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Grothendieck-Witt groups of quadrics and sums-of-squares formulas

by

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Contents

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Declarations

Chapter 2 reviews the history of sums-of-squares formulas, most of which are based on [76] and [78]. Chapter 3 reviews the paper [73] for readers' convenience, and I do not claim originality. Charles Walter announced some results of Chapter 4 in a conference talk at IHP in 2004, cf. [84]. However, his work has not been published. Chapter 5 has been published in [86].

Abstract

This thesis studies Grothendieck-Witt spectra of quadric hypersurfaces. In particular, we compute Witt groups of quadrics. Besides, by calculating Grothendieck-Witt groups of a deleted quadric over an algebraically closed field of characteristic different from 2, we improve Dugger and Isaksen's condition (some powers of 2 dividing some binomial coefficients) on an old problem Hurwitz concerning the existence of sums-of-squares formulas.

Chapter 1

Introduction

1.1 The Hurwitz problem

In 1898, Hurwitz asked the following question:

When does a sums-of-squares formula of type $[r, s, n]$ exist?

Recall that a sums-of-squares formula of type $[r, s, n]$ over a field F of characteristic $\neq 2$ is the formula

$$\Big(\sum_{i=1}^{r} x_i^2\Big) \cdot \Big(\sum_{i=1}^{s} y_i^2\Big) = \Big(\sum_{i=1}^{n} z_i^2\Big) \in F[x_1, ..., x_r, y_1, ..., y_s] \tag{1.1}$$

where $X = (x_1, ..., x_r)$ and $Y = (y_1, ..., y_s)$ are systems of coordinates and $z_i = z_i(X, Y)$ for each $1 \leq i \leq n$ is a bilinear form in X and Y (with coefficients in F) i.e. $z_i \in F[x_1, ..., x_r, y_1, ..., y_s]$ is homogeneous of degree 2 and F-linear in X and Y. To be specific, z_i is of the form $\sum_{k,j} c_{kj}^{(i)} x_k y_j$ for $c_{kj}^{(i)} \in F$.

The problem itself is easy to understand. However, it turns out that the proofs of some theorems towards the problem require advanced modern cohomology methods. Thus, the subject is valuable as a demonstration of the power of cohomology.

The problem has significant influence on other branches of mathematics, e.g. vector fields on spheres, composition algebras and embeddings of open manifolds, cf. [76]. During the past one hundred years, many prestigious mathematicians have been working on the problem for various purposes. Until now, it remains unsolved. Thus, the Hurwitz problem seems to be extremely hard.

It is also probably very difficult to determine if a sums-of-squares formula depends on the choice of base fields of characteristic different from 2. The "independent field" property of the Hurwitz problem is known as the Shapiro's conjecture. Recently, some classical theorems (towards sums-of-squares formulas) over real numbers $\mathbb{R}$ have been generalized to an arbitrary field of characteristic different from 2. Chapter 5 of this thesis is invested to recall the following result:

Theorem 1.1.1 (Xie [86]). *If $[r, s, n]$ exists over F, then $2^{\delta(s)-i+1} | \binom{n}{i}$ for $n - r < i \le \delta(s)$. Here, the number $\delta(s)$ denotes the cardinality of the set*

$$\{l \in \mathbb{Z} : 0 < l < s, l \equiv 0, 1, 2 \text{ or } 4 \mod 8\}.$$

The proof of Theorem 1.1.1 over $\mathbb{R}$ was provided by [5] and [90]. It involves computations of topological KO-theory of real projective spaces and γ^i-operations. The statement of Theorem 1.1.1 over $\mathbb{R}$ can be extended to any field of characteristic 0 by an algebraic remark of T.Y. Lam and K.Y. Lam, cf. [76, Theorem 3.3]. By using algebraic K-theory, D. Dugger and D. Isaksen proved a similar result over an arbitrary field of characteristic $\neq 2$, where $\delta(s)$ in the above theorem is replaced by $\lfloor \frac{s-1}{2} \rfloor$, cf. [22, Theorem 1.1]. They actually conjectured the above statement. Since $\delta(s) \ge \lfloor \frac{s-1}{2} \rfloor$, our theorem generalizes theirs. One may wish to look at the following table.

s	2	3	4	5	6	7	8	9	10	$\cdots$
$\delta(s)$	1	2	2	3	3	3	3	4	5	$\cdots$
$\lfloor \frac{s-1}{2} \rfloor$	0	1	1	2	2	3	3	4	4	$\cdots$

It is not hard to see

$$\delta(s) = \begin{cases} \lfloor \frac{s-1}{2} \rfloor & \text{if } s \equiv 7, 8, 9 \mod 8; \\ \lfloor \frac{s-1}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

1.2 Grothendieck-Witt groups of quadrics

In [66], Quillen computed algebraic K groups of projective spaces. In [80], Swan studied algebraic K groups of quadric hypersurfaces. The aim of Chapter 4 is to study Grothendieck-Witt and Witt groups of quadric hypersurfaces.

In [83], Walter defined Grothendieck-Witt groups of triangulated categories and in [82] he announced Grothendieck-Witt groups of projective bundles (see also [73, Section 9]). In a conference talk [84], Walter announced the computation of Grothendieck-Witt groups and Witt groups of quadrics. However, his work has not been published. Calmès [18] kindly informed me about Walter's results. Walter probably used the Gersten-Witt spectral sequence (introduced in [13]) to compute Witt groups of quadrics. At the end of the lecture [84], he mentioned the computation of

Grothendieck-Witt groups of quadrics.

Let k be a commutative ring with $\frac{1}{2} \in k$ throughout Chapter 4, unless otherwise specified. Let $Q_d \subset \mathbb{P}^n_k$ be the quadric hypersurface defined by a non-degenerate quadratic form (P, q) over k of rank $n = d + 2$.

Theorem 1.2.1. *Let $d = 2m + 1$ be an odd integer. There is a stable equivalence of Grothendieck-Witt spectra*

$$GW^{[i]}(Q_d) \approx \bigoplus_{i=1}^{m} K(k) \oplus GW^{[i]}(\mathscr{A})$$

where $\mathscr{A}$ is the dg category defined in Section 4.3. Moreover, the spectrum $GW^{[i]}(\mathscr{A})$ fits into a homotopy fibration sequence

$$GW^{[i]}(k) \to GW^{[i]}(\mathscr{A}) \to GW^{[i+1]}(C_0(q), \sigma)$$

where $C_0(q)$ is the even part of the Clifford algebra of q and where σ is the canonical involution of $C_0(q)$.

Theorem 1.2.2. *Let $d = 2m$ be an even integer. If P is free, then there exists a homotopy fibration sequence of Grothendieck-Witt spectra*

$$\oplus_{j=1}^{m-1} K(k) \oplus GW^{[i]}(\mathscr{A}) \to GW^{[i]}(Q_d) \to GW^{[i-d]}(k)$$

where $GW^{[i]}(\mathscr{A})$ fits into a homotopy fibration sequence

$$GW^{[i]}(k) \to GW^{[i]}(\mathscr{A}) \to GW^{[i+1]}(C_0(q), \sigma)$$

Remark 1.2.1. The above theorems apply also to the case of $\mathbb{G}W$-spectra (see [73, Section 9]).

Hermitian K-theory (aka. Grothendieck-Witt theory) as a generalization of algebraic K-theory was first studied by Bass, Karoubi and others in terms of algebras with involution, cf. [14]. Schlichting generalized their work to exact categories and dg categories, the results of which make Hermitian K-theory bear a heavier burden, cf. [70], [71] and [73]. Schlichting used GW to mean Grothendieck-Witt theory, while some authors may have written K^h. As proved by Schlichting, the 0-th homotopy groups of Grothendieck-Witt spectra of schemes are Grothendieck-Witt groups of schemes. Moreover, the negative homotopy group of Grothendieck-Witt spectra are just Balmer's Witt groups. Our computations are based on Schlichting's framework.

As an application of Theorem 1.2.1, we prove the following theorem in Section 4.4.

Theorem 1.2.3. *Let k be a regular local ring with $\frac{1}{2} \in k$. If d is odd, then*

$$\begin{aligned} W^0(Q_d) &\approx \operatorname{coker}(F, \operatorname{tr}) \\ W^1(Q_d) &\approx W^2(C_0(q), \sigma) \\ W^2(Q_d) &\approx 0 \\ W^3(Q_d) &\approx \ker(F, \operatorname{tr}) \end{aligned}$$

where the map $(F, \operatorname{tr}) : W^0(C_0(q), \sigma) \longrightarrow W^0(k)$ is defined in Section 4.4.

As an application of Theorem 1.2.2, we deduce the following results in Section 4.7.

Theorem 1.2.4. *Let k be a regular local ring with $\frac{1}{2} \in k$. If d is even, then there is a 12-term long exact sequence*

$$\begin{tikzcd}
& \operatorname{coker}(F,\operatorname{tr}) \arrow[r] & W^0(Q) \arrow[r] & W^{-d}(k) \arrow[dr] & \\
W^{3-d}(k) \arrow[ur] & & & & W^2(C_0(q),\sigma) \arrow[d] \\
W^3(Q) \arrow[u] & & & & W^1(Q) \arrow[d] \\
\ker(F,\operatorname{tr}) \arrow[u] & & & & W^{1-d}(k) \arrow[dl] \\
& W^{2-d}(k) \arrow[ul] & W^2(Q) \arrow[l] & 0 \arrow[l] &
\end{tikzcd}$$

Recall the notation of the Pfister form

$$\langle\!\langle a_1, \ldots, a_n \rangle\!\rangle := \langle 1, a_1 \rangle \otimes \langle 1, a_2 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle.$$

As an application of Theorem 1.2.3 and 1.2.4, we prove the following result in Theorem 4.4.2 and Theorem 4.7.2.

Theorem 1.2.5. *Let k be a field with $\frac{1}{2} \in k$. Let Q_d be the smooth quadric associated to the quadratic form $\langle a_1, \ldots, a_n \rangle$ with $a_i \in k^\times$. Then, we have*

$$\langle\!\langle a_1a_2, \ldots, a_1a_n \rangle\!\rangle W^0(k) \subset \ker(p^* : W^0(k) \to W^0(Q_d)).$$

Moreover, assume $C_0(q)$ is a division algebra. If $d \not\equiv 0 \mod 4$, then $W^0(Q_d) \approx W^0(k)/\langle\!\langle a_1a_2, \ldots, a_1a_n \rangle\!\rangle W^0(k)$. If $d \equiv 0 \mod 4$, then there exists an exact sequence

$$0 \longrightarrow W^0(k)/\langle\!\langle a_1a_2, \ldots, a_1a_n \rangle\!\rangle W^0(k) \longrightarrow W^0(Q_d) \longrightarrow W^0(k).$$

As another application of Theorem 1.2.3 and 1.2.4, we also prove the following result in Theorem 4.4.3 and Theorem 4.7.3.

Theorem 1.2.6. *Let k be a field in which -1 is not a sum of two squares. Let Q_d be the smooth quadric associated to the quadratic form $n\langle 1\rangle$ of dimension $d = n-2$. If $d \not\equiv 0 \mod 4$, then $W^0(Q_d) \approx W^0(k)/2^{\delta(n)}W^0(k)$. If $d \equiv 0 \mod 4$, then there exists an exact sequence*

$$0 \longrightarrow W^0(k)/2^{\delta(n)}W^0(k) \longrightarrow W^0(Q_d) \longrightarrow W^0(k).$$

Moreover, if k is euclidean, then $W^0(Q_d) \approx \mathbb{Z}/2^{\delta(n)}\mathbb{Z}$. Recall that $\delta(n) := \#\{l \in \mathbb{Z} : 0 < l < n, l \equiv 0, 1, 2 \text{ or } 4 \mod 8\}$.

In [60], Witt groups of *split* quadrics were studied. In [93], Grothendieck-Witt (and Witt) groups of quadrics over $\mathbb{C}$ were computed. Chapter 4 focuses on the case of a general projective quadric over an arbitrary commutative ring k with $\frac{1}{2} \in k$. Besides, the paper [21] studied Witt groups of certain kind of quadrics by forgetting the 2 primary torsions. Chapter 4 determines the 2 primary torsion for the case of 'projective cones' $\mathbb{P}C^n$ (terminology in [21]) over an arbitrary Euclidean field.

Chapter 2

On the history of the Hurwitz problem

The 2-square identity

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$$

had been discovered in an ancient time. A similar 4-square identity was found by Euler (1748).

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2$$

where

$$\begin{aligned}
z_1 &= x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 \\
z_2 &= x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 \\
z_3 &= x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2 \\
z_4 &= x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1
\end{aligned}$$

Euler's motivation was to prove Fermat's conjecture that every positive integer is a sum of four integer squares. After Euler's discovery of 4-square identity, mathematicians soon realized a 3-square identity is impossible.

