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A nowhere convergent series of functions converging somewhere after every non-trivial change of signs

Abstract

We construct a sequence of continuous functions (h_n) on any given uncountable Polish space, such that $\sum h_n$ is divergent everywhere, but for any sign sequence $(\varepsilon_n) \in \{-1,+1\}^{\mathbb{N}}$ which contains infinitely many -1 and $+1$ the series $\sum \varepsilon_n h_n$ is convergent at at least one point. We can even have $h_n \to 0$, and if we take our given Polish space to be any uncountable closed subset of $\mathbb{R}$, we can require that every h_n be a polynomial. This strengthens a construction of Tamás Keleti and Tamás Mátrai.

1 Introduction

Let X be a topological space, $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of continuous functions. One can ask about a condition on this sequence which guarantees that for a "typical" choice of signs $\varepsilon_n = \pm 1$ the series $\sum \varepsilon_n f_n$ diverges everywhere on X.

By "typical" choice of signs we mean that the set of the proper sign sequences is a residual (or dense G_δ) subset of $S = \{-1,+1\}^{\mathbb{N}}$. Here we consider S as a product of discrete topological spaces, which is clearly a Baire space. By $\mathbb{N}$ we denote the set of the positive integers. By Polish space we mean complete separable metric space.

In [1, Theorem 4.1] for σ-compact X spaces a condition was given on the divergence of the partial sums of $\sum f_n$ implying that $\sum \varepsilon_n f_n$ diverges everywhere for a typical sign sequence $(\varepsilon_n) \in S$. Motivated by this result, S. Konyagin asked whether in case of compact metric spaces X, the pure fact that $\sum f_n$ diverges everywhere could imply that $\sum \varepsilon_n f_n$ diverges everywhere for a typical sign sequence. Tamás Keleti and Tamás Mátrai (see [2]) gave a negative answer to this question by

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showing an example of a sequence of continuous functions (f_n) on any uncountable Polish space, such that $\sum f_n$ is divergent everywhere, but for a typical sign sequence $(\varepsilon_n) \in S$, the series $\sum \varepsilon_n f_n$ is convergent at at least one point.

This paper strengthens this construction by showing a sequence of continuous functions f_n such that $\sum f_n$ is divergent everywhere but for every sign sequence $(\varepsilon_n) \in S_0 = \{(\varepsilon'_n) \in S \mid (\varepsilon'_n)$ contains infinitely many -1 and $+1\}$, the series $\sum \varepsilon_n f_n$ is convergent at at least one point. Clearly S_0 is the largest subset of S for which this could be true.

We will also construct an other series of continuous functions with the same properties which satisfies even that $f_n \to 0$. Providing that the uncountable Polish space is $\mathbb{R}$ (or a closed subset of $\mathbb{R}$) we can require every f_n to be a polynomial, see Remark 1.

2 The example

Theorem 1. [1] *Let P be an uncountable Polish space. There exists a sequence of continuous functions $h_n : P \to \mathbb{R}$ such that $\sum h_n$ diverges everywhere on P, but for any $(\varepsilon_n) \in \{-1, +1\}^{\mathbb{N}}$ sign sequence containing infinitely many -1 and $+1$ digits $\sum \varepsilon_n h_n$ converges at at least one point of P.*

Proof. At first we define continuous functions $f_n : S = \{-1, +1\}^{\mathbb{N}} \to [-1, +1]$ such that $\sum f_n$ is divergent everywhere, but for any $(\varepsilon_n) \in S_0 = \{(\varepsilon'_n) \in S \mid (\varepsilon'_n)$ contains infinitely many -1 and $+1\}$ the series $\sum \varepsilon_n f_n$ is convergent at at least one point, in fact, at (ε_n).

Consider a fix $x \in S$ as the sequence of the -1 and $+1$ digits. Divide this sequence into blocks of type $AAA \dots AB$ (where A and B stand for -1 and $+1$ in some order), with the property of containing at least one A and containing exactly one B at the end. We start the division in the beginning of the sequence. Occasionally we make one infinite block of type $AAA \dots$. Thus, the division is well defined.

