quartic form with degree at most 1 in \( x \). The harmonic quartic \( H(q) \) of \( q \) is divisible by \( u^2 \).

Proof. Write \( q = xq_3(y, z) + q_4(y, z) \), where \( q_3, q_4 \) are binary forms of degree 3 and 4 respectively. Using the definition of \( H \) it is clear that \( H(q) \) has degree at most 2 as a polynomial in the variables \( v, w \). Therefore, the harmonic quartic \( H(q) \) is divisible by \( u^2 \). \( \Box \)

In the case of a plane quartic curve \( C \), let \( H(C) \subset P_k^4 \) denote the vanishing set of the harmonic form associated to an equation of \( C \). Suppose that \( C \) is smooth. The point in \( P_k^4 \) corresponding to a line \( \ell \subset P_k^2 \) tangent to \( C \) is contained in \( H(C) \) if and only if the line \( \ell \) is an inflection line to \( C \). In general, a line \( \ell \) has intersection multiplicity at least 2 at some point \( p \) of \( C \) and corresponds to a point of \( H(C) \) if and only if the line \( \ell \) has multiplicity at least 3 at the point \( p \).

Remark 2.4. With notation as in Lemma 2.2, observe that if the conic defined by the equation \( c = 0 \) is smooth, then the only lines in \( P_k^2 \) having a point of intersection multiplicity at least 3 with the quartic \( q = 0 \) are the tangent lines to the conic \( c = 0 \). This shows that \( H(q) \) vanishes only on the conic dual to \( c = 0 \).
The indeterminacy locus of the rational map \( H : \mathbb{P}^4_k \to \mathbb{P}^4_k \) assigning to a plane quartic the corresponding harmonic quartic is supported on the set of totally harmonic forms. In the following proposition, we see that being totally harmonic for a ternary quartic form over a field of characteristic coprime with 6 implies that one of the variables of the form is redundant.

**Proposition 2.5.** Let \( k \) be an algebraically closed field of characteristic \( p \) and let \( q(x, y, z) \) be ternary quartic form with coefficients in \( k \). The form \( q \) is totally harmonic if and only if after a linear change of variables in \( x, y, z \) the pair \((p, q)\) is equal to one of following:

- \((p, f(y, z))\), where \( f \) is one of the binary forms mentioned in Remark 1.3;
- \((p, x^4 + y^2z)\), where \( p \in \{2, 3\} \) (singularity of type \( E_6 \));
- \((3, x(x^2y + z^3))\) (singularity of type \( E_7 \));
- \((3, x^4 + y^4 + z^4)\) (Fermat curve or Klein curve, they are isomorphic over algebraically closed fields of characteristic 3).

**Proof.** We reduce to the case in which \( q \) does not vanish identically.

If the plane quartic curve \( Q \) with equation \( q = 0 \) does not have a smooth point, then it follows that \( q(x, y, z) \) is the square of a not necessarily irreducible quadratic form \( c(x, y, z) \). If the quadratic form \( c \) defines a conic with a smooth point, then the quartic equation \( q = c^2 = 0 \) restrict to a general line in \( \mathbb{P}^2_k \) to a polynomial with two distinct double roots. In particular, the invariant \( I \) of such a binary form is non-zero, and we conclude that \( q \) is not totally harmonic. Otherwise, the conic \( c = 0 \) has no smooth points and hence its equation is the square of a linear form \( \ell \), and clearly \( q = \ell^4 \) is totally harmonic and of the required form.

We have therefore reduced to the case in which the quartic curve \( Q \) has smooth points: choose coordinates in \( \mathbb{P}^2_k \) so that the point \([1, 0, 0]\) lies in the smooth locus of \( Q \) and the tangent line to \( Q \) at \([1, 0, 0]\) is the line with equation \( y = 0 \). An equation of the curve \( Q \) has the form

\[
x^3y + x^2q_2(y, z) + xq_3(y, z) + q_4(y, z) = 0,
\]

where \( q_2, q_3, q_4 \) are binary forms of degrees 2, 3, 4, respectively.

Suppose first that the characteristic of the field is different from 3. Write \( q_2(y, z) = yq_1(y, z) + \alpha z^2 \), where \( q_1 \) is a binary linear form and \( \alpha \in k \) is a constant. Changing coordinates by \((x, y, z) \mapsto (x - \frac{1}{\alpha}q_1, y, z)\), the equation for \( Q \) simplifies to

\[
x^3y + \alpha x^2z^2 + xq_3(y, z) + q_4(y, z) = 0,
\]

for some binary forms \( q_3, q_4 \) of respective degrees 3, 4. Using the equation

\[
H(x^3y + \alpha x^2z^2 + xq_3(y, z) + q_4(y, z)) = 0,
\]

and formula \((4)\), we collect together the expressions with respect to the power of \( u \) appearing in the monomials. We find

\[
\begin{align*}
    u^0 : & \quad \langle x^3y, xq_3(y, z) \rangle_3 + \alpha^2 I(x^2z^2) = 0 \\
    u^1 : & \quad \langle x^3y, q_4(y, z) \rangle_3 + \alpha \langle x^2z^2, xq_3(y, z) \rangle_3 = 0 \\
    u^2 : & \quad \alpha \langle x^2z^2, q_4(y, z) \rangle_3 + H(xq_3(y, z)) = 0 \\
    u^3 : & \quad \langle xq_3(y, z), q_4(y, z) \rangle_3 = 0 \\
    u^4 : & \quad H(q_4(y, z)) = I(q_4(y, z)) = 0.
\end{align*}
\]

We analyze the condition on \( u^0 \): we deduce the equalities \( \alpha = 0 \) and hence also \( \langle x^3y, xq_4(y, z) \rangle_3 = 3wq_3(-w, v) = 0 \), so that \( q_3(y, z) = 0 \). We are left with the equalities \( \langle x^3y, q_4(y, z) \rangle_3 = -3\partial_v q_4(-w, v) = 0 \) and \( I(q_4(y, z)) = 0 \). If the characteristic of the ground field \( k \) is not only different from 3, but also different from 2, then we deduce that \( q_4(y, z) = \lambda y^4 \), for some constant \( \lambda \in k \) and we are done, since \( q = x^3y + \lambda y^4 \).

If the characteristic of the ground field \( k \) is equal to 2, then we obtain that \( q_4(y, z) = \lambda y^4 + \mu y^2z^2 + \nu z^4 \), for some constants \( \lambda, \mu, \nu \in k \). The condition \( I(q_4) = 0 \) becomes \( \mu^2 = 0 \), so that \( q(x, y, z) = x^3y + \lambda y^4 + \nu z^4 \). If \( \nu = 0 \), then \( q \) is of the required form; otherwise, choose \( \lambda', \nu' \in k \) satisfying \( \lambda'^4 = \lambda \) and \( \nu'^4 = \nu \) and change coordinates by \((x, y, z) \mapsto (x, y, (z + \lambda'y)/\nu')\) to transform \( q \) into \( x^3y + z^4 \), and we conclude after a cyclic permutation of the coordinates.
We now assume that the characteristic of the ground field is 3. By Equation (6) the condition \( I(q) = 0 \) implies that \( q \) is of the form
\[
q(x, y, z) = a_{11}x^4 + a_{12}x^3y + a_{13}x^3z + a_{21}xy^3 + a_{22}y^4 + a_{23}y^3z + a_{31}xz^3 + a_{32}yz^3 + a_{33}z^4,
\]
for some constants \( a_{ij} \in k \). Denote by \( A \) the \( 3 \times 3 \) matrix of the coefficients of \( q \):
\[
q(x, y, z) = (x^3 \quad y^3 \quad z^3) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\]

For any matrix \( M \) with coefficients in \( k \), denote by \( \text{Fr}(M) \) the matrix obtained from \( M \) by applying the Frobenius automorphism to all the entries of \( M \). Explicitly, if \( M \) has entries \((m_{ij})\), then \( \text{Fr}(M) \) has entries \((m_{ij}^3)\). If \( M \) is a \( 3 \times 3 \) matrix, then the identity
\[
q \left( M \cdot (x \quad y \quad z)^t \right) = (x^3 \quad y^3 \quad z^3) \text{Fr}(M)^t A M (x \quad y \quad z)^t
\]
holds. We are going to use the transformation rule \( A \mapsto \text{Fr}(M)^t A M \) to find a simpler form for the matrix \( A \), using a suitable matrix \( M \).