In 1843, Hamilton wrote a letter to John Graves about his discovery of the quaternion algebra, which allows a non-commutative multiplication. In fact, the 4-square identity can be interpreted by the law of moduli of the quaternion. John Graves found an algebra of 8 basis elements within two months after he received Hamilton's letter about the quaternion algebra. Nowadays, this algebra is widely known as the octonion, which is neither commutative nor associative. Independently, Cayley found the octonion in 1845. The multiplication of the octonion provides an 8-square identity, which is displayed as follows:

$$(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+x_7^2+x_8^2)(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2+y_6^2+y_7^2+y_8^2) = \\ z_1^2+z_2^2+z_3^2+z_4^2+z_5^2+z_6^2+z_7^2+z_8^2$$

where

$$\begin{aligned}
z_1 &= x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 - x_8y_8\\
z_2 &= x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3 + x_5y_6 - x_6y_5 - x_7y_8 + x_8y_7\\
z_3 &= x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2 + x_5y_7 + x_6y_8 - x_7y_6 - x_8y_6\\
z_4 &= x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1 + x_5y_8 - x_6y_7 + x_7y_5 - x_8y_5\\
z_5 &= x_1y_5 - x_2y_6 - x_3y_7 - x_4y_8 + x_5y_1 + x_6y_1 + x_7y_3 + x_8y_4\\
z_6 &= x_1y_6 + x_2y_5 - x_3y_8 + x_4y_7 - x_5y_2 + x_6y_2 - x_7y_4 + x_8y_3\\
z_7 &= x_1y_7 + x_2y_8 + x_3y_5 - x_4y_6 - x_5y_3 + x_6y_4 + x_7y_1 - x_8y_2\\
z_8 &= x_1y_8 - x_2y_7 + x_3y_6 + x_4y_5 - x_5y_4 - x_6y_3 + x_7y_2 + x_8y_1
\end{aligned}$$

Actually, such a formula had already been found in 1818 by Degen in Russia, but his work was not widely known.

Afterward, mathematicians attempted to achieve a 16-square identity, and they came to realize that a 16-square identity is impossible. In 1898, A. Hurwitz published a celebrated paper which provided a definitive answer on this topic, cf. [38]. Hurwitz proved that there exists an n-square identity over the complex numbers if and only if n is 1, 2, 4 or 8. At the end of this paper, Hurwitz asked a general question: For which positive integers r, s, n does there exist a formula

$$(x_1^2+\cdots+x_r^2)\cdot(y_1^2+\cdots+y_s^2) = z_1^2+\cdots+z_n^2$$

where $z_i = z_i(X,Y)$ is bilinear in X's and Y's where $X = (x_1,...,x_r)$ and $Y = (y_1,...,y_s)$ are systems of coordinates with coefficients in the base field of characteristic $\neq 2$? Nowadays, mathematicians denote such a formula by $[r,s,n]$.

It is easy to see a formula $[r,s,n]$ implies a formula $[r,s,n+1]$ by just adding the '0-term', which is bilinear. Thus, the Hurwitz problem is equivalent to the problem: Given positive integers r and s, what is the smallest n such that $[r,s,n]$ exists? The problem is attractive, because results towards this problem influence several branches of mathematics.

In 1922, Radon [67] was able to show that $[r,n,n]$ exists over $\mathbb{R}$ if and only if $r \leq \rho(n)$ where $\rho(n) := 8a+2^b$ if we write $n = \text{odd}\cdot 2^{4a+b}$ with $0 \leq b \leq 3$. In 1923, Hurwitz [39] independently found that the same result holds over the field of complex numbers $\mathbb{C}$. Thus, this result is widely known as the *Hurwitz-Radon theorem*, and the number $\rho(n)$ is usually called the *Hurwitz-Radon number*. Some authors also provided their own proofs towards the Hurwitz-Radon theorem, cf. [4], [20] [26], [58], [85] and [76].

Mathematicians also found that the Hurwitz problem highly relates to some topics in Topology. For example, a formula $[\rho(n), n, n]$ gives $\rho(n) - 1$ linearly independent vector fields on the sphere S^{n-1}. Finally, Adams [1] showed that this number is optimal in 1960s by introducing Adams operations and by calculating KO-theory of real projective spaces. Besides, a formula $[r, s, n]$ gives a non-singular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$. However, a non-singular bilinear map does not necessarily give a sums-of-squares. Thus, finding the lower bounds of non-singular bilinear maps only helps to find the lower bounds of sums-of-squares formulas. Around 1940, Stiefel and Hopf observed that some non-singular bilinear maps could be eliminated by cohomology methods, which provides certain lower bounds, cf. [37] and [79]. Stiefel used his theory of characteristic classes of vector bundles, and Hopf applied cohomology theory of real projective spaces. They proved that: If a non-singular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ exists, then $\binom{n}{i}$ is even whenever $s > i > n - r$. In 1980s, an argument of Atiyah [5] was applied by Yuzvinsky ([90]) to prove that: If a non-singular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^n$ exists, then $2^{\delta(s)+i-1} | \binom{n}{i}$ whenever $\delta(s) \geq i > n - r$ where $\delta(s) := \#\{l \in \mathbb{Z} : 0 < l < s, l \equiv 0, 1, 2 \text{ or } 4 \mod 8\}$. Indeed, non-singular bilinear maps are very important in the study of Topology and non-associative algebras. Using topological methods, K. Y. Lam constructed some interesting non-singular bilinear maps, cf. [52], [53], [54] and [55].

It is also valuable to consider formulas over $\mathbb{Z}$ because coefficients can only be taken over the set $\{-1, 0, 1\}$. Then the problem becomes combinatorial in nature. Around 1980s, Yuzvinsky, Yiu and others studied the framework of "intercalate" matrices. By this method, Yiu announced formulas $[r, s, n]$ over $\mathbb{Z}$ for $r, s \leq 16$ and other interesting formulas, and lower bounds, cf. [87], [88] and [89].

What about $[r, s, n]$ over an arbitrary field of characteristic $\neq 2$? Does it depend on the choice of base fields of characteristic $\neq 2$? Yuzvinsky developed the theory of monomial pairings which is useful for finding upper bounds, cf. [90], [91] and [92]. In [2] and [3], Adem reduced the existence of a formula of type $[r, n-1, n]$ to the Hurwitz-Radon theorem by matrices reduction. Adem also found a formula of type $[r, n-2, n]$ could be manipulated in a similar way. In a conference around 1980,

T. Y. Lam and K. Y. Lam discovered that a formula of type $[r, s, n]$ over a field of characteristic 0 implies a non-singular bilinear map $[r, s, n]$ over $\mathbb{R}$. This observation extends some classical results over $\mathbb{R}$ (e.g. the Hopf-Stiefel's condition) to any arbitrary field of characteristic 0. In the meantime, some mathematicians attempted to extend the Hopf-Stiefel's condition to an arbitrary field of characteristic different from 2, and they got some partial results, cf. [77] and [91]. In 2007, Dugger and Isaksen finally proved the Hopf-Stiefel's condition over any field of char. $\neq 2$ by using motivic cohomology, cf. [23]. Indeed, if $r \leq 9$, the Hopf-Stiefel condition implies a formula $[r, s, n]$ over any field of characteristic $\neq 2$, cf. [78]. Thus, the existence of $[r, s, n]$ for $r \leq 9$ is exactly given by the Hopf-Stiefel's condition. Dugger and Isaksen wrote another two papers that generalize some classical theorems towards sums-of-squares formulas over $\mathbb{R}$ to an arbitrary field of characteristic $\neq 2$, cf. [22] and [24].

For further information about histories and applications of sums-of-squares formulas, one may refer to [76] and [78].

Chapter 3

Review: Hermitian K-theory

In this chapter, we review some definitions and results in [73] which will be used later. This chapter does not contain anything essentially new. I do not claim originality.

3.1 Categories with duality

3.1.1 Definitions and examples

Let $\mathscr{C}$ be a category. Recall from [7], [47], [73] that

Definition 3.1.1. A *category with duality* is a triplet $(\mathscr{C}, *, \eta)$ where $(*, \eta)$ is a *duality on* $\mathscr{C}$, that is a functor $* : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ together with a natural transformation $\eta : 1 \to * \circ *^{\mathrm{op}}$ such that $1_{A^*} = (\eta_A^{\mathrm{op}})^* \circ \eta_{A^*}$, i.e. the following diagram

$$\begin{tikzcd} A^* \arrow[r, "id_{A^*}"] \arrow[d, "\eta_{A^*}"'] & A^* \\ A^{***} \arrow[ur, "(\eta_A^{\mathrm{op}})^*"'] & \end{tikzcd} \tag{3.1}$$

commutes where we write $A^* := *(A)$ and $f^* := *(f)$ for convenience. If η is an isomorphism (resp. the identity), we call the duality $(*, \eta)$ *strong* (resp. *strict*).

Remark 3.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. Recall that the quintuple

$$(\mathcal{A}, \mathcal{B}, F \dashv G, \eta, \epsilon)$$

consists of a functor $F : \mathcal{A} \to \mathcal{B}$ (*the left adjoint*), a functor $G : \mathcal{B} \to \mathcal{A}$ (*the right adjoint*) and two natural transformations $\eta : 1_{\mathcal{A}} \to GF$ (the *unit*) and $\epsilon : FG \to 1_{\mathcal{B}}$ (the *counit*), which satisfy the *triangular equation*

$$1_F = \epsilon F \circ F\eta \qquad 1_G = G\epsilon \circ \eta G$$

with the symbol $\circ$ representing the *horizontal composition* of natural transformations. If $(\mathcal{A}, \mathcal{B}, F \dashv G, \eta, \epsilon)$ satisfies further condition that η and ϵ are natural isomorphisms, we say $\mathcal{A}$ and $\mathcal{B}$ are *adjoint equivalent.* As observed in [19, Definition 2.1.1], deducing a category with duality is equivalent to specifying the data $(\mathscr{C}, \mathscr{C}^{\mathrm{op}}, *^{\mathrm{op}} \dashv *, \eta, \eta^{\mathrm{op}})$. Then, to define a category with strong duality is to give the information $(\mathscr{C}, \mathscr{C}^{\mathrm{op}}, *^{\mathrm{op}} \dashv *, \eta, \eta^{\mathrm{op}})$ with η and η^{op} natural isomorphisms.

To understand categories with duality better, we give the following examples which may be used later.

Example 3.1.1 (Example 2.2.1, Chapter II [47])**.** Let R be a ring with an involution σ. Consider the category $\mathcal{M}(R)$ of left R-modules. Then, for any object M in $\mathcal{M}(R)$, the set $\mathrm{Hom}_R(M, R)$ of left R-module morphisms may be considered as a left R-module via the involution σ. To illustrate, the action is given by

$$R \times \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(M, R), (r, f) \mapsto r \cdot f$$

where $r \cdot f(x) := f(x)\sigma(r)$. Clearly, $r \cdot f$ is a left R-module homomorphism. Then, we check $(r_1 r_2) \cdot f = r_1 \cdot (r_2 \cdot f)$ for all $r_1, r_2 \in R$. For any $x \in M$, we have $(r_1 r_2) \cdot f(x) = f(x)\sigma(r_1 r_2) = f(x)(\sigma(r_2)\sigma(r_1))$ which equals $r_1 \cdot (r_2 \cdot f)(x) = (r_2 \cdot f(x))\sigma(r_1) = (f(x)\sigma(r_2))\sigma(r_1)$ by associativity. We write $\sigma(\mathrm{Hom}_R(M, R))$ for this left R-module. There is a functor $\# : \mathcal{M}(R)^{\mathrm{op}} \to \mathcal{M}(R)$ given by

$$M \mapsto \sigma(\mathrm{Hom}_R(M, R)), \quad (f : M \to N) \mapsto (f^{\#} : N^{\#} \to M^{\#})$$

where $f^{\#}(g) := g \circ f$. Construct a natural transformation $\mathrm{can} : 1_{\mathcal{M}(R)} \to \# \circ \#$ by

$$\mathrm{can}_M : M \to M^{\#\#}, x \mapsto \hat{x},$$

where $\hat{x}(g) := \sigma(g(x))$ for all $g \in M^{\#}$. One checks that

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\mathrm{can}_M"'] & N \arrow[d, "\mathrm{can}_N"] \\ M^{\#\#} \arrow[r, "f^{\#\#}"'] & N^{\#\#} \end{tikzcd}$$

is commutative. Since the functor $\#$ depends on the involution σ, we may write $\#_\sigma$ for $\#$ to emphasize the role of the involution σ. Then, in order to say the triplet

$$(\mathcal{M}(R), \#_\sigma, \mathrm{can})$$

is a category with duality, one has to do an easy exercise to check the diagram (3.1) is commutative. Note that this is not a strong duality in general (even when R is a commutative field), cf. [17, Proof of Theorem 6 in p. 300, Chapter II 7.5]. However, let $\mathcal{P}(R)$ be the category of finitely generated projective left R-modules. We equip $\mathcal{P}(R)$ with the duality $(\#_\sigma, \mathrm{can})$ defined above. Then, the triplet

$$(\mathcal{P}(R), \#_\sigma, \mathrm{can})$$

is a category with strong duality (but not strict duality), since can_M is an isomorphism (rather than the identity) for all $M \in \mathcal{P}(R)$, cf. [17, Corollary 4, Chapter II 2.7].