For example,

$$\boxed{-1\ +1}\ \boxed{-1\ -1\ -1\ +1}\ \boxed{+1\ +1\ -1}\ \boxed{+1\ -1}\ \boxed{-1\ +1}\ \boxed{+1\ +1\ +1 \dots}$$

Let n be a positive integer. We are going to define the real number $f_n(x)$. Suppose that the n^{th} digit of x is in the k^{th} block of x and this digit is the i^{th} number in this block. Denote the size of the k^{th} block by l. (Thus $1 \le i \le l$ and $l \ge 2$.)

If l is even, then let $f_n(x) = \frac{(-1)^{i+1}}{k}$ if $1 \le i \le l-1$ and let $f_n(x) = \frac{+1}{k}$ if $i = l$.

[1] Independently from the author, Gergely Zábrádi gave almost the same construction on $\mathbb{R}$ at the same time.

If l is odd, then let $f_n(x) = \frac{(-1)^{i+1}}{k}$ if $1 \le i \le l-2$, let $f_n(x) = 0$ if $i = l-1$ and let $f_n(x) = \frac{+1}{k}$ if $i = l$.

If $l = \infty$, then let $f_n(x) = \frac{(-1)^{i+1}}{k}$.

For example (writing $f_n(x)$ below the n^{th} digit of x),

$$\boxed{-1\ +1}\ \boxed{-1\ -1\ -1\ +1}\ \boxed{+1\ +1\ -1}\ \boxed{+1\ -1}\ \boxed{-1\ +1}\ \boxed{+1\ +1\ +1 \ldots}$$

$$\boxed{\tfrac{+1}{1}\ \tfrac{+1}{1}}\ \boxed{\tfrac{+1}{2}\ \tfrac{-1}{2}\ \tfrac{+1}{2}\ \tfrac{+1}{2}}\ \boxed{\tfrac{+1}{3}\ 0\ \tfrac{+1}{3}}\ \boxed{\tfrac{+1}{4}\ \tfrac{+1}{4}}\ \boxed{\tfrac{+1}{5}\ \tfrac{+1}{5}}\ \boxed{\tfrac{+1}{6}\ \tfrac{-1}{6}\ \tfrac{+1}{6} \ldots}$$

Claim 1. *The function f_n is an $S \to [-1,+1]$ continuous function for every $n \in \mathbb{N}$.*

Proof. It is easy to see that $f_n(x)$ depends only on the first $n+1$ digits of x. This implies continuity. $\square$

Claim 2. *The series $\sum f_n(x)$ is divergent for every $x \in S$.*

Proof. For a fixed x consider those positive integers n for which the n^{th} digits of x are in the fixed k^{th} block. For these n the sum of $f_n(x)$ equals to $2/k$ if this block is finite. Hence $\sum_{n\in\mathbb{N}} f_n(x) = \infty$ if x has infinitely many blocks. Otherwise x has an infinite block so the terms of the series $\sum_{n\in\mathbb{N}} f_n(x)$ are not converging to 0. $\square$

Claim 3. *For every $(\varepsilon_n) \in S_0$ there exists $x \in S$ for which $\sum \varepsilon_n f_n(x)$ is convergent, namely $x = (\varepsilon_n)$.*

Proof. The sequence $x = (\varepsilon_n) \in S_0$ has only blocks of finite size. Consider those positive integers n for which the n^{th} digits of x are in the same fixed block. For these n the sum of $\varepsilon_n f_n(x)$ is exactly zero. The sequence of partial sums converges to 0, hence the series $\sum \varepsilon_n f_n(x)$ is convergent. $\square$

It is well known (see [3, Corollary 6.5]) that P contains a homeomorphic copy of the Cantor set, denote it by C. Clearly S is homeomorphic to the Cantor set, let φ be a $C \to S$ homeomorphism. Let $g_n : P \to [-1,+1]$ be a continuous extension of $f_n \circ \varphi : C \to [-1,+1]$ for every n. On P let $h_n(p) = g_n(p) + n \cdot d(p, C)$, where $d(p, C)$ denotes the distance of p from the closed set C. Clearly for $p \notin C$ the series $\sum h_n(p)$ diverges. On C we have $h_n = f_n \circ \varphi$, hence by Claim 2 and Claim 3 we obtain that (h_n) satisfies all required properties. $\square$