**Case 1: the matrix \( A \) is invertible.** Let \( f_1, f_2, f_3 \) be homogeneous coordinates on \( \mathbb{P}^2_k \). First, we look for equations on the coordinates of the vector \( f = (f_1, f_2, f_3)^t \) so that the conditions \( \text{Fr}(x, y, z) A f = 0 \) and \( \text{Fr}(\text{Fr}(f)^t A(x, y, z)^t)^t = 0 \) coincide. Denote by \( \alpha_1, \alpha_2, \alpha_3 \) the linear forms in \( f \) appearing as entries of \( A(f_1, f_2, f_3)^t \) and by \( \beta_1, \beta_2, \beta_3 \) the linear forms in \( f_1^3, f_2^3, f_3^3 \) appearing as entries of \( (f_1^3, f_2^3, f_3^3)\text{Fr}(A) \). Imposing the proportionality of the vectors \( A f \) and \( \text{Fr}(A^t)\text{Fr}^t(f) \) is equivalent to imposing the condition that the rank of the \( 2 \times 3 \) matrix
\[
K = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}
\]
is at most 1. Let \( \Delta_A \) be the zero locus of the three \( 2 \times 2 \) minors of \( K \)
\[
\Delta_A: \quad \left\{ \begin{array}{c} \alpha_2 \beta_3 - \alpha_3 \beta_2 = 0, \\ -\alpha_1 \beta_3 + \alpha_3 \beta_1 = 0, \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 = 0 \end{array} \right\}
\]
with associated Jacobian matrix
\[
J = \begin{pmatrix} \beta_3 \nabla \alpha_2 - \beta_2 \nabla \alpha_3 \\ -\beta_3 \nabla \alpha_1 + \beta_1 \nabla \alpha_3 \\ \beta_2 \nabla \alpha_1 - \beta_1 \nabla \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix} A = BA
\]
with respect to the variables \( f_1, f_2, f_3 \). Observe that the Jacobian matrix \( J \) is the product of the skew-symmetric matrix \( B \) and the matrix \( A \). As long as \( A \) is non-zero, the matrix \( B \) is non-zero, and hence of rank 2. Note that the scheme \( \Delta_A \) is not empty, as we wanted to show.

In this case, we prove that, among the points of \( \Delta_A \), there are some on which \( q \) does not vanish.

Since \( A \) is invertible, the rank of \( B \) is 2 and hence also the rank of \( J \) is 2 for all choices of \([f_1, f_2, f_3] \in \mathbb{P}^2_k\). It follows that the scheme \( \Delta_A \) is reduced of dimension 0. Moreover, any two of the equations defining \( \Delta_A \) imply the third, unless one among \( \alpha_1, \alpha_2, \alpha_3 \) vanishes. Thus, away from the union of the three lines \( \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0 \), any two linearly independent combinations of the equations defining \( \Delta_A \) imply the third. The intersection of \( \Delta_A \) with any line in \( \mathbb{P}^2_k \) consists of a scheme of dimension 0 and length at most 10, since \( \Delta_A \) is defined by equations of degree 10. It follows that \( \Delta_A \) contains at least \( 10^2 - 3 \cdot 10 = 70 \) points. On the other hand, the intersection of \( \Delta_A \) with the vanishing set of \( q \) consists of at most 40 points, since the degree of \( q \) is 4 and \( \Delta_A \) is defined by equations of degree 10. We finally obtain that there are points \( l \) of \( \Delta_A \) satisfying \( q(l) \neq 0 \). Choose vectors \( l, m, n \in k^3 \) as follows: \( l \) lies in \( \Delta_A \) and \( q(l) = 1 \); \( m, n \) form a basis of the kernel of \( \text{Fr}(l^t)A \). Observe that \( l, m, n \) form a basis of \( k^3 \) and that, using coordinates \( x', y', z' \) with respect to this basis, the form \( q \) becomes
\[
q(x', y', z') = x'^4 + q'(y', z'),
\]
where the binary form \( q'(y', z') \) satisfies \( I(q') = 0 \). Using Remark 1.3, we obtain that, after a change of coordinates, the form \( q \) is equal to one of the following: \( x^4 + y^4 + z^4, x^4 + y^4 z, x^4 + y^4, x^4 \), as we wanted to show.
Case 2: the matrix $A$ is not invertible. In this case, there is a non-zero linear combination of the rows of $A$ that vanishes: let $N$ be an invertible $3 \times 3$ matrix such that the first row of $NA$ is the zero row. Let $M$ be the matrix $Fr^{-1}(N^t)$ and evaluate the quartic form $q$ at $M \cdot (x, y, z)^t$, to obtain the quartic form with matrix

$$NAM = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$ 

If $a_1 = b_1 = 0$, then we obtained a binary form in $y, z$ and we conclude using Remark 1.3. Suppose therefore that $a_1, b_1$ are not both zero and, exchanging if necessary the last two rows and the last two columns of this matrix, we reduce to the case in which $b_1$ is non-zero. After the substitution

$$(x, y, z) \mapsto (x, y, z - Fr^{-1}(a_1/b_1) y),$$

we reduce to the case in which $a_1$ vanishes.

Suppose that $a_2$ does not vanish. After the substitution

$$(x, y, z) \mapsto (x, y - a_3/a_2 z, z),$$

we reduce to the case in which $a_3$ vanishes. Finally, after the substitution

$$(x, y, z) \mapsto (x - b_2/b_1 y - b_3/b_1 z, y, z),$$

we reduce to the case in which $b_2$ and $b_3$ also vanish. After these reductions, we are left with the quartic form $b_1 x z^3 + a_2 y^4$; rescaling and permuting the variables, we obtain the form $x^4 + y^4 z$, as required.

We are still left with the case in which $a_1 = a_2 = 0$ and $b_1 \neq 0$. Therefore, the quartic $q$ takes the form $z(a_3 y^3 + b_1 x z^2 + b_2 y z^2 + b_3 z^3)$. Repeating the substitution in (8), we obtain the quartic form $z(a_3 y^3 + b_1 x z^2)$; rescaling and permuting the variables, we obtain the form $x(x^2 y + z^3)$, as required. □

**Corollary 2.6.** Let $C \subset P_k^2$ be a smooth plane quartic curve over an algebraically closed field $k$. The following conditions are equivalent:

1. the curve $C$ does not have finitely many inflection lines;
2. the Gauss map of $C$ is not birational;
3. the characteristic of the field is 3 and the curve $C$ is isomorphic to the Fermat curve $x^4 + y^4 + z^4 = 0$.

**Proof.** Let $C^\vee \subset P_k^2$ denote the image of the Gauss map of $C$. By [Kaj89, Corollary 4.4], the Gauss map is purely inseparable. It follows that the curves $C$ and $C^\vee$ are birational and we deduce that $C^\vee$ has the same geometric genus as $C$. In particular, $C^\vee$ is a plane curve of degree at least 4.

(1) $\iff$ (2). The Gauss map is separable if and only if a general tangent line to $C$ is not an inflection line, and the equivalence of (1) and (2) follows.

(2) $\implies$ (3). Suppose that the Gauss map $\gamma: C \to P_k^2$ of $C$ is not birational. Let $c$ denote an equation of the curve $C$; the curve $C^\vee$ is contained in the vanishing set of the harmonic quartic $H(c)$. We obtain that either $H(c)$ is an equation of $C^\vee$ and the degree of $C^\vee$ is 4, or $H(c)$ vanishes and $c$ is totally harmonic. If $H(c)$ does not vanish, then the degree of $C^\vee$ is 4, the degree of inseparability of the Gauss map is 3 and hence the characteristic of the field is 3. We already observed that the harmonic quartic in characteristic 3 is always the square of a quadratic form. Nevertheless, the curve $C^\vee$ cannot simultaneously have the same geometric genus as $C$ and be contained in a conic. Thus, the only remaining possibility is that $c$ is totally harmonic. From Proposition 2.5 we conclude that $C$ must be isomorphic to the Fermat quartic curve in characteristic 3.

(3) $\implies$ (1). Let $\ell$ be a tangent line to $C$, choose a parameterization of $\ell$ and denote by $F_\ell$ the restriction of the Fermat equation to the line $\ell$ under the chosen parameterization. Since the harmonic quartic $H(x^4 + y^4 + z^4)$ vanishes, the formulas in (5) imply that the invariant $I$ of $F_\ell$ vanishes. Since $F_\ell$ has a repeated root, Remark 1.3 shows that $F_\ell$ has at least a triple root and we conclude that $\ell$ is an inflection line for $C$ and we are done. □
We can rephrase Corollary 2.6 saying that the dimension of the inflection scheme of a smooth plane quartic is 0 if and only if the quartic is not projectively equivalent to the Fermat quartic in characteristic 3.

We conclude with two examples showing that there is no direct link between the smoothness of a plane quartic $C$ and the smoothness of its harmonic $H(C)$. Still, in the next section we shall prove results about singularities of (limit) harmonic quartics (Propositions 4.5 and 4.6).

**Example 2.7.** Smooth $C$ and singular $H(C)$. The quartic with equation $x^4 + y^4 + yz^3 = 0$ is smooth: its harmonic quartic $12w(w^3 - v^3) = 0$ is the union of 4 concurrent lines and in particular it is singular.

**Example 2.8.** Singular $C$ and smooth $H(C)$. The quartic with equation $x^2yz + xy^3 + xz^3 + y^4 = 0$ has a node at $[1,0,0]$; its harmonic quartic $-12uwv + 9u^2vw + 3uw^3 + 3uw^3 + v^2w^2 = 0$ is smooth.

3. **Configurations of inflection lines of plane quartics**

Let $k$ be a field of characteristic different from 3. We maintain our standard notation: $\mathbb{P}^2_k$ is the projective plane with homogeneous coordinates $x, y, z$; $\mathbb{P}^2_k^\vee$ is the dual plane with coordinates $u, v, w$ dual to $x, y, z$. A plane quartic is the subscheme of $\mathbb{P}^2_k$ defined by the vanishing of a nonzero homogeneous polynomial of degree four. The space of plane quartics is a projective space $\mathbb{P}^2_{24}^\vee$ of dimension 14.

We denote by $\text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$ the Hilbert scheme parameterizing closed subschemes of dimension zero and degree 24 of $\mathbb{P}^2_k^\vee$. We often switch between thinking of a point in $\mathbb{P}^2_k^\vee$ and the corresponding line in $\mathbb{P}^2_k$; thus, we may refer to the Hilbert scheme $\text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$ as a space parameterizing configurations of 24 lines in the plane $\mathbb{P}^2_k$. If $h$ is a point in $\text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$ and $L$ is a line corresponding to a point in $h$, then we call the multiplicity of $L$ in $h$ the degree of the component of $h$ containing $L$.