Example 3.1.2. Let R be a ring with an involution σ. Consider the category $\underline{e}$ with one object e and with morphisms $\mathrm{Hom}_{\underline{e}}(e, e) := \mathrm{End}_R(R) = R^{\mathrm{op}}$. Define a functor $* : \underline{e}^{\mathrm{op}} \to \underline{e}$ by $*(e) := e, *(f) := \sigma(f)$ for any $f \in R^{\mathrm{op}}$. The natural transformation $\mathrm{can} : 1_{\underline{e}} \to * \circ *$ is given by $\mathrm{can}_e : e \to e^{**} = e$ corresponding to the identity. Since $\sigma\sigma(f) = f$, we have a category with strict duality

$$(\underline{e}, *_\sigma, \mathrm{can})$$

where we write $*_\sigma$ to emphasize the role of the involution σ.

3.1.2 Symmetric forms and form functors

Definition 3.1.2. Let $(\mathscr{C}, *, \eta)$ be a category with duality. A *symmetric form* in $(\mathscr{C}, *, \eta)$ is a pair (X, φ) where X is an object in $\mathscr{C}$ and where $\varphi : X \to X^*$ is a morphism such that $\varphi^* \circ \eta_X = \varphi$. A *morphism of symmetric forms* $f : (X, \varphi) \to (Y, \psi)$ is a map $f : X \to Y$ in $\mathscr{C}$ such that $\varphi = f^* \circ \psi \circ f$.

In light of Definition 3.1.2, we can talk about the category of symmetric forms in $(\mathscr{C}, *, \eta)$ which is denoted by $(\mathscr{C}, *, \eta)_h$.

Definition 3.1.3. Let $(\mathcal{B}, \vee, \omega)$ and $(\mathcal{C}, *, \eta)$ be categories with duality. A *form functor* from $(\mathcal{B}, \vee, \omega)$ to $(\mathcal{C}, *, \eta)$ is a pair (F, ϵ) where $F : \mathcal{B} \to \mathcal{C}$ is a functor and where $\epsilon : F \circ \vee \to * \circ F$ is a natural transformation such that $\epsilon \vee \circ F(\omega) = *\epsilon \circ \eta_F$.

3.2 Dg categories with duality

3.2.1 The category of complexes

Let k be a commutative ring in this section and let $C(k)$ be the category of dg k-modules (*aka.* complexes). Its objects are the pairs (E, d) where E are all direct sum decompositions $\bigoplus_{i \in \mathbb{Z}} E^i$, and where $d = \{d^i\}_{i \in \mathbb{Z}}$ are sequences of k-linear maps (called differential) $d^i : E^i \to E^{i+1}$ such that $d^{i+1} \circ d^i = 0$. A morphism $f : X \to Y$ of dg

k-modules consists of maps $\{f^i : X^i \to Y^i\}_{i\in\mathbb{Z}}$ satisfying $f^i((d^X)^i(x)) = (d^Y)^i f^i(x)$ for each $x \in X^i$. There are functors $Z^0, H^0 : C(k) \to Mod_k$ to the category of k-modules defined by

$$Z^0(M) = \mathrm{Ker}(d^0 : M^0 \to M^1), \qquad H^0(M) = Z^0(M)/\mathrm{Im}(d^{-1} : M^{-1} \to M^0).$$

For $X, Y \in C(k)$, the *tensor product dg k-module* $X \otimes Y$ in degree n is given by

$$(X \otimes Y)^n = \bigoplus_{i+j=n} X^i \otimes Y^j$$

with differential $d(x\otimes y) = (d^X x)\otimes y + (-1)^{|x|} x \otimes (d^Y y)$. It gives a *symmetric monoidal category* $(C(k), \otimes, \mathbb{1}, a, l, r, c)$ in the sense of [43] where the unit $\mathbb{1}$ is the dg k-module consisting of k concentrated in degree 0 and where natural isomorphisms a, l, r and c are

$$\begin{aligned}
&a_{XYZ} : (X \otimes Y) \otimes Z \xrightarrow{=} X \otimes (Y \otimes Z) : (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z),\\
&l_X : \mathbb{1} \otimes X \to X : a \otimes x \mapsto ax,\\
&r_X : X \otimes \mathbb{1} \to X : x \otimes a \mapsto xa,\\
&c_{XY} : X \otimes Y \to Y \otimes X : x \otimes y \mapsto (-1)^{|x||y|} y \otimes x.
\end{aligned}$$

The *function dg k-module* $[X, Y]$ in degree n is given by

$$[X, Y]^n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_k(X^i, Y^{i+n}) \tag{3.2}$$

with differential $d(f) = d^Y \circ f - (-1)^n f \circ d^X$. Note that an element in $Z^0[A, B]$ is exactly a morphism of complexes. This makes the symmetric monoidal category $(C(k), \otimes, \mathbb{1}, a, l, r, c)$ *closed*, i.e. for each object $Y \in C(k)$, the functor $- \otimes Y$ has a right adjoint $[Y, -]$ with unit and counit

$$d_X : X \to [Y, X \otimes Y] : x \mapsto (f_x : y \mapsto x \otimes y), \qquad e_Z : [Y, Z] \otimes Y \to Z : f \otimes x \mapsto f(x).$$

Remark 3.2.1. One should be cautious that the function object $[X, Y]$ may be endowed with another choice of differential by other authors, which is given by $d(f) = f \circ d^X - (-1)^n d^Y \circ f$. We may denote this by $[X, Y]_\dagger$ to distinguish from $[X, Y]$ defined above. Note that $[-, -]_\dagger$ also gives a closed structure on $(C(k), \otimes, \mathbb{1}, a, l, r, c)$ with unit and counit

$$\begin{aligned}
&d^\dagger_X : X \to [Y, X \otimes Y] : x \mapsto (f_x : y \mapsto (-1)^{\frac{|x|(|x|+1)}{2}} x \otimes y),\\
&e^\dagger_Z : [Y, Z] \otimes Y \to Z : f \otimes x \mapsto (-1)^{\frac{|f|(|f|+1)}{2}} f(x).
\end{aligned}$$

3.2.2 Dg categories and co-dg categories

Definition 3.2.1 ([43])**.** A *dg category* $\mathcal{A}$ *over* k is a category enriched in the mononidal category $(C(k), \otimes, \mathbb{1}, a, l, r)$ in the sense of [43, Section 1.2]. That is given by the following data:

- a set of objects $\mathrm{ob}(\mathcal{A})$,
- for any two objects A and B, a dg k-module $\mathcal{A}(A,B) \in C(k)$,
- for any object A, a unit morphism $e_A : \mathbb{1} \to \mathcal{A}(A,A)$,
- for any three objects A, B and C, a composition morphism

$$\mu_{ABC} : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \to \mathcal{A}(A,C)$$

satisfying the following conditions:

1. (Associativity) for any four objects A, B, C and D the following diagram

$$\begin{tikzcd} \mathcal{A}(C,D) \otimes \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \arrow[r, "\mu_{BCD}\otimes id"] \arrow[d, "id\otimes\mu_{ABC}"'] & \mathcal{A}(B,D) \otimes \mathcal{A}(A,B) \arrow[d, "\mu_{ABD}"] \\ \mathcal{A}(C,D) \otimes \mathcal{A}(A,C) \arrow[r, "\mu_{ACD}"'] & \mathcal{A}(A,D) \end{tikzcd}$$

   commutes.

2. (Unit) for any two objects A and B, the two morphisms

$$\mathcal{A}(A,B) \xrightarrow{l^{-1}} \mathbb{1} \otimes \mathcal{A}(A,B) \xrightarrow{e_B\otimes id} \mathcal{A}(B,B) \otimes \mathcal{A}(A,B) \xrightarrow{\mu_{ABB}} \mathcal{A}(A,B)$$
$$\mathcal{A}(A,B) \xrightarrow{r^{-1}} \mathcal{A}(A,B) \otimes \mathbb{1} \xrightarrow{id\otimes e_A} \mathcal{A}(A,B) \otimes \mathcal{A}(A,A) \xrightarrow{\mu_{AAB}} \mathcal{A}(A,B)$$

   are equal to the identities.

Definition 3.2.2 ([43])**.** Let $\mathcal{A}, \mathcal{B}$ be two dg categories. A dg functor $F : \mathcal{A} \to \mathcal{B}$ consists of the following data:

- a map of sets $F : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$,
- for any two objects A and B, a morphism of dg k-module

$$F_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(FA, FB)$$

These data should satisfy the following:

1. for any three objects A, B and C, the following diagram

$$\begin{tikzcd}
\mathcal{A}(B,C)\otimes\mathcal{A}(A,B) \arrow[rr, "\mu^{\mathcal{A}}_{ABC}"] \arrow[d, "F_{BC}\otimes F_{AB}"'] & & \mathcal{A}(A,C) \arrow[d, "F_{AC}"] \\
\mathcal{B}(FB,FC)\otimes\mathcal{B}(FA,FB) \arrow[rr, "\mu^{\mathcal{B}}_{FA,FB,FB}"'] & & \mathcal{B}(FA,FC)
\end{tikzcd}$$

commutes.

2. for any object A, the following diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "e^{\mathcal{A}}_{A}"] \arrow[dr, "e^{\mathcal{B}}_{FA}"'] & \mathcal{A}(A,A) \arrow[d, "F_{AA}"] \\
& \mathcal{B}(FA,FA)
\end{tikzcd}$$

commutes.

Let $F:\mathcal{A}\to\mathcal{B}$ and $G:\mathcal{B}\to\mathcal{C}$ be two dg functors. One may define their composition $G\circ F:\mathcal{A}\to\mathcal{C}$ as the following:

- composition of maps of sets $G\circ F:\mathrm{ob}(\mathcal{A})\to\mathrm{ob}(\mathcal{B})\to\mathrm{ob}(\mathcal{C})$
- for any two objects A and B, a morphism of dg k-module

$$G_{FA,FB}\circ F_{AB}:\mathcal{A}(A,B)\to\mathcal{B}(FA,FB)\to\mathcal{B}(GFA,GFB)$$

Lemma 3.2.1. *The datum $G\circ F:\mathcal{A}\to\mathcal{C}$ defined above is a dg functor. Moreover, we have constructed a category of dg categories with morphisms dg functors. We record this category by* ***DgCat****.*

Proof. Easy verification and exercise. □

Compare the definition of dg categories and dg functors to the following definitions of co-dg categories and co-dg functors.

Definition 3.2.3 ([81])**.** A *co-dg category $\mathcal{A}$ over k* is given by the following data:

- a set of objects $\mathrm{ob}(\mathcal{A})$,
- for any two objects A and B, a dg k-module $\mathcal{A}(A,B)\in C(k)$,
- for any object A, a unit morphism $e_A:\mathbb{1}\to\mathcal{A}(A,A)$,

- for any three objects A, B and C, a co-composition morphism

$$\mu_{ABC} : \mathcal{A}(A,B) \otimes \mathcal{A}(B,C) \to \mathcal{A}(A,C)$$

satisfying the following conditions:

1. (Associativity) for any four objects A, B, C and D, the following diagram

$$\begin{tikzcd}
\mathcal{A}(A,B) \otimes \mathcal{A}(B,C) \otimes \mathcal{A}(C,D) \arrow[r, "id\otimes\mu_{BCD}"] \arrow[d, "\mu_{ABC}\otimes id"'] & \mathcal{A}(A,B) \otimes \mathcal{A}(B,D) \arrow[d, "\mu_{ABD}"] \\
\mathcal{A}(A,C) \otimes \mathcal{A}(C,D) \arrow[r, "\mu_{ACD}"'] & \mathcal{A}(A,D)
\end{tikzcd}$$

   commutes,

2. (Unit) for any two objects A and B, the two morphisms

$$\mathcal{A}(A,B) \xrightarrow{l^{-1}} \mathbb{1} \otimes \mathcal{A}(A,B) \xrightarrow{e_A \otimes id} \mathcal{A}(A,A) \otimes \mathcal{A}(A,B) \xrightarrow{\mu_{AAB}} \mathcal{A}(A,B)$$
$$\mathcal{A}(A,B) \xrightarrow{r^{-1}} \mathcal{A}(A,B) \otimes \mathbb{1} \xrightarrow{id \otimes e_B} \mathcal{A}(A,B) \otimes \mathcal{A}(B,B) \xrightarrow{\mu_{ABB}} \mathcal{A}(A,B)$$

   are equal to the identities.