Theorem 2. *Requiring that $h_n \to 0$, Theorem 1 remains true.*

Proof. Just like in the proof of Theorem 1, at first we define functions f_n on S. Let $x \in S$ be fixed. Consider the same blocks. Suppose that the k^{th} block is finite and contains the a^{th}, $(a+1)^{th}, \ldots, b^{th}$ digits of x ($a, b \in \mathbb{N}$, $b - a \geq 2$). Define $f_a(x)$, $f_{a+1}(x), \ldots, f_b(x)$ to be respectively

$$\frac{+1}{k}\ \frac{-1}{2k}\ \frac{-1}{2k}\ \frac{+1}{3k}\ \frac{+1}{3k}\ \frac{+1}{3k}\ \cdots \underbrace{\frac{+1}{(2m+1)k}\ \cdots\ \frac{+1}{(2m+1)k}}_{2m+1}\ \underbrace{0\ 0\ \ldots\ 0}_{<4m+5}\ \frac{+1}{k}$$

where the number of zeros is less than $4m+5$ and maybe there are no zeros at all. This properly defines the value of m ($m \in \{0, 1, 2, \ldots\}$). Note that $\sum_{n=a}^{b} f_n(x) = \frac{2}{k}$ and if $x = (\varepsilon_n) \in S_0$ then $\sum_{n=a}^{b} \varepsilon_n f_n(x) = 0$.

If the k^{th} block is infinite and contains the a^{th}, $(a+1)^{th}, \ldots$ digits of x then define $f_a(x)$, $f_{a+1}(x), \ldots$ to be respectively

$$\frac{+1}{k}\ \frac{-1}{2k}\ \frac{-1}{2k}\ \frac{+1}{3k}\ \frac{+1}{3k}\ \frac{+1}{3k}\ \frac{-1}{4k}\ \frac{-1}{4k}\ \frac{-1}{4k}\ \frac{-1}{4k} \cdots$$

Note that $\sum_{n=a}^{\infty} f_n(x)$ diverges.

One can easily check that $f_n(x)$ depends only on the first $2n+2$ digits of x, so these functions are continuous. It is clear that Claim 2 and Claim 3 also hold for this sequence of functions f_n, and $-1 \leq f_n \leq +1$ for every $n \in \mathbb{N}$. Define φ and g_n the same way as in the proof of Theorem 1. We modify the definition of function h_n, put

$$h_n(p) = (\max(1 - d(p, C),\ 0))^n\, g_n(p) + \frac{d(p, C)}{n}.$$

If $p \notin C$ then $h_n(p) \sim \frac{1}{n}$, hence $\sum h_n(p)$ diverges and $h_n(p) \to 0$. For $p \in C$ we have $h_n(p) = f_n \circ \varphi(p)$. Hence by Claim 2 and Claim 3 we obtain that (h_n) satisfies all required properties. □

Remark 1. *Let P be an uncountable closed subset of $\mathbb{R}$ (hence P is a Polish space). There exists a sequence of polynomials $p_n : P \to \mathbb{R}$ such that $p_n \to 0$ and $\sum p_n$ diverges everywhere on P, but for any sign sequence $(\varepsilon_n) \in \{-1, +1\}^{\mathbb{N}}$ containing infinitely many -1 and $+1$, the series $\sum \varepsilon_n p_n$ converges at at least one point of P.*

Proof. Consider the continuous functions h_n given by Theorem 2 for P. Let p_n be a polynomial on $\mathbb{R}$ for which $|p_n(x) - h_n(x)| \leq \frac{1}{n^2}$ for every $x \in P \bigcap [-n, n]$. Clearly $p_n(x) \to 0$ for every $x \in P$. Since the series $\sum \frac{1}{n^2}$ converges, for every $(\varepsilon_n) \in S$ the series $\sum \varepsilon_n p_n$ converges if and only if $\sum \varepsilon_n h_n$ converges. This completes the proof. □

References

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[3] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag New York, 1995.