For every smooth plane quartic $C$ we abuse the notation and keep denoting by $\text{Fl}(C)$ the point of $\text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$ corresponding to the inflection scheme of $C$, defined as the intersection of $H(C)$ and $K(C)$. Recall that the characteristic of $k$ is different from 3, so that $\text{Fl}(C)$ is indeed 0-dimension and of degree 24. Let

$$\mathcal{F} : \mathbb{P}^2_{14} \to \text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$$

be the rational map associating to a point representing a smooth plane quartic $C$ the inflection scheme $\text{Fl}(C)$. Let $\mathbb{P}^2_{14}$ be the closure in $\mathbb{P}^2_{14} \times \text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$ of the incidence variety $\{(C, \text{Fl}(C)) : C \text{ is a smooth plane quartic}\}$ and let

$$\tilde{\mathcal{F}} : \mathbb{P}^2_{14} \to \text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$$

be the restriction to $\mathbb{P}^2_{14}$ of the projection of $\mathbb{P}^2_{14} \times \text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$ onto the second factor. If $C$ is a plane quartic and $F$ is a point in $\text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$, we say that $\text{Fl}$ is a configuration of inflection lines associated to $C$ if the pair $(C, \text{Fl})$ is in $\mathbb{P}^2_{14}$.

We now come to the central definition of our paper: reconstructibility of plane quartics from their configurations of inflection lines.

**Definition 3.1.** Let $C \subset \mathbb{P}^2_k$ be a plane quartic. We say that $C$ is

- **reconstructible from a point $\text{Fl} \in \text{Hilb}_{24}(\mathbb{P}^2_k^\vee)$** if $\text{Fl}$ is a configuration of inflection lines associated to $C$ and no other plane quartic admits $\text{Fl}$ as configuration of inflection lines;
- **reconstructible from its inflection lines** if the morphism $\mathcal{F}$ is defined at the curve $C$ and $C$ is reconstructible from $\mathcal{F}(C)$;
- **reconstructible among smooth curves from its inflection lines** if $C$ is smooth and it is the only smooth quartic with $\mathcal{F}(C)$ as configuration of inflection lines.

There is a weaker notion of reconstructibility, where we require two plane quartics with the same configuration of inflection lines to be projectively equivalent and not necessarily equal. We will work with the stricter notion (Definition 3.1), although the weaker one is well-suited for the moduli space of curves of genus 3.

To analyze configurations of inflection lines associated to quartics in the indeterminacy locus of $\mathcal{F}$, we introduce deformations. Let $S$ be a connected $k$-scheme with a $k$-point $\{0\} \hookrightarrow S$. Let $\mathcal{C} \subset \mathbb{P}^2_S$ be a plane
quartic curve over $S$ and let $\pi: \mathcal{C} \to S$ be the structure morphism. Denote by $C \subset \mathbb{P}^2_k$ the plane quartic $C = \pi^{-1}(0)$ over $k$. We say that $\mathcal{C} \to S$ is a deformation of $C$ over $S$ if moreover there is a dense open subset of $S$ over which the morphism $\pi$ is smooth, then we say that $\mathcal{C} \to S$ is a smoothing of $C$ over $S$.

Let $\pi: \mathcal{C} \to S$ be a deformation of a plane quartic over $S$. The deformation $\pi$ determines uniquely a morphism $\text{hilb}_\pi: S \to \mathbb{P}^{14}_k$, since $\mathbb{P}^{14}_k$ is the Hilbert scheme of plane quartics. If the composition of the morphism $\text{hilb}_\pi$ and the rational map $\mathcal{F}$ extends to a morphism $S \to \text{Hilb}_{24}(\mathbb{P}^{24}_k)$, then we say that $\pi$ is inflective. In this case, we write $\text{Fl}_{\pi}(C)$ for the image of $\pi(C)$ in $\text{Hilb}_{24}(\mathbb{P}^{24}_k)$.

Let $\pi$ be a smoothing of a plane quartic over the spectrum $S$ of a DVR. Since the characteristic of the ground field is different from 3, the rational map $\mathcal{F}$ is defined on the image of the generic point of $S$ in $\mathbb{P}^{14}_k$. By the valuative criterion of properness, we deduce that $\pi$ is inflective. Moreover, for any plane quartic $C$, each configuration of inflection lines associated to $C$ arises in this way, from some smoothing of $C$ over some DVR. Even more, we can assume that the DVR is complete. Since we work over a fixed algebraically closed field $k$, this means that we only need to consider smoothings over the spectrum of the formal power series ring $k[[t]]$ in one variable $t$.

We emphasize that all the points of $\mathbb{P}^{14}_k$ are of the form $(C, \text{Fl}_{\pi}(C))$ for some plane quartic $C$ and some smoothing $\pi$ of $C$ over the spectrum of $k[[t]]$. This observation appears, sometimes implicitly, in our arguments.

Let $C$ be a plane quartic curve with equation $q = 0$. A deformation of $C$ over the power series ring $k[[t]]$ is defined by the vanishing of a ternary quartic form $f_t(x, y, z) \in k[[t]][x, y, z]$, whose coefficients are formal power series in $t$ reducing to the coefficients of $q$ modulo the ideal $(t)$.

We use interchangeably the notion of deformation for plane quartics or ternary quartic forms.

From now on, we exclude also the case of fields of characteristic 2, so that the characteristic of the ground-field is coprime with 6.

Let $U$ be the open subset of $\mathbb{P}^{14}_k$ parameterizing reduced plane quartics whose singular points have multiplicity two.

**Lemma 3.2.** The rational map $\mathcal{F}$ is defined over $U$.

**Proof.** Let $C$ be a plane quartic; as usual, denote by $\text{Fl}(C)$ the intersection of $H(C)$ and $K(C)$. It suffices to show that if $C$ is in the open set $U$, then the scheme $\text{Fl}(C)$ is finite. The lines corresponding to points in $\text{Fl}(C)$ are lines with a point of intersection multiplicity at least 3 with $C$. Let $L$ be a line in $\text{Fl}(C)$.

Suppose first that $L$ contains a singular point $p$ of $C$. In this case, $L$ must be in the tangent cone to $C$ at $p$ and there are only a finite number of such lines. This in particular takes care of the case in which $L$ is a component of $C$.

Suppose now that $L$ and $C$ meet entirely at smooth points. Let $p$ be the point of multiplicity at least 3 in $L \cap C$ and let $C_p \subset C$ denote the irreducible component of $C$ containing $p$. The curve $C_p$ is not a line, by the first part of the argument. Therefore, the line $L$ corresponds to a singular point of the curve $C'_p$, dual to $C_p$. We deduce that the Gauss map associated to $C_p$ is not constant and that $L$ is in its branch locus. Since the characteristic of the ground-field is coprime with 6, the Gauss map is separable: the line $L$ is contained in a finite set and we are done. \(\square\)

The proof of Lemma 3.2 reduces to the one of [PT13, Proposition 2.5] in the case of fields of characteristic 0. In particular, for the quartics parametrized by points in $U$, there is a well-defined configuration of inflection lines $\mathcal{F}(C)$ of degree 24 in $\mathbb{P}^{24}_k$ and it coincides with $\text{Fl}(C)$. This is not true in general.

**Remark 3.3.** Suppose that $C$ is a plane quartic curve and that the line $\ell$ has intersection multiplicity at least 3 with $C$ at a point $p \in C$. Let $C_{\ell, p}$ be a smooth plane quartic curve admitting $\ell$ as an inflection line at the point $p$. The general curve in the pencil generated by $C$ and $C_{\ell, p}$ is smooth and admits $\ell$ as an inflection line at the point $p$. We deduce that the limiting configuration of inflection lines as we approach the curve $C$ within the pencil must contain $\ell$. It follows that the locus in $\mathbb{P}^{24}_k$ swept by all the configurations of inflection lines associated to $C$ is the support of the inflection scheme $\text{Fl}(C)$ of $C$. Thus, if $\text{Fl}(C)$ is not finite, then there cannot be a well-defined configuration of inflection lines for the curve $C$. Examples of curves for which this happens are double conics and curves with a triple point; the closure of this locus is the complement of $U$. See Table 2 for computations of dimensions of inflection schemes for all plane quartics.
Using the definition of multiplicity 4. We call $L$ a simple inflection line if $p$ has multiplicity 3; we call $L$ a hyperinflection line if $p$ has multiplicity 4.

Remark 3.3 shows that, for every plane quartic $C$, simple inflection lines and hyperinflection lines correspond to points in the inflection scheme $\text{Fl}(C)$. In the next lemma, we prove that, for quartics in the open set $U$, simple inflection lines and hyperinflection lines are characterized by their multiplicity in the inflection scheme.

**Lemma 3.5.** Let $C$ be a plane quartic curve in $U$ and let $L$ be a line in $\text{Fl}(C)$.

1. The multiplicity of $L$ in $\text{Fl}(C)$ is 1 if and only if $L$ is a simple inflection line.
2. The multiplicity of $L$ in $\text{Fl}(C)$ is 2 if and only if the line $L$ is a hyperinflection line.
3. The multiplicity of $L$ in $\text{Fl}(C)$ is at least 3 if and only if the line $L$ is in the tangent cone to a singular point of $C$.