Definition 3.2.4 ([43])**.** Let $\mathcal{A}, \mathcal{B}$ be two co-dg categories. A co-dg functor $F : \mathcal{A} \to \mathcal{B}$ consists of the following data:

- a map of sets $F : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$,
- for any two objects A and B, a morphism of dg k-module

$$F_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(FA, FB)$$

These data should satisfy the following:

1. for any three objects A, B and C, the following diagram

$$\begin{tikzcd}
\mathcal{A}(A,B) \otimes \mathcal{A}(B,C) \arrow[r, "\mu^{\mathcal{A}}_{ABC}"] \arrow[d, "F_{AB}\otimes F_{BC}"'] & \mathcal{A}(A,C) \arrow[d, "F_{AC}"] \\
\mathcal{B}(FA,FB) \otimes \mathcal{B}(FB,FC) \arrow[r, "\mu^{\mathcal{B}}_{FA,FB,FC}"'] & \mathcal{B}(FA,FC)
\end{tikzcd}$$

   commutes.

2. for any object A, the following diagram

$$\begin{tikzcd} \mathbb{1} \arrow[r, "e^{\mathcal{A}}_{A}"] \arrow[dr, "e^{\mathcal{B}}_{FA}"'] & \mathcal{A}(A,A) \arrow[d, "F_{AA}"] \\ & \mathcal{B}(FA,FA) \end{tikzcd}$$

commutes.

Remark 3.2.2 (Caution)**.** In literature, some authors define our 'co-dg categories' and 'co-dg functors' by the term 'dg categories' and 'dg functors'. This shouldn't be confused with our definition of dg categories and dg functors. One can see directly the difference between dg categories and co-dg categories is on the data of composition and co-composition and on the two conditions. Co-dg functors (resp. dg functor) only make sense when we talk about co-dg categories (resp. dg categories).

Lemma 3.2.2. *Let $G : \mathcal{A} \to \mathcal{B}, F : \mathcal{B} \to \mathcal{C}$ be two co-dg functors, and let $G \circ F$ be the data given by Lemma 3.2.1. Then, $G \circ F : \mathcal{A} \to \mathcal{C}$ defined above is a co-dg functor. Moreover, we construct a category of co-dg categories with morphisms co-dg functors. We denote this category by **coDgCat**.*

Proposition 3.2.1. *There is an equivalence of categories*

$$\Psi : \boldsymbol{coDgCat} \to \boldsymbol{DgCat} \qquad \Phi : \boldsymbol{DgCat} \to \boldsymbol{coDgCat}$$

Proof. Let $\mathcal{A}$ be a co-dg category. We define a dg category $\Psi(\mathcal{A})$ by the following data. Objects of $\Psi(\mathcal{A})$ are the same with $\mathcal{A}$, i.e. ob$(\mathcal{A})$. Morphisms of $\Psi(\mathcal{A})$ are also the same as $\mathcal{A}$, i.e. $\Psi(\mathcal{A})(A,B) = \mathcal{A}(A,B)$ for any pair $A, B \in ob(\mathcal{A})$. Unit of $\Psi(\mathcal{A})$ is just the unit of $\mathcal{A}$. Finally, composition in $\Psi(\mathcal{A})$ is given by

$$\mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \xrightarrow{c} \mathcal{A}(A,B) \otimes \mathcal{A}(B,C) \xrightarrow{\mu_{ABC}} \mathcal{A}(A,C)$$

where c is the symmetry. One checks that $\Psi(\mathcal{A})$ is a dg category. Let $F : \mathcal{A} \to \mathcal{B}$ be a co-dg functor, i.e. a morphism in **coDgCat**. Then, define $\Psi(F) : \Psi(\mathcal{A}) \to \Psi(\mathcal{B})$ by the same data of F. One carefully checks that $\Psi(F)$ becomes a dg functor. Composition formula $\Psi(G \circ F) = \Psi(G) \circ \Psi(F)$ for dg functors is trivial.

Let $\mathcal{B}$ be a dg category and F a dg functor. One may define $\Phi(\mathcal{B})$ and $\Phi(F)$ analogously to the above. One checks that $\Phi(\mathcal{B})$ is a co-dg category and $\Phi(F)$ is a co-dg functor. Note that $\Psi \circ \Phi(\mathcal{A}) = \mathcal{A}$ and $\Phi \circ \Psi(\mathcal{B}) = \mathcal{B}$, because $c \circ c$ is the identity. So, Ψ and Φ are even inverse to each other. $\square$

For later reference, it is necessary to discuss the following examples.

Example 3.2.1. Let $\mathcal{A}$ be a dg category. Its opposite category is a dg category given by the following data. Objects and unit are the same as $\mathcal{A}$. For any $A, B \in \mathcal{A}$ morphisms $\mathcal{A}^{\text{op}}(A, B) \coloneqq \mathcal{A}(B, A)$. Compositions are given by

$$\mu^{\text{op}} : \mathcal{A}^{\text{op}}(B, C)^n \otimes \mathcal{A}^{\text{op}}(A, B)^m \longrightarrow \mathcal{A}^{\text{op}}(A, C)^{m+n} : f^{\text{op}} \otimes g^{\text{op}} \mapsto (-1)^{|f||g|}(g \circ f)^{\text{op}}$$

Therefore, we conclude that the opposite category of a dg category is again a dg category rather than a co-dg category. One could also define the opposite category of a co-dg category, which can be shown as a co-dg category. Note that $(\mathcal{A}^{\text{op}})^{\text{op}} = \mathcal{A}$.

Example 3.2.2. Let $F : \mathcal{A} \to \mathcal{B}$ be a dg functor. Its opposite functor $F^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{B}^{\text{op}}$ is defined by the map $F : \text{ob}(\mathcal{A}) \to \text{ob}(\mathcal{B})$ on objects, and a morphism $F^{\text{op}}_{AB} : \mathcal{A}^{\text{op}}(A, B) \to \mathcal{B}^{\text{op}}(FA, FB), f^{\text{op}} \mapsto F(f)^{\text{op}}$. Check that F^{op} is a dg functor and $F^{\text{op}} \circ G^{\text{op}} = (F \circ G)^{\text{op}}$.

Example 3.2.3. There is a dg category $\underline{C(k)}$ whose objects are the same as $C(k)$ and for any two objects A, B the morphism is defined by the function object $[A, B]$ in $C(k)$. The composition in degree $m + n$

$$\mu_{ABC} : [B, C]^n \otimes [A, B]^m \longrightarrow [A, C]^{m+n}$$

is defined by component-wise compositions $g \otimes f \mapsto g \circ f$, i.e.

$$\{(g_n)^i : B^i \to B^{i+n}\}_{i \in \mathbb{Z}} \otimes \{(f_m)^i : A^{i-m} \to B^i\}_{i \in \mathbb{Z}} \mapsto \{(g_n)^i \circ (f_m)^i : A^{i-m} \to B^{i+n}\}_{i \in \mathbb{Z}}.$$

Note that $\underline{C(k)}$ is not a co-dg category.

Example 3.2.4. There is a co-dg category $\underline{C(k)}_\dagger$ with objects are the same as $C(k)$. For any two objects A, B morphism is defined by $[A, B]_\dagger \in C(k)_\dagger$. The composition in degree $m + n$

$$\mu_{ABC} : [A, B]^m_\dagger \otimes [B, C]^n_\dagger \longrightarrow [A, C]^{m+n}_\dagger$$

is defined by $f \otimes g \mapsto g \circ f$, i.e.

$$\{(f_m)^i : A^{i-m} \to B^i\}_{i \in \mathbb{Z}} \otimes \{(g_n)^i : B^i \to B^{i+n}\}_{i \in \mathbb{Z}} \mapsto \{(g_n)^i \circ (f_m)^i : A^{i-m} \to B^{i+n}\}_{i \in \mathbb{Z}}.$$

(Check μ_{ABC} is a morphism of dg k-modules). Note that $\underline{C(k)}_\dagger$ itself is not a dg category. However, we can take its associated dg category $\Psi(\underline{C(k)}_\dagger)$. One may check the composition in $\Psi(\underline{C(k)}_\dagger)$ actually coincides (1.28) in [43, Section 1.6] from the closed symmetric monoidal structure of $C(k)$ with the internal hom $[-, -]_\dagger$. It is the unique shape in the sense of [44].

Remark 3.2.3 (Caution)**.** For simplicity, we may abuse the notation $\underline{C(k)}_\dagger$ to mean the dg category $\Psi(\underline{C(k)}_\dagger)$ if there is no confusion on the compositions.

Lemma 3.2.3. *The data $F : \underline{C(k)}_\dagger \to \underline{C(k)}$ given by the identity map on sets*

$$\mathrm{id} : \mathrm{ob}(\underline{C(k)}_\dagger) \to \mathrm{ob}(\underline{C(k)})$$

and by a morphism of dg k-modules

$$F_{AB} : [A,B]_\dagger \to [A,B] : f \mapsto (-1)^{\frac{|f|(|f|+1)}{2}} f$$

specify a dg functor.

Proof. Check that F_{AB} is a morphism of dg k-modules. This is to check

$$F_{AB}(d^{[A,B]_\dagger} f) = d^{[A,B]}(F_{AB}(f))$$

Left hand side equals

$$(-1)^{\frac{(|f|+1)(|f|+2)}{2}} (f \circ d^A + (-1)^{|f|+1} d^B \circ f). \tag{3.3}$$

Right hand side equals

$$(-1)^{\frac{(|f|)(|f|+1)}{2}} (d^B \circ f + (-1)^{|f|+1} f \circ d^A). \tag{3.4}$$

To see (3.3) = (3.4), one checks the signs match. We verify the diagram

$$\begin{tikzcd}
{[B,C]_\dagger \otimes [A,B]_\dagger} \arrow[r, "\mu_{ABC} \circ c"] \arrow[d, "F_{BC} \otimes F_{AB}"'] & {[A,C]_\dagger} \arrow[d, "F_{AC}"] \\
{[B,C] \otimes [A,B]} \arrow[r, "\mu_{ABC}"'] & {[A,C]}
\end{tikzcd}$$

commutes. This is to check the identity

$$F_{AC}(\mu_{ABC} \circ c(f \otimes g)) = \mu_{ABC}(F_{BC}(f) \otimes F_{AB}(g)).$$

Left hand side of this identity equals

$$(-1)^{|f||g|}(-1)^{\frac{(|f|+|g|)(|f|+|g|+1)}{2}} f \circ g.$$

Right hand side of this identity equals

$$(-1)^{\frac{|f|(|f|+1)}{2}}(-1)^{\frac{|g|(|g|+1)}{2}} f \circ g$$

One checks that the signs match. The unit condition simply holds because $\mathbb{1}$ concentrates in degree 0. □

Proposition 3.2.2. *Let A be any object in $C(k)$. The following data*

$$[-,A] : \underline{C(k)}^{\mathrm{op}} \to \underline{C(k)} \quad (\text{resp. } [-,A]_\dagger : \underline{C(k)}_\dagger^{\mathrm{op}} \to \underline{C(k)}_\dagger)$$

given by the map

$$C(k) \to C(k) : X \mapsto [X,A]$$

of sets and by the morphism

$$\epsilon : [X,Y]_{\underline{C(k)}^{\mathrm{op}}} \to [[X,A],[Y,A]] : f^{\mathrm{op}} \mapsto (\epsilon(f^{\mathrm{op}}) : g \mapsto (-1)^{|f||g|} g \circ f)$$

of dg k-modules (resp. by the morphism

$$\epsilon_\dagger : [X,Y]_{\underline{C(k)}_\dagger^{\mathrm{op}}} \to [[X,A]_\dagger,[Y,A]_\dagger]_\dagger : f^{\mathrm{op}} \mapsto (\epsilon_\dagger(f^{\mathrm{op}}) : g \mapsto (-1)^{\frac{|f|(|f|+1)}{2}} g \circ f)$$

of dg k-modules) specify a dg functor.

Proof. The functors $[-,A]$ and $[-,A]_\dagger$ are *contravariant representable* dg functors in the sense of [43, Section 1.6], so we can deduce all the maps under the procedure introduced in *loc. cit.*. □

Definition 3.2.5. Let $\mathcal{A}$ be a dg category. The *underlying category* $Z^0\mathcal{A}$ (resp. the *homotopy category* $H^0\mathcal{A}$) of the dg category $\mathcal{A}$ is a k-linear category which has objects same as $\mathcal{A}$ and for any two objects A, B morphism $(Z^0\mathcal{A})(A,B)$ (resp. morphism $(H^0\mathcal{A})(A,B)$) is defined by the k-module $Z^0(\mathcal{A}(A,B))$ (resp. $H^0(\mathcal{A}(A,B))$).