Moreover, if $L$ is a hyperinflection line, then the harmonic quartic $H(C)$ is smooth at $L$, the tangent space to $H(C)$ at $L$ coincides with the Zariski tangent space to $\text{Fl}(C)$ at the point $L$ and it corresponds to the pencil of lines through the point $p$.

**Proof.** If the line $L$ is not contained in $C$, then let $p$ be the unique point of $L \cap C$ with intersection multiplicity at least 3. If the line $L$ is contained in $C$, then let $p$ be a singular point of $C$ on the line $L$. Choose coordinates $x,y,z$ in $\mathbb{P}^2_k$ so that the line $L$ is the line with equation $x = 0$ and the point $p$ is the point $[0,0,1]$. We write an equation of $C$ as

$$q(x,y,z) = y^3q_1(y,z) + xq_3(x,y,z),$$

where $q_1$ and $q_3$ are forms of respective degrees 1 and 3. Let $\alpha$ be the coefficient of the monomial $z^3$ in $q_3$; this coefficient vanishes if and only if the curve $C$ is singular at the point $[0,0,1]$. Moreover, we also assume that

- $q_1(y,z) = z$ if $L \cap C$ has intersection multiplicity exactly 3 at $[0,0,1]$;
- $q_1(y,z) = y$ if $L \cap C$ has intersection multiplicity exactly 4 at $[0,0,1]$;
- $q_1(y,z) = 0$ if $L$ is contained in $C$.

With our reductions, the harmonic quartic $H(q)$ and the harmonic sextic $K(q)$ vanish as the point $[1,0,0] \in \mathbb{P}^2_k$. We compute the expansion of $H$ to first order near the point $[1,0,0]$ and the expansion of $K$ to at most second order near the same point. Using the definition of $H$, we find

$$H(q)(u,v,w) = u^4 I \left( y^3q_1(y,z) + \left(-\frac{u}{w}y - \frac{w}{u}z\right)q_3 \left(-\frac{v}{u}y - \frac{w}{u}z, y, z\right) \right).$$

Using the definition of $I$, we obtain the congruences

1. if $q_1(y,z) = z$, then $H(q) \equiv 3\alpha u^3v \mod (v,w)^2 + (w)$;
2. if $q_1(y,z) = y$, then $H(q) \equiv -12\alpha u^3w \mod (v,w)^2$.
Analogously, we compute $K(q)$ and obtain the congruences
\[
\begin{align*}
\text{if } q_1(y, z) &= z, & K(q) &\equiv 27\alpha u^5w &\mod (v, w)^2; \\
\text{if } q_1(y, z) &= y, & K(q) &\equiv -27\alpha u^4v^2 &\mod (v, w)^3 + (vw, w^2); \\
\text{if } q_1(y, z) &= 0, & K(q) &\equiv 0 &\mod (v, w)^2.
\end{align*}
\]

Suppose that the curve $C$ is smooth at $p$. This implies that the coefficient $\alpha$ does not vanish, that $L$ is not contained in $C$ by our choice of $p$, and that $L$ is either a simple inflection line or a hyperinflection line. If $L$ is a simple inflection line, then the curves $H(C)$ and $K(C)$ are transverse at $[1, 0, 0]$ and hence the multiplicity of $L$ in $F_l(C)$ is 1. If $L$ is a hyperinflection line, then the curve $H(C)$ is smooth at $[1, 0, 0]$, the curve $K(C)$ has a double point and the tangent line to $H(C)$ at $[1, 0, 0]$ is not in the tangent cone to $K(C)$ at $[1, 0, 0]$. In this case, the multiplicity of $L$ in $F_l(C)$ is 2. Observe that the tangent line to $H(C)$ at $L$ is the line with equation $w = 0$ which corresponds to the pencil of lines through the point $p$.

Suppose that the curve $C$ is singular at the point $p$. Thus, $\alpha$ vanishes and since the intersection multiplicity of $L$ and $C$ is at least 3 at $p$, it follows that $L$ is in the tangent cone to $C$ at $p$. By the computation of the harmonic sextic, the curve $K(C)$ has a point of multiplicity at least 3 at $[1, 0, 0]$, as required. \hfill \Box

For fields of characteristic 0, there is a more refined computation of the multiplicities of Lemma 3.5 in [PT13, Proposition 2.10].

**Lemma 3.6.** Let $C$ be the union of four distinct lines through a point $p$ in $\mathbb{P}_k^2$ and let $F_l$ be a configuration of inflection lines associated to $C$. All the lines corresponding to points in $F_l$ contain $p$. In particular, $F_l$ is not the configuration of inflection lines of any smooth quartic.

**Proof.** The first part of the lemma is a consequence of the observation that every line with intersection multiplicity at least 3 at some point of $C$ must contain the point $p$.

For the second part, let $D$ a smooth quartic and let $q$ be a point in $\mathbb{P}_k^2$. Projecting $D$ away from $q$ and using the Riemann-Hurwitz we find that there are at most 8 inflection lines through $q$. Thus, there are inflection lines of $D$ not containing the point $q$, as required. \hfill \Box

**Remark 3.7.** Let $C$ be a smooth plane quartic admitting at least 8 inflection lines through a point $p$, counted with multiplicity. It follows that $C$ has exactly 4 hyperinflection lines through $p$ and that it is projectively equivalent to $x^4 = yz(y-z)(y-\lambda z)$, where $\lambda \in k \setminus \{0, 1\}$ is a constant and $p$ is the point $[1, 0, 0]$. This assertion is an easy consequence of the Riemann-Hurwitz formula.

### 4. Limits and Valuations

In this section, we assume that the characteristic of the field $k$ is coprime with 6. We analyze the harmonic quartic associated to a quartic form $q(x, y, z)$ to obtain information on the configuration of inflection lines to the curve $q = 0$. We make no assumptions on the singularities of the quartic form $q$ and it is crucial for us to analyze the case in which $q$ is totally harmonic. In such cases, the harmonic quartic vanishes and we use deformations of $q$ to examine the limiting configurations of inflection lines.

In this section, we use the symbol $q(x, y, z)$ to denote a ternary quartic form with coefficients in $k$.

Let $f_t(x, y, z)$ be a deformation of a ternary quartic form $q$ over $k[t]$. Write $H(f_t) = t^\delta\gamma(u, v, w) + t^{\delta+1}\gamma_t(u, v, w)$, where $\delta$ is a non-negative integer, $\gamma(u, v, w)$ is a ternary quartic form with coefficients in $k$ and $\gamma_t(u, v, w)$ is a ternary quartic form whose coefficients are formal power series in $t$. If $\gamma(u, v, w)$ does not vanish, then we say that $f_t$ admits a limit harmonic quartic and we call the form $\gamma(u, v, w)$ the limit harmonic quartic of $f_t$.

The existence of the limit harmonic of $f_t$ is equivalent to the non-vanishing of the harmonic quartic $H(f_t)$. Even in the case of a totally harmonic quartic form $q$, every smoothing $f_t$ of $q$ admits a limit harmonic quartic. Properties of the limit harmonic quartics of deformations of totally harmonic forms are the central goal of this section.

**Example 4.1.** Let $C$ be the (reducible) plane quartic with equation $x^4 - xy^3 = 0$; the quartic $C$ is totally harmonic, that is, $H(x^4 - xy^3) = 0$. The polynomial $f_t = (x^4 - xy^3) + tz^4 + y^4$ defines a smoothing of $C$. The limit harmonic quartic of $f_t$ is $w^3v + v^4 + w^4$, a form defining a smooth quartic curve and vanishing on the limits of the coordinates of the inflection lines of the smooth quartics in the deformation $f_t$. Indeed,
the limit configuration of inflection lines $F_l$ associated to this smoothing is the finite subscheme of $\mathbb{P}_k^2$ with equations $w^3 + v v^4 + w^4 = w^5 = 0$ and length 24. The support of $F_l$ consists of 4 lines in $\mathbb{P}_k^2$ through the point $[0,0,1]$, each counted with multiplicity 6.

**Definition 4.2.** Let $n$ be a non-negative integer. Let $a(t) = \sum_{i \geq 0} a_i t^i \in k[[t]]$ be a formal power series; we call the polynomial $\sum_{i=0}^n a_i t^i$ the truncation of $a(t)$ at order $n$. Let $p(x, y, z) \in k[[t]][x, y, z]$ be a polynomial in $x, y, z$ with coefficients in $k[[t]]$. Denote by $\tau_n p(x, y, z) \in k[t][x, y, z]$ the polynomial with coefficients in $k[t]$ obtained by truncating the coefficients of $p$ at order $n$. We call $\tau_n p(x, y, z)$ the truncation of $p$ at order $n$, or simply the truncation of $p$, if $n$ is clear from the context.

Clearly, the truncation of a deformation is still a deformation, while the truncation of a smoothing need not be a smoothing.

**Remark 4.3.** Let $f_t$ be a deformation of $q$ over $k[[t]]$ and let $n$ be a non-negative integer. The two harmonic quartics $H(f_t)$ and $H(\tau_n f_t)$ have the same truncation to order $n$; in formulas, the identity $\tau_n H(f_t) = \tau_n H(\tau_n f_t)$ holds.

Let $\xi_t(x, y, z), \eta_t(x, y, z), \zeta_t(x, y, z)$ be linear forms in $x, y, z$ with coefficients in $k[[t]]$. We think of the function $G : (x, y, z) \mapsto (\xi_t, \eta_t, \zeta_t)$ as a one-parameter family of linear transformations on $x, y, z$, depending on the parameter $t$. Denote by $\xi_0, \eta_0, \zeta_0$ the truncations of $\xi_t, \eta_t, \zeta_t$ at order 0. We let $G_0$ denote the linear transformation

$$G_0 : (x, y, z) \mapsto (\xi_0(x, y, z), \eta_0(x, y, z), \zeta_0(x, y, z)).$$

Let $f_t$ be a deformation of a quartic form $q$ over $k[[t]]$ with non-vanishing harmonic quartic and write

$$H(f_t) \equiv t^5 \gamma(u, v, w) \mod (t^4 + 1),$$

where $\gamma(u, v, w)$ is the limit harmonic quartic of $f$. The quartic $G(f_t) = f_t(\xi_t, \eta_t, \zeta_t)$ is a deformation of the quartic $G_0(q) = q(G_0(x, y, z))$. Suppose that the linear transformation $G_0$ is invertible. It follows from the contravariance of $H$ that

$$H(G(f_t)) \equiv t^5 \gamma(G_0(u, v, w)) \mod (t^4 + 1),$$

where $G_0(u, v, w)$ is the adjoint transpose of the linear transformation $G_0$. Thus $\gamma(G_0(u, v, w))$ is the limit harmonic quartic of $G(f_t)$.

From the previous discussion, we deduce that if we make a linear transformation depending on the parameter $t$ that is invertible for $t = 0$, then limit harmonic quartics change to projectively equivalent quartics. We now exploit changes of coordinates depending on $t$ that are invertible for $t = 0$ to simplify our computations.

We say that a quartic form $f \in k[[t]][x, y, z]$ with coefficients in $k[[t]]$ is *depressed* if it is of the form

$$f(x, y, z) = x^4 + t(x^3(ay^2 + bz^2) + \varphi_3(y, z)x + \varphi_4(y, z)),$$

where $a, b$ are power series in $k[[t]]$ and $\varphi_3, \varphi_4$ are forms in $k[[t]][y, z]$ of respective degrees 3, 4 in $y, z$.

In the following proofs, we use the notion of valuation. For a non-zero power series $a(t) \in k[[t]]$, we call valuation of $a$ the largest power of $t$ dividing $a$ and we denote it by $p(a)$. Note that if $a, b$ are non-zero polynomials in $t$, then $p(ab) = p(a) + p(b)$, and if also $a + b \neq 0$, then $p(a + b) \geq \min\{p(a), p(b)\}$, with equality if $p(a) \neq p(b)$. The ring $k[[t]]$ is a DVR with respect to the valuation $p$.

**Lemma 4.4.** Let $f_t$ be a deformation of $x^4$ over $k[[t]]$. There is a linear change of coordinates $g(x, y, z)$ with coefficients in $k[[t]]$ such that $f_t \circ g$ is depressed and for $t = 0$ the change $g(x, y, z)$ is invertible and $g(x) = x$.

**Proof.** Write $f_t$ as

$$f_t(x, y, z) = \psi_0 x^4 + t(\psi_1(y, z)x^3 + \psi_2(y, z)x^2 + \psi_3(y, z)x + \psi_4(y, z)),$$

where $\psi_0, \psi_1, \ldots, \psi_4$ are forms in $k[[t]][y, z]$ of respective degrees 0, 1, \ldots, 4 in $y, z$. Since $f_t$ is a deformation of $x^4$, we have $\psi_0 \equiv 1 \pmod{(t)}$.

Let $\psi \in k[[t]]$ be the power series expansion of $\psi_0^{-1}$: this makes sense, since we are assuming that the characteristic of $k$ is different from 2. The change of coordinates $(x, y, z) \mapsto (\psi x, y, z)$ is linear in $x, y, z$, is the identity modulo $t$, and allows us to reduce to the case in which $f_t$ is the deformation

$$f_t(x, y, z) = x^4 + t(\psi_1(y, z)x^3 + \psi_2(y, z)x^2 + \psi_3(y, z)x + \psi_4(y, z)).$$
Observe that $t\psi_1$ is linear in $y, z$; hence the change of coordinates $(x, y, z) \mapsto (x - \frac{1}{2}t\psi_1, y, z)$ is linear (again, recall that $\text{char} k \neq 2$) and it is the identity for $t = 0$. After this change of coordinates, the deformation has the form

$$f_t(x, y, z) = x^4 + t(\varphi_2(y, z)x^2 + \varphi_3(y, z)x + \varphi_4(y, z)),$$

where $\varphi_2, \varphi_3, \varphi_4 \in k[[t]][y, z]$ are forms of respective degrees 2, 3, 4. We write $\varphi_2(y, z) = a'y^2 + b'yz + c'z^2$, where $a', b', c'$ are power series in $k[[t]]$. We still want to reduce to the case in which the coefficient $b'$ vanishes. Thus, suppose that $b' \neq 0$. After a general, invertible, linear change of coordinates in $y, z$ with coefficients in $k$, we reduce to the case in which the valuation $\alpha = \rho(a')$ of $a'$ is less than or equal to the valuation of $b'$, so that the ratio $\frac{b'}{a'}$ is a power series in $k[[t]]$. Completing the square with the linear change of variables $(x, y, z) \mapsto (x - \frac{b'}{a'}z, z)$, invertible for $t = 0$, we reduce to the case in which $\varphi_2(y, z)$ is of the form $ay^2 + bz^2$, as required.

**Proposition 4.5.** Let $f_t$ be a deformation of $x^4$ and suppose that the harmonic quartic $H(f_t)$ is non-zero. The limit harmonic quartic of $f_t$ is singular at the point $[1, 0, 0]$.

**Proof.** First, using Lemma 4.4, we reduce to the case in which $f_t$ is depressed. We write $q_2(y, z) = a_{20}y^2 + a_{202}z^2,$

$$f_t(x, y, z) = x^4 + t\varphi_2(y, z)x + \varphi_3(y, z)x + \varphi_4(y, z)) + \left(a_{40}y^4 + a_{031}y^3z + a_{022}y^2z^2 + a_{013}yz^3 + a_{004}z^4 - \frac{q_2(y, z)^2}{12}\right),$$

where the coefficients $a_{ijk}$ are power series in $k[[t]]$ of valuation at least 1 and we offset the coefficients not divisible by $x$ in $f_t$ by $\frac{q_2}{12}$ to simplify the upcoming computations. Recall that the characteristic of the ground field is coprime with 6.

We now evaluate the harmonic quartic of $f_t$ and collect the results in Table 3. In each row of Table 3, we write the coefficient of some of the monomials in the expression for $H(f_t)$; we do not need the coefficients of the monomials $u^2vw, u^3w^2$. The polynomials $A_0, A_1, A_2$ are linear combinations of monomials of degree at least two, involving at least one of the variables $a_{040}, a_{031}, a_{022}, a_{013}, a_{004}$.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Monomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{3}a_{220}^2 + A_0$</td>
<td>$u^4$</td>
</tr>
<tr>
<td>$\frac{1}{3}a_{220}a_{202}a_{112} + a_{202}a_{130} + A_1$</td>
<td>$u^3v$</td>
</tr>
<tr>
<td>$a_{220}^2a_{103} + \frac{1}{3}a_{220}a_{202}a_{121} + A_2$</td>
<td>$u^2w$</td>
</tr>
<tr>
<td>$-\frac{1}{3}a_{220}^2a_{202} + 12a_{220}a_{004} + 2a_{202}a_{022} - 3a_{121}a_{103} + a_{112}^2$</td>
<td>$u^4v^2$</td>
</tr>
<tr>
<td>$-2a_{202}a_{112}$</td>
<td>$uv^4$</td>
</tr>
<tr>
<td>$-6a_{220}a_{103} + 4a_{202}a_{121}$</td>
<td>$uv^2w$</td>
</tr>
<tr>
<td>$4a_{220}a_{112} - 6a_{202}a_{130}$</td>
<td>$uw^2v$</td>
</tr>
<tr>
<td>$-2a_{220}a_{121}$</td>
<td>$uw^3$</td>
</tr>
<tr>
<td>$12a_{004}$</td>
<td>$v^4$</td>
</tr>
<tr>
<td>$-12a_{013}$</td>
<td>$v^3w$</td>
</tr>
<tr>
<td>$12a_{022}$</td>
<td>$v^2w^2$</td>
</tr>
<tr>
<td>$-12a_{031}$</td>
<td>$vw^3$</td>
</tr>
<tr>
<td>$12a_{040}$</td>
<td>$w^4$</td>
</tr>
</tbody>
</table>

Table 3. Some terms of the harmonic quartic of a deformation of $x^4$

Our goal is to show that if there is a non-zero coefficient for one of the monomials $u^4, u^3v, u^2w^2$ in $H(f_t)$ with valuation $\rho$, then there is a non-zero coefficient for one of the remaining monomials with valuation
strictly smaller than \( \rho \). This is enough for our purposes, since it means that the limit harmonic quartic is a combination of monomials having degree at least 2 in \( v, w \), that is, having multiplicity at least 2 at the point \([1, 0, 0]\).