Example 3.2.5. The category $\mathscr{C}_k$ of bounded chain complexes of finitely generated free modules over k may be considered as a full dg subcategory of $\underline{\mathscr{C}(k)}$ which is denoted by $\underline{\mathscr{C}_k}$ (also denoted by $\mathscr{C}$ for short).

Example 3.2.6. Let R be an associative and unital k-algebra. Consider the category $C(R)$ of complexes of left (or right) R-modules. There is a dg category $\underline{C(R)}$ which generalizes $\underline{C(k)}$. For any two complexes $A, B \in C(R)$ the function object $[A,B]$ is regarded as a dg k-module. Then, $Z^0C(R)$ is just the category $C(R)$ itself and $H^0C(R)$ is the homotopy category of $C(R)$.

Example 3.2.7. The category $Ch^b(R)$ of bounded chain complexes of finitely generated projective left R-modules may be reinterpreted as a full dg subcategory of $\underline{C(R)}$. Denote it by sPerf(R).

Example 3.2.8. Let X be a scheme. Consider the category $C(X)$ of complexes of coherent sheaves over X. There is a dg category $\underline{C(X)}$ with the same objects as in $C(X)$. For any $\mathscr{F}, \mathscr{G} \in C(X)$, morphisms are given by the function object $[\mathscr{F}, \mathscr{G}] \in C(X)$ which is similar to Equation 3.2. Compositions are defined in a similar way as in Example 3.2.3.

Example 3.2.9. The category $Ch^b(X)$ of bounded chain complexes of finitely generated vector bundles over X may be regarded as a full dg subcategory of $\underline{C(X)}$. Denote it by $\mathrm{sPerf}(X)$.

Example 3.2.10. Let R be a k-algebra. The category $*$ with one object e and with morphisms $\mathrm{Hom}_*(e,e) = \mathrm{End}_R(R) \cong R^{\mathrm{op}}$ (of left R-module homomorphisms from R to itself) can be viewed as a dg category $\underline{*}_R$ as follows. The unique object is e. Morphisms $[e,e]_{\underline{*}} \in C(k)$ are simply the complex consisting of R^{op} concentrated in degree 0. Compositions are just multiplications in degree 0. When $R = k$, we call $\underline{*}_k$ the *unit dg category*.

Example 3.2.11. Let $\mathscr{C}$ be any small category. One can spell out a dg category $k[\mathscr{C}]$ with objects of $\mathscr{C}$ and internal hom $k[\mathscr{C}](A,B)$ is the free k-module $k[\mathscr{C}(A,B)]$ concentrated in degree 0.

Remark 3.2.4. The above examples of dg categories $\mathcal{A}$ with morphisms involving the function complexes $[A,B]$ can be adapted to define other dg categories $\mathcal{A}_\dagger$ which are the same as $\mathcal{A}$ on objects and units, but morphisms are the complexes $[A,B]_\dagger$ and compositions are component-wise composed with the symmetry map c. Note that $Z^0\mathcal{A}$ (resp. $H^0\mathcal{A}$) is the same as $Z^0\mathcal{A}_\dagger$ (resp. $H^0\mathcal{A}_\dagger$).

3.2.3 Tensor categories and functor categories

Let $\mathcal{A}, \mathcal{B}$ be dg categories.

Definition 3.2.6. The *tensor product dg category* $\mathcal{A} \otimes \mathcal{B}$ is given by the following

- objects are the pair $(A,B) \in \mathrm{ob}(\mathcal{A}) \times \mathrm{ob}(\mathcal{B})$,
- for any two pairs (A,B) and (A', B') the morphism is defined by

$$\mathcal{A} \otimes \mathcal{B}\big((A,B),(A',B')\big) := \mathcal{A}(A,A') \otimes \mathcal{B}(B,B'),$$

- for any three pairs $E_i = (A_i, B_i) : 1 \le i \le 3$ the composition

$$\big(\mathcal{A} \otimes \mathcal{B}(E_2,E_3)\big) \otimes \big(\mathcal{A} \otimes \mathcal{B}(E_1,E_2)\big) \to \mathcal{A} \otimes \mathcal{B}(E_1,E_3)$$

  is given by $(f_1 \otimes g_2) \circ (f_2 \otimes g_2) \mapsto (-1)^{|g_1||f_2|}(f_1 \circ f_2) \otimes (g_1 \circ g_2)$.

- unit is the pair $(e_{\mathcal{A}}, e_{\mathcal{B}})$.

Definition 3.2.7. The *functor dg category* $[\mathcal{A}, \mathcal{B}]$ is given by

- objects are dg form functors $F : \mathcal{A} \to \mathcal{B}$,

- for any two dg form functors F and G the morphism $[F,G]$ in degree n is defined by the set $[F,G]^n$ of collections $\alpha = (\alpha_A)_{A\in\mathcal{A}}$ with $\alpha_A \in \mathcal{B}(FA,GA)^n$ such that $G(f)\circ\alpha_A = (-1)^{|\alpha||f|}\alpha_B \circ F(f)$.
- composition is deduced by composing termwise.
- unit is the collection $(e_{FA})_{A\in\mathcal{A}}$ with the units e_{FA} in $[FA,FA]^0$.

3.2.4 Natural transformations of dg functors

Definition 3.2.8 ([43])**.** Let $F,G : \mathcal{A} \to \mathcal{B}$ be two dg functors. A *natural transformation of dg functors* $\alpha : F \to G$ consists of the data

$$\alpha_A : \mathbb{1} \to \mathcal{B}(FA,GA)$$

i.e. $\alpha_A(1) \in Z^0\mathcal{B}(FA,GA)$ for each $A \in ob(\mathcal{A})$ such that for any two objects $A,B \in ob(\mathcal{A})$ the following diagram

$$\begin{tikzcd}
 & \mathbb{1}\otimes\mathcal{A}(A,B) \arrow[r, "\alpha_B\otimes F"] & \mathcal{B}(FB,GB)\otimes\mathcal{B}(FA,FB) \arrow[dr, "\mu"] & \\
\mathcal{A}(A,B) \arrow[ur, "l^{-1}"] \arrow[dr, "r^{-1}"'] & & & \mathcal{B}(FA,GB) \\
 & \mathcal{A}(A,B)\otimes\mathbb{1} \arrow[r, "G\otimes\alpha_A"] & \mathcal{B}(GA,GB)\otimes\mathcal{B}(FA,GA) \arrow[ur, "\mu"'] &
\end{tikzcd} \tag{3.5}$$

commutes.

Example 3.2.12. Let $\alpha : F \to G : \mathcal{A} \to \mathcal{B}$ be a natural transformation of dg functors. We define its *opposite natural transformation* $\alpha^{\mathrm{op}} : G^{\mathrm{op}} \to F^{\mathrm{op}}$ by the data

$$(\alpha^{\mathrm{op}})_A := \alpha_A : \mathbb{1} \to \mathcal{B}^{\mathrm{op}}(GA,FA) = \mathcal{B}(FA,GA)$$

Check that α^{op} is a natural transformation of dg functors.

Example 3.2.13 ([73])**.** Let $A \in C(k)$. Let $\vee_A$ denote the dg functor $[-,A]$. Recall

$$[-,A] : \underline{C(k)}^{\mathrm{op}} \to \underline{C(k)}$$

is defined in Proposition 3.2.2. Bear in mind we often use $X^{\vee_A}$ to mean $[X,A]$. There is a natural transformation

$$\mathrm{can} : 1 \to \vee_A \circ (\vee_A)^{\mathrm{op}}$$

given by the morphism of complexes

$$\mathrm{can}_X : X \to X^{\vee\vee} : x \mapsto \Big(\mathrm{can}_X(x) : f \mapsto (-1)^{|x||f|} f(x)\Big)$$

Here, the symbol 1 represents the identity dg functor of $C(k)$ to itself. The map $\mathrm{can} : X \to X^{\vee\vee}$ corresponds to the composition

$$X \xrightarrow{d_{X^\vee}} [X^\vee, X \otimes X^\vee] \xrightarrow{[1,c]} [X^\vee, X^\vee \otimes X] \xrightarrow{[1,e]} [X^\vee, A] = X^{\vee\vee} \tag{3.6}$$

Example 3.2.14. Let $A \in C(k)$. Denote $*_A$ the dg functor $[-, A]_\dagger$. Recall the dg functor

$$*_A : \underline{C(k)}_\dagger^{\mathrm{op}} \to \underline{C(k)}_\dagger.$$

in Proposition 3.2.2. There is a natural transformation

$$\mathrm{can}_\dagger : 1 \to *_A \circ (*_A)^{\mathrm{op}}$$

given by the morphism of complexes (Note that $|f(x)| = |f| + |x|$)

$$(\mathrm{can}_\dagger)_X : X \to X^{\vee\vee} : x \mapsto \Big((\mathrm{can}_\dagger)_X(x) : f \mapsto (-1)^{\frac{(|f(x)|)(|f(x)|+1)}{2}} f(x)\Big)$$

One may use the composition (3.6) to work out the sign. Note that the data of $(\mathrm{can}_\dagger)_X$ has already been captured by Gille in [30].

3.2.5 Dg categories with duality

Definition 3.2.9 ([73])**.** A *dg category with duality* is a triplet $(\mathcal{A}, *, \eta)$ where $\mathcal{A}$ is a dg category, $* : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ is a dg functor and $\eta : 1_{\mathcal{A}} \to * \circ *^{\mathrm{op}}$ is a natural transformation of dg functors such that $(\eta_A^{\mathrm{op}})^* \circ \eta_{A^*} = 1_{A^*}$ for all $A \in \mathrm{ob}(\mathcal{A})$.

Remark 3.2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories. Recall from [43, Section 1.11] that the data

$$(\mathcal{A}, \mathcal{B}, F \dashv G, \eta, \epsilon)$$

consists of a dg functor $F : \mathcal{A} \to \mathcal{B}$ (*the left adjoint*), a dg functor $G : \mathcal{B} \to \mathcal{A}$ (*the right adjoint*) and two natural transformations of dg functors $\eta : 1_{\mathcal{A}} \to GF$ (the *unit*) and $\epsilon : FG \to 1_{\mathcal{B}}$ (the *counit*) satisfying the *triangular equation*

$$1_F = \epsilon F \circ F\eta \qquad 1_G = G\epsilon \circ \eta G$$

where the symbol $\circ$ represents the *horizontal composition* of natural transformations of dg functors. Therefore, we conclude that deducing a dg category with duality is equivalent to specifying the data $(\mathcal{A}, \mathcal{A}^{\mathrm{op}}, *^{\mathrm{op}} \dashv *, \eta, \eta^{\mathrm{op}})$.

Example 3.2.15 (Section 1 [73]). The triplet

$$(\underline{C(k)}, \vee_A, \mathrm{can})$$

gives a dg category with duality. Recall $\vee_A$ and can in Example 3.2.13.

Similarly, one shows that $(\mathscr{C}, \vee_{k[n]}, \mathrm{can})$ is a dg category with duality, which is denoted by $\mathscr{C}^{[n]}$.

Example 3.2.16. The triplet

$$(\underline{C(k)}_\dagger, *_A, \mathrm{can}_\dagger)$$

provides a dg category with duality. Recall $*_A$ and $\mathrm{can}_\dagger$ in Example 3.2.14.

Definition 3.2.10 (Definition 1.16 [73]). Let $\mathscr{A} = (\mathcal{A}, \vee, \mathrm{can})$ be a dg category with duality. The *n-th shifted dg category with duality* is

$$\mathscr{A}^{[n]} = \mathscr{C}^{[n]} \otimes \mathscr{A}.$$

3.2.6 Dg form functors

Definition 3.2.11 ([73]). A *dg form functor* $(\mathcal{A}, *, \eta) \to (\mathcal{B}, \#, \beta)$ contains

- a dg functor $F : \mathcal{A} \to \mathcal{B}$,
- a natural transformation of dg functors $\varphi : F \circ * \to \# \circ F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$,

subject to the condition

$$\varphi_{A^*} \circ F(\eta_A) = (\varphi_A^{\mathrm{op}})^{\#} \circ \beta_{FA} \tag{3.7}$$

for every $A \in \mathrm{ob}(\mathcal{A})$.

The following example will provide a way to see Balmer's Witt groups (resp. Gille's coherent Witt groups) of a scheme X (resp. a scheme X with a dualizing complex $I_\bullet$) are isomorphic to Schlichting's Witt groups (resp. coherent Witt groups) of the scheme X (resp. the scheme X with the dualizing complex $I_\bullet$).

Example 3.2.17. There is a dg form functor

$$(F, \varphi) : (\underline{C(k)}_\dagger, *_A, \mathrm{can}_\dagger) \to (\underline{C(k)}, \vee_A, \mathrm{can})$$

where $F : \underline{C(k)}_\dagger \to \underline{C(k)}$ is the dg functor given in Example 3.2.3. We define the natural transformation of dg functors

$$\varphi : F \circ *_A \to \vee_A \circ F^{\mathrm{op}} : (\underline{C(k)}_\dagger)^{\mathrm{op}} \to \underline{C(k)}$$

by

$$\varphi_X : [X, A]_\dagger \to [X, A] : f \mapsto (-1)^{\frac{|f|(|f|+1)}{2}} f$$

for each $X \in C(k)$. Note that φ_X is exactly F_{XA} discussed in Lemma 3.2.3. We have already examined it is a morphism of complexes, and it is a natural transformation by [43, Section 1.8]. Check Equation 3.7 holds. This is to verify

$$\varphi_{X^*} \circ F\big((\mathrm{can}_\dagger)_X\big) = (\varphi_X^{\mathrm{op}})^\vee \circ \mathrm{can}_{FX}$$

which holds if and only if for any $x \in X$

$$\varphi_{X^*}\big((\mathrm{can}_\dagger)_X(x)\big) = \epsilon(\varphi_X^{\mathrm{op}})\big(\mathrm{can}_X(x)\big) = \mathrm{can}_X(x) \circ \varphi_X \in [[X, A]_\dagger, A]$$

(Note that $(\mathrm{can}_\dagger)_X$ and φ^{op} are morphisms of complexes, so of degree 0. Hence, there are no signs in the equation). Let $g \in [X, A]_\dagger$. Put g into the left hand side.

$$\begin{aligned}
\varphi_{X^*}\big((\mathrm{can}_\dagger)_X(x)\big)(g) &= (-1)^{\frac{|x|(|x|+1)}{2}}\Big((\mathrm{can}_\dagger)_X(x)\Big)(g) \\
&= (-1)^{\frac{|x|(|x|+1)}{2}}(-1)^{\frac{|g(x)|(|g(x)|+1)}{2}} g(x) \\
&= (-1)^{\frac{|g|(|g|+1)}{2}+|g||x|} g(x) \\
&\quad \text{(Use } |g(x)| = |g| + |x|\text{)}
\end{aligned}$$

Put g into the right hand side.

$$\begin{aligned}
\mathrm{can}_X(x)(\varphi_X(g)) &= (-1)^{\frac{|g|(|g|+1)}{2}} \mathrm{can}_X(x)(g) \\
&= (-1)^{\frac{|g|(|g|+1)}{2}} (-1)^{|g||x|} g(x)
\end{aligned}$$

This finishes the verification.

Lemma 3.2.4. *Assume that $(\mathcal{A}, *_\mathcal{A}, \mathrm{can}_\mathcal{A}) \otimes (\mathcal{B}, *_\mathcal{B}, \mathrm{can}_\mathcal{B}) \to (\mathcal{C}, *_\mathcal{C}, \mathrm{can}_\mathcal{C})$ is a dg form functor. Then, any symmetric form (A, φ) on $(\mathcal{A}, *_\mathcal{A}, \mathrm{can}_\mathcal{A})$ induces a dg form functor $(A, \varphi)\otimes ? : (\mathcal{B}, *_\mathcal{B}, \mathrm{can}_\mathcal{B}) \longrightarrow (\mathcal{C}, *_\mathcal{C}, \mathrm{can}_\mathcal{C})$.*

Proof. A dg form functor $(\mathcal{A}, *_\mathcal{A}, \mathrm{can}_\mathcal{A}) \otimes (\mathcal{B}, *_\mathcal{B}, \mathrm{can}_\mathcal{B}) \to (\mathcal{C}, *_\mathcal{C}, \mathrm{can}_\mathcal{C})$ consists of the data, a dg functor

$$F : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C} : (A, B) \mapsto A \otimes B$$

and a duality compatibility natural transformation $\epsilon_{A,B} : A^* \otimes B^* \to (A \otimes B)^*$ such that the diagram

$$\begin{tikzcd}
A \otimes B \arrow[r] \arrow[d] & A^{**} \otimes B^{**} \arrow[d] \\
(A \otimes B)^{**} \arrow[r] & (A^* \otimes B^*)^*
\end{tikzcd} \tag{3.8}$$

commutes. Given a symmetric form (A,φ) on $(\mathcal{A},*_{\mathcal{A}},\mathrm{can}_{\mathcal{A}})$, we can define a dg form functor

$$(A,\varphi)\otimes ? : (\mathcal{B},*_{\mathcal{B}},\mathrm{can}_{\mathcal{B}}) \longrightarrow (\mathcal{C},*_{\mathcal{C}},\mathrm{can}_{\mathcal{C}})$$

by the data, a dg functor $A\otimes ? : \mathcal{B}\to\mathcal{C} : B\mapsto A\otimes B$ and a duality compatibility map

$$\varphi : A\otimes B^* \xrightarrow{\varphi\otimes 1} A^*\otimes B^* \xrightarrow{\epsilon_{A,B}} (A\otimes B)^*$$

Consider the diagram

$$\begin{tikzcd}
& A\otimes B \arrow[rr, "1\otimes\mathrm{can}_B"] \arrow[dl] \arrow[dd, "\mathrm{can}\otimes\mathrm{can}"] \arrow[ddrr, phantom, "\square_1"] & & A\otimes B^{**} \arrow[dd, "\varphi\otimes 1"] \\
(A\otimes B)^{**} \arrow[dd] \arrow[dr, phantom, "\square_2"] & & & \\
& A^{**}\otimes B^{**} \arrow[rr, "\varphi^*\otimes\mathrm{can}_B"] \arrow[dl] \arrow[dr, phantom, "\square_3"] & & A^*\otimes B^{**} \arrow[dl] \\
(A^*\otimes B^*)^* \arrow[rr] & & (A\otimes B^*)^* &
\end{tikzcd}$$

We conclude the commutativity of $\square_1$ by the symmetry of the form φ, the commutativity of $\square_2$ by the commutative diagram (3.8) and the commutativity of $\square_3$ by the naturality of ϵ. $\square$

Remark 3.2.6. The multiplication $m : k[2]\otimes k[2]\to k[4]$ gives a symmetric form $(k[2],m)$ in $\mathscr{C}^{[4]}$. The dg form functor $\mathscr{C}^{[4]}\otimes\mathscr{A}^{[n]}\to\mathscr{A}^{[n+4]}$ induces an isomorphism of categories with duality $(k[2],m)\otimes ? : \mathscr{A}^{[n]}\xrightarrow{\sim}\mathscr{A}^{[n+4]}$, which induces the '4-periodicity' on Grothendieck-Witt groups. For more details, one may refer to [73, Remark 1.18].

3.2.7 Pretriangulated hull and weak equivalences

Every dg category $\mathcal{A}$ has a pretriangulated-hull $\mathcal{A}^{\mathrm{ptr}}$ (which is *pretriangulated*) such that $\mathcal{A}$ is a full dg subcategory of $\mathcal{A}^{\mathrm{ptr}}$, $Z^0\mathcal{A}^{\mathrm{ptr}}$ is Frobenius exact and $H^0\mathcal{A}^{\mathrm{ptr}}$ is triangulated, cf. [73, Section 1]. Moreover, $(-)^{\mathrm{ptr}}$ is functorial.

Definition 3.2.12. A *dg category with weak equivalences* is a pair $(\mathcal{A},w)$ consisting of the data: a full dg subcategory $\mathcal{A}^w\subset\mathcal{A}$ and a set of morphisms w in $\mathcal{A}^{\mathrm{ptr}}$ such that $f\in w$ if and only if f is an isomorphism in the quotient triangulated category $\mathcal{T}(\mathcal{A},w) := H^0\mathcal{A}^{\mathrm{ptr}}/H^0(\mathcal{A}^w)^{\mathrm{ptr}}$.

Definition 3.2.13. An *exact dg functor* $F : (\mathcal{A}, w) \to (\mathcal{C}, v)$ of dg categories with weak equivalences consists of a dg functor $F : \mathcal{A} \to \mathcal{C}$ preserving weak equivalences, i.e. $F(\mathcal{A}^w) \subset \mathcal{C}^v$.

Definition 3.2.14. A quadruple $\mathscr{A} = (\mathcal{A}, w, *, \mathrm{can})$ is called a *dg category with weak equivalences and duality* if $(\mathcal{A}, w)$ is a dg category with weak equivalences, if $(\mathcal{A}, *, \mathrm{can})$ is a dg category with duality, if $\mathcal{A}^w$ is invariant under the duality and if $\mathrm{can}_A \in w$ for all objects A in $\mathcal{A}$.

Definition 3.2.15. An *exact dg form functor* $(F, \varphi) : (\mathcal{A}, w, *, \mathrm{can}) \to (\mathcal{C}, v, \#, \mathrm{can})$ consists of a dg form functor $(F, \varphi) : (\mathcal{A}, *, \mathrm{can}) \to (\mathcal{C}, \#, \mathrm{can})$ and an exact dg functor $F : (\mathcal{A}, w) \to (\mathcal{C}, v)$.

Remark 3.2.7. Any dg category $\mathscr{A}$ with weak equivalences and duality gives a pretriangulated dg category $\mathscr{A}^{\mathrm{ptr}}$ with weak equivalences and duality.

Remark 3.2.8. For any dg category $\mathscr{A}$ with weak equivalences and duality, one could also define the n-th shifted version $\mathscr{A}^{[n]}$ by putting appropriate weak equivalences, cf. [73, Section 1.10].

3.3 Triangulated categories with duality

Our reference for this part is [73, Section 3]. Note that the translation functor $T : \mathcal{T} \to \mathcal{T}$ in the triangulated category $\mathcal{T}$ is only assumed to be an *auto-equivalence* (rather than an *auto-morphism*).

Definition 3.3.1 (Definition 3.1 [73])**.** A *triangulated category with duality* is a quadruple $(\mathcal{T}, \#, \eta, \lambda)$ with a triangulated category $\mathcal{T}$, with an additive functor $\# : \mathcal{T}^{\mathrm{op}} \to \mathcal{T}$, and with natural isomorphisms $\eta : 1 \to \#\#$ and $\lambda : \# \to T\#T$ such that $(\mathcal{T}, \#, \eta)$ is a category with duality, that λ and η are compatible and that $\#$ is compatible with exact triangles.

Remark 3.3.1. This definition is slightly different from the one given in [9]. The framework in [9] only assumes that $T : \mathcal{T} \to \mathcal{T}$ is an automorphism, that $T^{-1}\# = \#T$ and that the duality is compatible with δ-exact triangles where $\delta = \pm 1$. In general, note that the translation functors in triangulated categories need not to be an automorphism, and that some triangulated categories with duality do not satisfy the condition $T^{-1}\# = \#T$. Thus, it is helpful to keep the data with a natural transformation $\lambda : \# \to T\#T$.

Definition 3.3.2 (Definition 3.4 [73])**.** A *morphism of triangulated categories with duality* is a triple

$$(F, \rho, \varphi) : (\mathcal{T}_1, \#_1, \eta_1, \lambda_1) \to (\mathcal{T}_2, \#_2, \eta_2, \lambda_2)$$

where $(F, \rho) : \mathcal{T}_1 \to \mathcal{T}_2$ is a triangle functor (with $\rho : FT \to TF$ a natural isomorphism) and where $\varphi : F\# \to \#F$ is also a natural isomorphism such that φ and ρ are compatible with the duality.

Remark 3.3.2. For every dg category $\mathscr{A} := (\mathcal{A}, w, *, \mathrm{can})$ with weak equivalences and duality, there is an associated triangulated category $\mathcal{T}\mathscr{A} = (\mathcal{T}(\mathcal{A}, w), *, \mathrm{can}, \lambda)$ with duality, cf. [73, Section 3]. Any exact dg form functor $\mathscr{A}_1 \to \mathscr{A}_2$ gives a duality preserving functor $\mathcal{T}\mathscr{A}_1 \to \mathcal{T}\mathscr{A}_2$ of triangulated categories with duality.

Remark 3.3.3. Consider the dg categories $(\underline{C(k)}, \vee)$ and $(\underline{C(k)}_{\dagger}, *)$ with duality discussed above. One can regard quasi-isomorphisms as weak equivalences, and make them into dg categories with weak equivalences and duality. The triangulated category with duality coming from $(\underline{C(k)}, \mathrm{quis}, \vee)$ (resp. $(\underline{C(k)}_{\dagger}, \mathrm{quis}, *)$) fits into Schlichting's framework [73] (resp. Balmer, Walter and Gille's framework). Example 3.2.17 gives a way to connect them.