We begin by analyzing the polynomials \( A_0, A_1, A_2 \). For each polynomial \( A_0, A_1, A_2 \), at least one of the coefficients of \( v^4, v^3w, v^2w^2, vwu, w^4 \) has smaller valuation. Thus, we only need to worry about the remaining monomials

\[
a_{220}a_{202}a_{112} \quad a_{220}a_{202}a_{130} \quad a_{220}a_{202}a_{103} \quad a_{220}a_{202}a_{121}.
\]

The monomial \( a_{220}a_{202}^2 \). The valuation of \( a_{220}a_{202}^2 \) is strictly larger than the valuation of \( a_{220}a_{202}^2 \). Moreover, looking at the coefficient \( \kappa \) of \( u^2v^2 \), we deduce that if the valuation of \( a_{220}a_{202}^2 \) is strictly smaller than the valuations of each one of the remaining non-zero terms in the same coefficient \( \kappa \), then we are done. Otherwise, we consider separately the four possibilities

\[
\rho(a_{220}a_{202}^2) \geq \rho(a_{220}a_{202}^2) \quad \rho(a_{220}a_{202}^2) \geq \rho(a_{2112}^2) \quad \rho(a_{220}a_{202}^2) \geq \rho(a_{2112}^2) \quad \rho(a_{220}a_{202}^2) \geq \rho(a_{2112}^2).
\]

In the two cases \( \rho(a_{220}a_{202}^2) \geq \rho(a_{220}a_{202}^2) \) and \( \rho(a_{220}a_{202}^2) \geq \rho(a_{202}a_{202}^2) \), we deduce that one of the coefficients of \( v^4, v^3w, v^2w^2 \) is non-zero, has strictly smaller valuation than \( a_{220}a_{202}^2 \) and we are done. In the case \( \rho(a_{220}a_{202}^2) \geq \rho(a_{2112}^2) \), we deduce the inequality \( \frac{3}{2}\rho(a_{220}) + \rho(a_{202}) \geq \rho(a_{2112}) \). From this, we find

\[
2\rho(a_{220}) + 2\rho(a_{202}) > \frac{1}{2}\rho(a_{220}) + 2\rho(a_{202}) \geq \rho(a_{202}) + \rho(a_{112}),
\]

showing that the coefficient of \( uv^3 \) has strictly smaller valuation than \( a_{220}a_{202}a_{112} \) and again we are done. Finally, the case \( \rho(a_{220}a_{202}^2) \geq \rho(a_{2112}a_{202}) \) implies \( \rho(a_{220}a_{202}^2) \geq \rho(a_{220}a_{2112}a_{103}) \) and is settled considering the monomial \( uv^3 \).

The monomial \( a_{220}a_{202}a_{112} \). The coefficient of \( uv^3 \) has strictly smaller valuation than \( a_{220}a_{202}a_{112} \).

The monomial \( a_{202}a_{130} \). Looking at the coefficient of \( uvu^3 \), we deduce that we need to address the case \( \rho(a_{202}a_{130}) \geq \rho(a_{220}a_{112}) \). Under this assumption, we find \( \rho(a_{202}a_{130}) > \rho(a_{220}a_{112}) \) and we conclude that the coefficient of \( uv^3 \) has strictly smaller valuation than \( a_{202}a_{130} \).

The monomial \( a_{220}a_{103} \). Looking at the coefficient of \( uv^2w \), we deduce that we need to address the case \( \rho(a_{220}a_{103}) \geq \rho(a_{220}a_{121}) \). Under this assumption, we find \( \rho(a_{220}a_{103}) > \rho(a_{220}a_{121}) \) and we conclude that the coefficient of \( uv^3 \) has strictly smaller valuation than \( a_{220}a_{103} \).

The monomial \( a_{220}a_{121} \). The coefficient of \( uv^3 \) has strictly smaller valuation than \( a_{220}a_{202}a_{121} \).

\( \square \)

**Proposition 4.6.** Let \( f_t \) be a deformation of \( x^3y \) and suppose that the harmonic quartic \( H(f_t) \) is non-zero. The limit harmonic quartic of \( f_t \) is singular at the point \([1, 0, 0]\).

**Proof.** First, let \( a \) be the coefficient of the monomial \( x^2y^2 \) and let \( b \) be the coefficient of the monomial \( x^2yz \) in \( f_t \). After the linear change of coordinates \((x, y, z) \mapsto (x - \frac{ay + bz}{3}, y, z)\), we reduce to the case in which \( f_t \) does not involve the monomials \( x^2y^2 \) and \( x^2yz \). Let \( c \) be the coefficient of \( x^3z \) in \( f_t \). Using the change of coordinates \((x, y, z) \mapsto (x, y - cz, z)\), we reduce further to the case in which \( f_t \) does not involve the term \( x^3z \) either. We write

\[
\pi(y, z) = \frac{a_{202}}{18}(3a_{112}y^4 + 18a_{130}y^2z^2 + (32a_{200}a_{031} + 8a_{121})yz^3 + 3a_{112}z^4),
\]

\[
f_t(x, y, z) = x^3y + a_{400}x^4 + a_{202}x^2z^2 + x(a_{130}y^3 + a_{121}y^2z + a_{112}yz^2 + a_{031}y^3) + (a_{103}y^2z + a_{022}y^2z^2 + a_{013}yz^3 + a_{004}z^4 - \pi),
\]

where the coefficients \( a_{ijk} \) are power series in \( k[[t]] \) of valuation at least 1 and we offset the coefficients not divisible by \( x \) in \( f_t \) by \( \pi \) to simplify the upcoming computations.

We now evaluate the harmonic quartic of \( f_t \) and we collect the results in Table 4. In each row of Table 4, we write the coefficient of some of the monomials in the expression for \( H(f_t) \); we do not need the coefficients of the monomials \( x^3z^2, u^2v, u^2w^2, w^4 \). The power series \( A_0, A_1, A_2 \) are combinations of the power series \( a_{031}, a_{022}, a_{013}, a_{004}, a_{121} \) with each coefficient involving at least one further coefficient of \( f_t \).
Our goal is to show that the minimum valuation of a non-zero coefficient of $u^4, u^3v, w^3$ above the break in Table 4 is strictly bigger than the valuation of some coefficient below the break in Table 4. This is enough for our purposes, since it means that the limit harmonic quartic of $f_t$ is a combination of monomials having degree at least 2 in $v, w$, that is, having multiplicity at least 2 at the point $[1, 0, 0]$.

Let $I$ be the ideal generated by the coefficients of the monomials below the break in Table 4, that is the ideal generated by the coefficients of the monomials not divisible by $u^3$. Clearly, the ideal $I$ is generated by

$$a_{004}, a_{022}, a_{031}, a_{202}(-2a_{400}a_{112} + a_{202}), a_{202}^2(8a_{400}^2a_{130} + 1),$$

where the element $\alpha$ is obtained by eliminating $a_{112}$ using the two generators in the fourth column of (9).

The minimum valuation of a coefficient below the break in Table 4 is therefore the valuation of one of the elements in (9). Thus, it suffices to show that the valuation of each coefficient appearing above the break in Table 4 is strictly bigger than the valuation of an element of (9).

By definition, the valuations of $A_0, A_1, A_2$ are strictly bigger than the valuation of some element in (9), as needed.

The remaining summands of the coefficient of $w^3$ have valuation strictly bigger than $\rho(a_{103})$ and they also do not cause problems, since $a_{103}$ is in $I$.

All remaining coefficients are divisible by $a_{202}$ and we therefore reduce to the case in which $a_{202}$ does not vanish. If the valuation of $a_{112}$ is at most equal to the valuation of $a_{202}$, then the bottom element in the fourth column of (9) has valuation $\rho(a_{112})$ and we are done. Otherwise, the valuation of $a_{112}$ is strictly bigger than the valuation of $a_{202}$ (possibly $a_{112}$ vanishes) and the remaining power series

$$a_{202}^2a_{130}^2, a_{202}^2a_{112}^2, a_{202}a_{112}a_{040}, a_{202}a_{130}a_{112}$$

have valuations that are strictly bigger than $2\rho(a_{202}) = \rho(\alpha)$ and again we are done.

\[\square\]

**Theorem 4.7.** The configuration of inflection lines of a smooth plane quartic with smooth harmonic quartic is different from the configuration of inflection lines associated to any singular plane quartic.

**Proof.** Let $F_1$ be the configuration of inflection lines of a singular plane quartic $C$ and let $D$ be a smooth plane quartic with smooth harmonic quartic $H(D)$. By Lemma 3.5 if the curve $C$ were contained in $U$, then $F_1$ would contain a point with multiplicity at least 3, while all multiplicities of $F(D)$ are at most 2. We therefore reduce to the case in which $C$ is not in $U$ and hence does not have isolated singular points of multiplicity 2.
It suffices to show that the configuration $F_l$ is contained in a singular plane quartic. Indeed, if $F_l$ coincided with $F_l(D)$ and $F_l$ were contained in a singular quartic curve $Q_C$, then the harmonic quartic $H(D)$ and $Q_C$ would define a scheme of dimension zero and length 16 containing $F_l(D)$. This is impossible, since the scheme $F_l(D)$ has degree 24.

If $C$ is not totally harmonic, then by Lemmas 2.2 and 2.3 the harmonic quartic of $C$ is non-reduced, thus singular, and we are done. Finally, suppose that $C$ is totally harmonic, so that by Proposition 2.5 the quartic $C$ has a quadruple point $p$. If $C$ is reduced, then $C$ consists of four distinct lines through a point and we exclude this case by Lemma 3.6. Finally, suppose that $C$ contains a line with multiplicity at least 3. The configuration $F_l$ arises from a smoothing $\pi$ of $C$ over the spectrum of $k[[t]]$. Since $\pi$ is a smoothing, it admits a limit harmonic quartic $Q_C$. By Proposition 4.5 or 4.6, the quartic $Q_C$ is singular and it contains $F_l$, as needed. □

5. THE RECONSTRUCTION OF THE GENERAL PLANE QUARTIC

In this section, we continue to assume that the characteristic of the ground field $k$ is coprime to 6. We prove our main reconstructibility result in Theorem 5.10. We begin with some results relating the various notions of reconstructibility (see Definition 3.1).