Remark 3.3.4. For any triangulated category with duality $\mathscr{T} = (\mathcal{T}, \#, \eta, \lambda)$, one could define the n-th shifted triangulated category with duality $\mathcal{T}^{[n]}$, cf. [73, Definition 3.10].

The following is proved in [73, Lemma 3.12]:

Lemma 3.3.1. *Let $\mathscr{A}$ be a pretriangulated dg category with weak equivalences and duality. There are equivalences of triangulated categories with duality*

$$(\mathcal{T}\mathscr{A})^{[n]} \to \mathcal{T}(\mathscr{A}^{[n]})$$

3.4 Grothendieck-Witt groups and Witt groups

3.4.1 Grothendieck-Witt groups of dg categories

Let $\mathscr{A} := (\mathcal{A}, w, *, \mathrm{can})$ be a dg category with weak equivalences and duality.

Definition 3.4.1 (Definition 1.18 [73])**.** The *Grothendieck-Witt group $GW_0(\mathscr{A})$* is defined to be the free abelian group generated by symmetric spaces (X, φ) in $(Z^0\mathscr{A}^{\mathrm{ptr}}, w, *, \mathrm{can})$ such that

1. $(X, \varphi) + (Y, \psi) = (X \oplus Y, \varphi \oplus \psi)$;

2. if $g : X \to Y$ is in w, then $(Y, \psi) = (X, g^* \psi g)$;

3. if the diagram of exact sequences

$$\begin{tikzcd} E_{-1} \arrow[r, tail, "i"] \arrow[d, "\varphi_{-1}"] & E_0 \arrow[r, two heads, "p"] \arrow[d, "\varphi_0"] & E_1 \arrow[d, "\varphi_1"] \\ E_1^* \arrow[r, tail, "p^*"] & E_0^* \arrow[r, two heads, "i^*"] & E_{-1}^* \end{tikzcd}$$

is commutative with $\varphi_i \in w$ and $\varphi_i = \varphi_{-i}^* \circ \mathrm{can}$, then

$$(E_0, \varphi_0) = (E_{-1} \oplus E_1, \begin{pmatrix} 0 & \varphi_1 \\ \varphi_{-1} & 0 \end{pmatrix}).$$

The *Witt group* $W_0(\mathscr{A})$ is defined to be $GW_0(\mathscr{A})$ modulo the following relation

3′. if the diagram of exact sequences

$$\begin{tikzcd} E_{-1} \arrow[r, tail, "i"] \arrow[d, "\varphi_{-1}"] & E_0 \arrow[r, two heads, "p"] \arrow[d, "\varphi_0"] & E_1 \arrow[d, "\varphi_1"] \\ E_1^* \arrow[r, tail, "p^*"] & E_0^* \arrow[r, two heads, "i^*"] & E_{-1}^* \end{tikzcd}$$

is commutative with $\varphi_i \in w$ and $\varphi_i = \varphi_{-i}^* \circ \mathrm{can}$, then

$$(E_0, \varphi_0) = 0.$$

The *n-th shifted Witt groups* and *n-th shifted Grothendieck-Witt groups* of the dg category $\mathscr{A}$ with weak equivalences and duality are defined as

$$W^{[n]}(\mathscr{A}) := W_0(\mathscr{A}^{[n]}) \qquad GW_0^{[n]}(\mathscr{A}) := GW_0(\mathscr{A}^{[n]})$$

3.4.2 Grothendieck-Witt groups of triangulated categories

Let $\mathscr{T} := (\mathcal{T}, *, \eta, \lambda)$ be a triangulated category with duality.

Definition 3.4.2 (Definition 3.5 [73])**.** The *Grothendieck-Witt group* $GW^0(\mathscr{T})$ is defined to be the free abelian group generated by the isometry classes of symmetric spaces (X, φ) in $\mathscr{T}$ such that

1. $(X, \varphi) + (Y, \psi) = (X \oplus Y, \varphi \oplus \psi)$;

2. if the diagram of exact triangles

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\varphi_{-1}"] & Y \arrow[r, "p"] \arrow[d, "\varphi_0"] & Z \arrow[r, "q"] \arrow[d, "\varphi_1"] & TX \arrow[d, "T\varphi_{-1}"] \\ Z^* \arrow[r, "p^*"] & Y^* \arrow[r, "i^*"] & X^* \arrow[r, "T(q^*)\circ\lambda_X"] & T(Z^*) \end{tikzcd}$$

is commutative with φ_i being isomorphisms and $\varphi_i = \varphi_{-i}^* \circ \eta$, then

$$(Y, \varphi_0) = (X \oplus Z, \begin{pmatrix} 0 & \varphi_1 \\ \varphi_{-1} & 0 \end{pmatrix}).$$

The *Witt group* $W^0(\mathscr{T})$ is defined to be $GW^0(\mathscr{T})$ modulo the relation

2′. if the diagram of exact triangles

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\varphi_{-1}"] & Y \arrow[r, "p"] \arrow[d, "\varphi_0"] & Z \arrow[r, "q"] \arrow[d, "\varphi_1"] & TX \arrow[d, "T\varphi_{-1}"] \\ Z^* \arrow[r, "p^*"] & Y^* \arrow[r, "i^*"] & X^* \arrow[r, "T(q^*)\circ\lambda_X"] & T(Z^*) \end{tikzcd}$$

is commutative with φ_i are isomorphisms and $\varphi_i = \varphi_{-i}^* \circ \eta$, then

$$(Y, \varphi_0) = 0.$$

The *n-th shifted Witt groups* and *n-th shifted Grothendieck-Witt groups* of the triangulated category $\mathscr{T}$ with duality are defined as

$$W^n(\mathscr{T}) := W_0(\mathscr{T}^{[n]}) \qquad GW^n(\mathscr{T}) := GW_0(\mathscr{T}^{[n]})$$

Recall that for any dg category $\mathscr{A}$ with weak equivalences and duality, there is an associated triangulated category $\mathcal{T}\mathscr{A}$ with duality. The following is proved in [73, Proposition 3.8] and [73, Corollary 3.13]:

Proposition 3.4.1. *If $\mathscr{A}$ is a pretriangulated dg category with weak equivalences and duality, then there are isomorphism of groups*

$$W^{[n]}(\mathscr{A}) \to W^n(\mathcal{T}\mathscr{A}) \qquad GW_0^{[n]}(\mathscr{A}) \to GW^n(\mathcal{T}\mathscr{A}).$$

3.5 Grothendieck-Witt spectra of dg categories

Recall from [73, Section 4 and 5] the $\mathcal{R}_\bullet$-construction. Let $\mathscr{A} := (\mathcal{A}, w, *, \eta)$ be a pointed dg category with weak equivalences and duality. Let $\underline{n}$ be the totally ordered set

$$\{n' < (n-1)' < \cdots < 0' < 0 < 1 < \cdots < n-1 < n\}.$$

This may be considered as a category with a unique strict duality $n \leftrightarrow n'$. Let $\mathrm{Ar}(\underline{n}) = \mathrm{Fun}([1], \underline{n})$ be the category of arrows. Recall $\mathcal{R}_n\mathcal{A} \subset [\mathrm{Ar}(\underline{n}), \mathcal{A}]$ is the full dg subcategory consisting of objects $A : \mathrm{Ar}(\underline{n}) \to \mathcal{A}$ for which $\mathcal{A}_{i,i} = 0$, and whenever $i \le j \le k \in \underline{n}$ the sequence $0 \to A_{i,j} \to A_{i,k} \to A_{j,k} \to 0$ is an exact sequence in the dg category $\mathcal{A}$.

The full dg subcategory $\mathcal{R}^{(n)}_{i_1,\cdots,i_n}\mathcal{A} \subset [\mathrm{Ar}(\underline{k_1}) \times \cdots \times \mathrm{Ar}(\underline{k_n}), \mathcal{A}]$ consists of dg functors $A : \mathrm{Ar}(\underline{k_1}) \times \cdots \times \mathrm{Ar}(\underline{k_n}) \to \mathcal{A}$ such that the restriction $\mathrm{Ar}(k_s) \to \mathcal{A}$ is an object of $\mathcal{R}_{k_s}\mathcal{A}$ whenever $s \in \{1, \cdots, n\}$ (pointed by $\mathrm{Ar}(k_1) \times \cdots \times \mathrm{Ar}(k_n) \to 0$). Recall $\mathcal{R}^{(n)}_{\bullet}\mathcal{A}$ is the diagonal of the multi-simplicial dg category $\mathcal{R}^{(n)}_{\bullet,\cdots,\bullet}\mathcal{A}$. There is an evident way to define weak equivalences and duality on $\mathcal{R}^{(n)}_{\bullet}\mathcal{A}$. Denote this dg category with weak equivalences and duality by $\mathcal{R}^{(n)}_{\bullet}\mathscr{A}$. Define $GW(\mathscr{A})_n$ to be the geometric realization $|i \mapsto (\mathcal{R}^{(n)}_i \mathscr{A}^{(n)})|$ where $\mathscr{A}^{(n)} = \mathbb{Z}^n((\mathscr{C}^{[1]})^{\otimes n} \otimes \mathscr{A})$ (recall the $\mathbb{Z}(\mathcal{A})$ construction in [73, Section 1]).

Definition 3.5.1 (Definition 5.2 [73])**.** For any pointed dg category $\mathscr{A}$ with weak equivalences and duality. The *Grothendieck-Witt spectrum* $GW(\mathscr{A})$ is the symmetric sequence

$$\{GW(\mathscr{A})_0, GW(\mathscr{A})_1, GW(\mathscr{A})_2, \cdots\}$$

where each space $GW(\mathscr{A})_n$ has a basepoint preserving continuous left Σ_n (symmetric group)-action with bonding maps

$$\epsilon_n : (S^1)^{\wedge n} \wedge GW(\mathscr{A})_m \to GW(\mathscr{A})_{n+m}$$

defined in [73, Section 5.2]. The *n-th shifted Grothendieck-Witt spectrum* $GW^{[n]}(\mathscr{A})$ is defined to be $GW(\mathscr{A}^{[n]})$, and $GW^{[n]}_i(\mathscr{A}) := \pi_i GW^{[n]}(\mathscr{A})$ which are the i-th stable homotopy groups. There is also a non-connective version of Grothendieck-Witt spectrum $\mathbb{G}W^{[n]}$ called *Karoubi-Grothendieck-Witt spectrum* analogue to the case of $\mathbb{K}$, cf. [73, Section 8]. Define $\mathbb{G}W^{[n]}_i(\mathscr{A}) := \pi_i \mathbb{G}W^{[n]}(\mathscr{A})$.

Theorem 3.5.1 (Agreement [73])**.** *Assume that $\frac{1}{2} \in \mathscr{A}$. There are group isomorphisms $\pi_0 GW^{[n]}(\mathscr{A}) \approx GW^{[n]}_0(\mathscr{A})$ and $\pi_i GW^{[n]}(\mathscr{A}) \approx W^{[n-i]}_0(\mathscr{A})$ for $i < 0$.*

Any exact dg form functor $\mathscr{A} \to \mathscr{C}$ gives maps of spectra $GW^{[n]}(\mathscr{A}) \to GW^{[n]}(\mathscr{C})$, $\mathbb{G}W^{[n]}(\mathscr{A}) \to \mathbb{G}W^{[n]}(\mathscr{C})$ and maps of groups $GW^{[n]}_0(\mathscr{A}) \to GW^{[n]}_0(\mathscr{C})$, $W^{[n]}_0(\mathscr{A}) \to W^{[n]}_0(\mathscr{C})$. The following theorem is proved in [73, Theorem 6.5].

Theorem 3.5.2 (Invariance of GW)**.** *Assume $\frac{1}{2} \in \mathscr{A}, \mathscr{C}$. If $F : \mathscr{A} \to \mathscr{C}$ induces an equivalence $\mathcal{TA} \to \mathcal{TC}$ of triangulated categories, then F gives an equivalence of spectra $GW^{[n]}(\mathscr{A}) \to GW^{[n]}(\mathscr{C})$.*

Recall that a triangle functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ is *cofinal* if it is fully faithful and every object in $\mathcal{T}_2$ is a direct summand of an object in $\mathcal{T}_1$. The following is proved in [73, Theorem 8.9].