**Theorem 5.1.** Let $C \subset \mathbb{P}^2_k$ be a smooth plane quartic. If $C$ is reconstructible among smooth curves from its inflection lines and the harmonic quartic $H(C)$ is smooth, then $C$ is reconstructible from its inflection lines.

**Proof.** Combine Theorem 4.7 and the assumption that $C$ is reconstructible among smooth curves from its inflection lines. □

Let $\varepsilon \in k$ satisfy $\varepsilon^4 = 1$ and let $q_\varepsilon(x, y, z)$ be the quartic form

$$q_\varepsilon(x, y, z) = (x - y)^4 + (x - z)^4 + (y - \varepsilon z)^4 - (x^4 + y^4 + z^4).$$

Observe that the three identities

$$q_\varepsilon(x, y, 0) = (x - y)^4 \quad q_\varepsilon(x, 0, z) = (x - z)^4 \quad q_\varepsilon(0, y, z) = (y - \varepsilon z)^4$$

hold.

**Lemma 5.2.** A smooth plane quartic $C$ has 3 non concurrent hyperinflection lines if and only if, up to a change of coordinates, there are constants $\varepsilon, \lambda, \mu, \nu \in k$ with $\varepsilon^4 = 1$ such that

$$q_\varepsilon(x, y, z) + xyz(\lambda x + \mu y + \nu z) = 0$$

is an equation of $C$.

**Proof.** Suppose first that $C$ is a smooth plane quartic having the three lines $\ell_1, \ell_2, \ell_3$ as non concurrent hyperinflection lines. Changing coordinates in $\mathbb{P}^2_k$ if necessary, we reduce to the case in which the three lines have equations

$$\ell_1: x = 0 \quad \ell_2: y = 0 \quad \ell_3: z = 0.$$ 

For $i \in \{1, 2, 3\}$ let $p_i \in C$ denote the hyperinflection point of $C$ corresponding to the hyperinflection line $\ell_i$: none of the three points $p_1, p_2, p_3$ is a coordinate point, since otherwise two coordinate lines would be tangent to $C$ at the same point. Rescaling the $x$ and the $y$ coordinate if necessary, we reduce further to the case in which $p_3 = [1, 1, 0]$ and $p_2 = [1, 0, 1]$. Write $p_1 = [0, 1, \varepsilon]$ for the remaining hyperinflection point, where $\varepsilon \in k$.

Let $q(x, y, z)$ be a ternary quartic form. We now impose the linear conditions implying that the coordinate lines are hyperinflection lines to the plane quartic defined by $q(x, y, z) = 0$ at the points $p_1, p_2, p_3$.

Restricting the polynomial $q$ to each line $z = 0, y = 0$ and $x = 0$ in succession we find the conditions

$$q(x, y, 0) = \alpha(x - y)^4 \quad q(x, 0, z) = \beta(x - z)^4 \quad q(0, y, z) = \gamma(y - \varepsilon z)^4,$$

where $\alpha, \beta, \gamma \in k$ are non-zero constants. Moreover, $\alpha$ and $\beta$ must coincide, since they are the coefficient of $x^4$ in $q$. Similarly, $\alpha$ and $\gamma$ must also coincide, since they are the coefficient of $y^4$ in $q$. Analogously, $\beta$ and $\gamma^4$ must also coincide, since they are the coefficient of $z^4$ in $q$. Rescaling $q$ if necessary, we reduce to the case in which the identities $\alpha = \beta = \gamma = \varepsilon^4 = 1$ hold.
Finally, observe that the only remaining coefficients of $q$ are the coefficients of the three monomials $x^3yz, xy^2z, xyz^2$, as required.

The converse is clear. \qed

Lemma 5.2 gives a standard form for a smooth plane quartic with the coordinate lines as hyperinflection lines. In the next lemma, we determine some of the conditions arising from imposing a further hyperinflection line.

**Lemma 5.3.** Let $q_c(x, y, z) + xyz(\lambda x + \mu y + \nu z) = 0$ be the equation of a quartic curve $C$, let $a, b \in k$ be constants and let $\ell$ be the line with equation $z = ax + by$. If $C$ and $\ell$ meet at a unique point $p$, then there are constants $d_p, e_p, f_p \in k$, depending on $p$, such that the equation

$$
\begin{pmatrix}
  a & 0 & a^2 \\
  b & a & 2ab \\
  0 & b & b^2 \\
\end{pmatrix}
\begin{pmatrix}
  \lambda \\
  \mu \\
  \nu \\
\end{pmatrix}
= 
\begin{pmatrix}
  d_p \\
  e_p \\
  f_p \\
\end{pmatrix}
$$

holds. \[\text{Proof.}\] Restricting the equation of $C$ to the line $\ell$ we find the binary quartic form

$$c_\ell(x, y) = q_c(x, y, ax + by) + xy(ax + by)(\lambda x + \mu y + \nu(ax + by)).$$

By assumption, the form $c_\ell$ is the fourth power of a non-zero linear form. This implies that at least one of the coefficients of $x^4$ and $y^4$ is non-zero. Exchanging the roles of $x$ and $y$ if necessary, we reduce to the case in which the coefficient $(1 - a)^4$ of $x^4$ is non-zero and hence there is a constant $\pi \in k$ such that $c_\ell = (1 - a)^4(x - \pi y)^4$. Equating the coefficients of $x^3y, x^2y^2, xy^3$ we obtain the required relations. \[\square\]

**Proposition 5.4.** Let $p_1, \ldots, p_5 \in \mathbb{P}_k^2$ be distinct points and let $\ell_1, \ldots, \ell_5 \subset \mathbb{P}_k^2$ be distinct lines such that for $i \in \{1, \ldots, 5\}$ the point $p_i$ lies on the line $\ell_i$. There is at most one smooth quartic curve $C$ admitting $\ell_1, \ldots, \ell_5$ as hyperinflection lines and the points $p_1, \ldots, p_5$ as corresponding hyperinflection points.

**Proof.** Suppose that $C$ is a smooth quartic curve with the required properties. First, the five lines $\ell_1, \ldots, \ell_5$ cannot contain a common point by Remark 3.7. Thus, there are three non concurrent lines among $\ell_1, \ldots, \ell_5$; relabeling the lines, if necessary, we assume that $\ell_1, \ell_2, \ell_3$ are non concurrent. Applying Lemma 5.2 to the quartic $C$ and the lines $\ell_1, \ell_2, \ell_3$, we obtain coordinates on $\mathbb{P}_k^2$ such that an equation of $C$ is

$$C: \quad q_c(x, y, z) + xyz(\lambda x + \mu y + \nu z) = 0.$$ 

The two hyperinflection lines $\ell_4$ and $\ell_5$ are therefore not coordinate lines. As a consequence, each of these two lines has an equation involving at least two of the variables $x, y, z$. Permuting if necessary the variables, we reduce to the case in which the equations of the two lines $\ell_4$ and $\ell_5$ involve the variable $z$: let $z = ax + by$ be an equation for $\ell_4$ and $z = cx + dy$ an equation for $\ell_5$. Applying Lemma 5.3 we find that the matrix $M_{abcd}$ of the not necessarily homogeneous linear system satisfied by the coefficients $\lambda, \mu, \nu$ of the equation of $C$ is

$$M_{abcd} = 
\begin{pmatrix}
  a & 0 & a^2 \\
  b & a & 2ab \\
  0 & b & b^2 \\
  d & c & 2cd \\
  0 & d & d^2 \\
\end{pmatrix}.$$

It is now easy to check that the matrix $M_{abcd}$ has rank 3 as soon as $(a, b)$ and $(c, d)$ are distinct and different from $(0, 0)$. We conclude that there is at most one solution to the system, as required. \[\square\]

**Corollary 5.5.** Let $C$ be a smooth quartic curve. If $C$ has at least 5 hyperinflection lines, then $C$ is reconstructible among smooth curves from its inflection lines.

**Proof.** Let $\text{Fl}(C)$ be the configuration of inflection lines of $C$, let $D$ be a smooth quartic having $\text{Fl}(C)$ as configuration of inflection lines and let $H(D)$ be the harmonic quartic of $D$. The two harmonic quartics $H(C)$ and $H(D)$ meet at the points corresponding to inflection lines. Moreover, at the points corresponding to hyperinflection lines, they also have a common tangent direction, thanks to Lemma 3.5. This implies that
$D$ must also go through the hyperinflection points of $C$. Since $C$ has at least 5 hyperinflection lines and $D$ shares these same lines as hyperinflection lines and also goes through the corresponding hyperinflection points, we conclude using Proposition 5.4 that $C$ and $D$ coincide, as required. \qed

In the next lemma, we use a one-parameter family of smooth plane quartic curves with at least 8 hyperinflection lines. Over fields of characteristic zero, such a family was studied by Vermeulen [Ver83] and Girard-Kohel [GK06]; we also used this family in [PT14]. For properties of this family, see the cited references.

**Lemma 5.6.** Let $k$ be a field of characteristic coprime to 6, let $t$ be a constant in $k$ and let $V_t$ be the plane quartic with equation $tx^4 + y^4 - z^4 - 2x^2y^2 - 4xyz^2 = 0$. For $t$ different from 0, 1, $\frac{1}{14}$ the quartic $V_t$ is smooth and has exactly 8 hyperinflection lines and exactly 8 simple inflection lines. The harmonic quartic of $V_t$ is

$$H(V_t) = -3u^4 - 3tv^4 + (3t + 1)u^4 + 10u^2v^2 + 8uvw^2 = 0;$$

for general $t$, the curve $H(V_t)$ is smooth.

**Proof.** It is easy to check that $V_t$ is smooth for $t$ different from 0, 1 and that the hyperinflection points of the curve $V_t$ are the intersection of $V_t$ with the two lines $x = 0$ and $y = 0$. The remaining inflection points are the points of intersection of the curve $V_t$ with the conic $2xy + 3z^2 = 0$. These remaining intersection points satisfy the equations

$$2xy + 3z^2 = 0 \quad \text{and} \quad tx^4 + \frac{2}{9}x^2y^2 + y^4 = 0.$$

The discriminant of the polynomial $tx^2 + \frac{2}{9}xy + y^2$ is $4(\frac{1}{14} - t)$. Thus, for $t$ different from $\frac{1}{14}$ and 0, there are 8 distinct inflection points on $V_t$, besides the 8 hyperinflection points already determined. We deduce that for the values of $t$ not equal to 0, 1, $\frac{1}{14}$, the curve $V_t$ is smooth and has exactly 8 hyperinflection lines and 8 hyperinflection points.

The statement about the harmonic quartic is a simple computation. In fact, the harmonic quartic $H(V_{-1})$ is non-singular for all fields of characteristic coprime to 6. \qed

The curves in the following remark appear in the main result of [PT13].

**Remark 5.7.** Let $k$ be an algebraically closed field of characteristic 13 and let $V_{-1}$ and $V'_{-1}$ be the smooth plane quartics with equations

$$V_{-1} : \quad -x^4 + y^4 - z^4 - 2x^2y^2 - 4xyz^2 = 0,$$

$$V'_{-1} : \quad x^4 - y^4 - z^4 - 2x^2y^2 - 4xyz^2 = 0.$$ 

These curves are projectively equivalent, under the exchange of the coordinates $x$ and $y$. The configurations of inflection lines associated to these two curves are different points in $\text{Hilb}_{24}(\mathbb{P}^2_k)$: they have distinct and smooth harmonic quartic curves. Nevertheless, these two configurations have the same image under the Chow morphism $\text{Hilb}_{24}(\mathbb{P}^2_k) \to \text{Sym}^{24}(\mathbb{P}^2_k)$. In fact, as zero-cycle, this image is $2(H_1 + \cdots + H_8) + (L_1 + \cdots + L_8)$, for some distinct lines $H_1, \ldots, H_8$ and $L_1, \ldots, L_8$.

For every line $L \subset \mathbb{P}^2_k$, we define the loci

$$\mathcal{V}_{L}^{\text{inf}} := \{ C \in \mathbb{P}^4_k : L \cdot C = 3p + q, \quad \text{for some} \quad p, q \in C \},$$

$$\mathcal{V}_{L}^{H} := \{ C \in \mathbb{P}^4_k : [L] \in H(C) \subset \mathbb{P}^2_k \}.$$ 

The locus $\mathcal{V}_{L}^{\text{inf}}$ is the codimension 2 locus of quartics in $\mathbb{P}^4_k$ having the line $L$ as an inflection line. The locus $\mathcal{V}_{L}^{H}$ is the codimension 1 locus of quartics in $\mathbb{P}^4_k$ whose harmonic quartic contains the point corresponding to the line $L$; equivalently, $\mathcal{V}_{L}^{H}$ is the locus of quartics in $\mathbb{P}^4_k$ whose restriction to the line $L$ is a binary quartic form with vanishing invariant $I$. Clearly, there is an inclusion $\mathcal{V}_{L}^{\text{inf}} \subset \mathcal{V}_{L}^{H}$, since the restriction of the equation of a quartic curve to an inflection line is a binary form with a triple root and hence vanishing invariant $I$. This inclusion implies an inclusion at the level of Zariski tangent spaces. In particular, if $C$ is a smooth quartic curve and $L$ is a hyperinflection line of $C$, then $\mathcal{V}_{L}^{\text{inf}}$ is singular at the point corresponding to $C$, while $\mathcal{V}_{L}^{H}$ is not. Hence, we still have one linear condition imposed by $L$ on tangent spaces, even at some singular points of $\mathcal{V}_{L}^{\text{inf}}$. 

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Lemma 5.8. Let $k$ be a field of characteristic different from 3. Let $C$ be a smooth plane quartic, let $L$ be an inflection line of $C$ and write $L \cdot C = 3p + q$, for some $p, q \in C$. The following tangent space computations hold:
\[
\begin{align*}
T_C \mathcal{V}^\text{inf}_L &\simeq H^0(C, \mathcal{O}_C(4) \otimes \mathcal{O}_C(-2p)), \quad \text{if } p \neq q; \\
T_C \mathcal{V}^H_L &\simeq H^0(C, \mathcal{O}_C(4) \otimes \mathcal{O}_C(-p)), \quad \text{if } \text{char}(k) \neq 2 \text{ and } p = q.
\end{align*}
\]

Proof. For the computation of the tangent space to $\mathcal{V}^\text{inf}_L$ see [PT13, Lemma 3.1] (although the stated reference works with fields of characteristic 0, the proof of the statement only requires characteristic different from 3). We still need to compute the tangent space to $\mathcal{V}^H_L$ at $C$ when $L$ is a hyperinflection line of $C$. Choose coordinates on $\mathbb{P}^2_k$ so that $L$ is the line $x = 0$ and the point $p$ is the point $[0, 0, 1]$. It follows that a polynomial $q_C$ defining the quartic $C$ is $q_C = y^4 + xc(x, y, z)$, where $c$ is a ternary cubic form. To each ternary quartic form $q \in H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2}(4))$, we associate a first order deformation of the quartic curve $C$ by $q_C + \varepsilon q$, where $\varepsilon^2 = 0$. In this way, we identify the tangent space to $\mathbb{P}^4_k$ at the quartic $C$ with $H^0(C, \mathcal{O}_C(4))$. The first order deformations tangent to $\mathcal{V}^H_L$ therefore are the ones for which the invariant $I((q_C + \varepsilon q)|_L)$ vanishes. This condition translates to
\[
I(y^4 + \varepsilon q(0, y, z)) = \varepsilon \langle y^4, q(0, y, z) \rangle_2 = \varepsilon \langle y^4, q(0, 0, z) \rangle_2 = 0.
\]
Letting $\gamma$ be the coefficient of $z^4$ in $q$, this condition translates to the equation $12\gamma = 0$. We deduce that $\gamma$ vanishes and hence $q$ vanishes at the hyperinflection point $[0, 0, 1]$, as needed. \hfill \Box

Lemma 5.9. The restriction of the morphism $\tilde{\mathcal{F}}: \mathbb{P}^4_k \to \text{Hilb}_{24}(\mathbb{P}^2_k)$ to the locus of smooth plane quartics with at most 10 hyperinflection lines is unramified.

Proof. Let $\text{Fib}$ be a fiber of $\tilde{\mathcal{F}}$ and let $C$ be a smooth plane quartic with at most 10 hyperinflection lines contained in $\text{Fib}$. Let $p_1, \ldots, p_{f+h}$ denote the inflection points of $C$ and let $L_1, \ldots, L_{f+h}$ denote the corresponding inflection lines. We label the points in such a way that for $i \in \{1, \ldots, f+h\}$ we have $L_i \cdot C = 4p_i$ if and only if $i \geq f + 1$. Let $\Delta_C$ denote the divisor $\Delta_C = \sum_{i=1}^f 2p_i + \sum_{j=1}^h p_{f+j}$ on $C$. By the discussion before Lemma 5.8, the sequence of inclusions
\[
\text{Fib} \subset \mathcal{V}^\text{inf}_{L_1} \cap \cdots \cap \mathcal{V}^\text{inf}_{L_{f+h}} \subset \left( \bigcap_{i=1}^f \mathcal{V}^\text{inf}_{L_i} \right) \cap \left( \bigcap_{j=1}^h \mathcal{V}^H_{L_{f+j}} \right)
\]
holds. Therefore, by Lemma 5.8, the tangent space to $\text{Fib}$ at $C$ is contained in the vector space $H^0(C, \mathcal{O}_C(4L - \Delta_C))$, where $L$ is the class on $C$ of a line in $\mathbb{P}^2_k$. By the assumption that $C$ has at most 10 hyperinflection lines, the degree of the divisor $4L - \Delta_C$ is $16 - 2(f + h) \leq -2$ and hence the tangent space to fiber $\text{Fib}$ at $C$ is zero. \hfill \Box

Theorem 5.10. Let $k$ be an algebraically closed field of characteristic coprime to 6. The general plane quartic is reconstructible from its inflection lines.

Proof. Let $V_t$ be a general quartic appearing in the family of Lemma 5.6. By Theorem 5.1, Corollary 5.5 and Lemma 5.6, the curve $V_t$ is reconstructible from its inflection lines. Thus, the fiber of $\tilde{\mathcal{F}}$ above $\tilde{\mathcal{F}}(V_t)$ only contains $V_t$ and hence, by Lemma 5.9, the morphism $\tilde{\mathcal{F}}$ is generically injective. \hfill \Box

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