Theorem 3.5.3 (Invariance of $\mathbb{G}W$)**.** *Assume $\frac{1}{2} \in \mathscr{A}, \mathscr{C}$. If $F : \mathscr{A} \to \mathscr{C}$ induces a cofinal triangle functor $\mathcal{T}\mathcal{A} \to \mathcal{T}\mathcal{C}$ of triangulated categories, then F gives a stable equivalence of spectra $\mathbb{G}W^{[n]}(\mathscr{A}) \to \mathbb{G}W^{[n]}(\mathscr{C})$.*

A sequence $\mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3$ of triangulated categories is called *exact* if $\mathcal{T}_1 \to \mathcal{T}_2$ makes $\mathcal{T}_1$ into a full subcategory which is closed under direct factor, and the induced functor $\mathcal{T}_2/\mathcal{T}_1 \to \mathcal{T}_3$ is an equivalence where $\mathcal{T}_2/\mathcal{T}_1$ is the Verdier quotient. A sequence $\mathscr{A}_1 \to \mathscr{A}_2 \to \mathscr{A}_3$ of dg categories with weak equivalences and duality is *quasi-exact* if the induced sequence $\mathcal{T}\mathscr{A}_1 \to \mathcal{T}\mathscr{A}_2 \to \mathcal{T}\mathscr{A}_3$ is an exact sequence of triangulated categories. The following is proved in [73, Lemma 6.6].

Theorem 3.5.4 (Localization for GW)**.** *Assume that $\mathscr{A}_1 \to \mathscr{A}_2 \to \mathscr{A}_3$ is quasi-exact. Then, there is a homotopy fibration of spectra*

$$GW^{[n]}(\mathscr{A}_1) \to GW^{[n]}(\mathscr{A}_2) \to GW^{[n]}(\mathscr{A}_3).$$

Consequently, there is a long exact sequence of groups

$$\cdots \to GW^{[n]}_{i+1}(\mathscr{A}_3) \to GW^{[n]}_{i}(\mathscr{A}_1) \to GW^{[n]}_{i}(\mathscr{A}_2) \to GW^{[n]}_{i}(\mathscr{A}_3) \to GW^{[n]}_{i-1}(\mathscr{A}_1) \to \cdots$$

A sequence $\mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3$ of triangulated categories is called *exact up to factors* if $\mathcal{T}_1 \to \mathcal{T}_2$ is fully faithful, and the induced functor $\mathcal{T}_2/\mathcal{T}_1 \to \mathcal{T}_3$ is cofinal. A sequence $\mathscr{A}_1 \to \mathscr{A}_2 \to \mathscr{A}_3$ of dg categories with weak equivalences and duality is *Morita exact* if the induced sequence $\mathcal{T}\mathscr{A}_1 \to \mathcal{T}\mathscr{A}_2 \to \mathcal{T}\mathscr{A}_3$ is exact up to factors. The following is proved in [73, Theorem 8.10].

Theorem 3.5.5 (Localization for $\mathbb{G}W$)**.** *Assume that $\mathscr{A}_1 \to \mathscr{A}_2 \to \mathscr{A}_3$ is Morita exact. Then, there is a homotopy fibration of spectra*

$$\mathbb{G}W^{[n]}(\mathscr{A}_1) \to \mathbb{G}W^{[n]}(\mathscr{A}_2) \to \mathbb{G}W^{[n]}(\mathscr{A}_3).$$

Consequently, there is a long exact sequence of groups

$$\cdots \to \mathbb{G}W^{[n]}_{i+1}(\mathscr{A}_3) \to \mathbb{G}W^{[n]}_{i}(\mathscr{A}_1) \to \mathbb{G}W^{[n]}_{i}(\mathscr{A}_2) \to \mathbb{G}W^{[n]}_{i}(\mathscr{A}_3) \to \mathbb{G}W^{[n]}_{i-1}(\mathscr{A}_1) \to \cdots$$

Recall that a *semi-orthogonal decomposition* of a triangulated category $\mathcal{T}$ is denoted by $\langle \mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n \rangle$ consisting of the data

1. $\mathcal{T}_i$ are full triangulated subcategories of $\mathcal{T}$;
2. $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_n$ generate $\mathcal{T}$;
3. $\mathrm{Hom}(\mathcal{T}_i, \mathcal{T}_j) = 0$ for all $j < i$.

Theorem 3.5.6 (Additivity [73]). *Let $\mathscr{A} = (\mathcal{A}, w, *, \mathrm{can})$ be a pretriangulated dg category with weak equivalences and duality, and let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be full dg subcategories of $\mathcal{A}$. Assume*

1. *$\mathcal{A}_2$ is fixed by the duality (that is $*(\mathcal{A}_2) \subset \mathcal{A}_2$),*
2. *$\mathcal{A}_1$ and $\mathcal{A}_3$ are exchanged by the duality, i.e. $*(\mathcal{A}_1) \subset \mathcal{A}_3$ and $*(\mathcal{A}_3) \subset \mathcal{A}_1$,*
3. *$\langle \mathcal{T}\mathcal{A}_1, \mathcal{T}\mathcal{A}_2, \mathcal{T}\mathcal{A}_3 \rangle$ is a semi-orthogonal decomposition of $\mathcal{T}\mathcal{A}$.*

Then, there are stable equivalences of spectra

$$K(\mathscr{A}_1) \oplus GW^{[n]}(\mathscr{A}_2) \xrightarrow{\sim} GW^{[n]}(\mathscr{A}) \ , \ \mathbb{K}(\mathscr{A}_1) \oplus \mathbb{G}W^{[n]}(\mathscr{A}_2) \xrightarrow{\sim} \mathbb{G}W^{[n]}(\mathscr{A})$$

Proof. We prove this for GW, and $\mathbb{G}W$ is an analog. Let v be the set of morphisms that become isomorphisms in the Verdier quotient $\mathcal{T}\mathscr{A}/\mathcal{T}\mathscr{A}_2$. Then, we have a quasi-exact sequence $(\mathscr{A}_2, w) \to (\mathscr{A}, w) \to (\mathscr{A}, v)$. By localization, we see that this sequence induces a homotopy fibration of spectra

$$GW^{[n]}(\mathscr{A}_2) \to GW^{[n]}(\mathscr{A}) \to GW^{[n]}(\mathscr{A}, v)$$

Recall the hyperbolic dg category $\mathcal{H}\mathscr{A}_1$, cf. [73, Section 4.7]. Define a map $\mathcal{H}\mathscr{A}_1 \to \mathscr{A}, (A, B) \mapsto A \oplus B^*$. Claim that the composition

$$\mathcal{H}\mathscr{A}_1 \to (\mathscr{A}, w) \to (\mathscr{A}, v)$$

induces an equivalence $GW^{[n]}(\mathcal{H}\mathscr{A}_1) \to GW^{[n]}(\mathscr{A}, v)$. The argument is similar to the one given in [73, Proof of Proposition 6.8]. Thus, we have

$$GW^{[n]}(\mathcal{H}\mathscr{A}_1) \oplus GW^{[n]}(\mathscr{A}_2) \xrightarrow{\sim} GW^{[n]}(\mathscr{A})$$

where $GW^{[n]}(\mathcal{H}\mathscr{A}_1)$ is equivalent to $GW(\mathcal{H}\mathscr{A}_1)$. That is exactly our model for the K-theory $K(\mathscr{A}_1)$, cf. [73, Section 6]. $\square$

Chapter 4

Grothendieck-Witt groups of quadric hypersurfaces

4.1 Review: Semi-orthogonal decomposition on $\mathscr{D}^b Q$

Let k be a commutative ring (possibly $\frac{1}{2} \notin k$ in this section). Let (P, q) be a non-degenerate quadratic form of rank n over k. One has a homogeneous ring $A = S(P^*)/(q)$ by regarding q as an element in $S^2(P^*)$, cf. [80, Section 2]. Define Q (or Q_d) to be the projective variety Proj A which is smooth of relative dimensional $d = n - 2$ over k, cf. [80, Proposition 2.2].

If (P, q) is a quadratic form over k, the Clifford algebra $C(q)$ is defined as $T(P)/I(q)$ where $T(P)$ is the tensor algebra of P and where $I(q)$ is the two-sided ideal generated by $v \otimes v - q(v)$ for all $v \in P$. Moreover, the Clifford algebra $C(q) = C_0(q) \oplus C_1(q)$ is $\mathbb{Z}/2\mathbb{Z}$-graded from the grading on $T(P)$. Let $\iota : P \to C(q)$ be the inclusion. The Clifford algebra $C(q)$ has the universal property in the following sense: for any k-algebra B, if $\varphi : P \to B$ satisfies $\varphi(v)^2 = q(v)$, then there exists a unique k-algebra homomorphism $\tilde{\varphi} : C(q) \to B$ such that $\tilde{\varphi} \circ \iota = \varphi$.

Let $\mathscr{D}^b Q$ be the derived category of bounded chain complexes of finite rank locally free sheaves over Q. It is well-known that Swan's computation of K-theory of quadric hypersurfaces ([80]) can be adapted to deduce a semi-orthogonal decomposition of $\mathscr{D}^b Q$. Meanwhile, $\mathscr{D}^b Q$ has been extensively studied, cf. [41], [50]. Since Swan's version is most related to what we are doing, I will explain how to adapt Swan's computation of K-theory of quadrics to a semi-orthogonal decomposition of $\mathscr{D}^b Q$. I thank Marco Schlichting for sharing me with his personal notes on this.

Certainly, I take the responsibility of any mistake written here.

Let $\Lambda^i := \Lambda^i(P^*)$ be the i-th exterior power. Recall the *Tate resolution*

$$\ldots \to \mathscr{O}(-1) \otimes (\Lambda^1 + \Lambda^3 + \Lambda^5) \to \mathscr{O}(-1) \otimes (\Lambda^0 + \Lambda^2) \to \mathscr{O} \otimes \Lambda^1 \to \mathscr{O}(1), \qquad (4.1)$$

for $\mathscr{O}(1)$, cf. [80, Section 7] or [80, Proof of Lemma 8.4]. Precisely, the resolution (4.1) is given by

$$T_i := \mathscr{O}(-i) \otimes \big(\bigoplus_{d \geq 0} \Lambda^{i+1-2d} \big)$$

where $\Lambda^i := 0$ whenever $i \geq n$ and $i < 0$. The differential

$$\partial_i : \mathscr{O}(-i) \otimes \big(\bigoplus_{d \geq 0} \Lambda^{i+1-2d} \big) \to \mathscr{O}(-i+1) \otimes \big(\bigoplus_{d \geq 0} \Lambda^{i-2d} \big)$$

is defined by $\partial_i^{'} + \partial_i^{''}$ where

$$\partial_i^{'} : f \otimes (p_1 \wedge \cdots \wedge p_k) \mapsto \sum_s (-1)^{s+1} f p_s \otimes (p_1 \wedge \cdots \wedge \hat{p}_s \wedge \cdots \wedge p_k)$$

and

$$\partial_i^{''} : f \otimes w \mapsto \gamma \wedge (f \otimes w) = \sum_i f \xi_i \otimes (\beta_i \wedge w)$$

where the element $\gamma = \sum \xi_i \otimes \beta_i \in P^* \otimes P^*$ lifts $q \in S_2(P^*)$ via the natural surjective map $P^* \otimes P^* \to S_2(P^*)$.

Recall from [80, Section 8] that there is a pairing $\mathscr{O}(i) \otimes P^* \to \mathscr{O}(i+1)$ induced by the multiplication in the symmetric algebra. It follows that one has a map $\mathscr{O}(i) \to \mathscr{O}(i+1) \otimes P$ by the following composition

$$\mathscr{O}(i) \to \mathrm{Hom}_k(P^*, \mathscr{O}(i+1)) \xrightarrow{\approx} \mathscr{O}(i+1) \otimes \mathrm{Hom}_k(P^*, k) \to \mathscr{O}(i+1) \otimes P$$

where the first map is induced by the pairing, where the middle map is the canonical isomorphism and where the last map is induced by the double dual identification.

If $M = M_0 \oplus M_1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded *left* $C(q)$-module, then there are natural maps $P \otimes M_j \to M_{j+1}$. Define

$$\ell = \ell_{i,j} : \mathscr{O}(i) \otimes M_j \to \mathscr{O}(i+1) \otimes M_{j+1}$$

by the composition

$$\mathscr{O}(i) \otimes M_j \to \mathscr{O}(i+1) \otimes P \otimes M_j \to \mathscr{O}(i+1) \otimes M_{j+1}.$$

It follows that there is a sequence ($s_i = i + d + 1 \in \mathbb{Z}/2\mathbb{Z}$)

$$\cdots \longrightarrow \mathscr{O}(-i-1) \otimes M_{s_{i+1}} \longrightarrow \mathscr{O}(-i) \otimes M_{s_i} \longrightarrow \mathscr{O}(-i+1) \otimes M_{s_{i-1}} \longrightarrow \cdots \tag{4.2}$$

which is called *Clifford sequence* in [80, Section 8]. Swan defines

$$\mathscr{U}_i(M) := \operatorname{coker}\Big[\mathscr{O}(-i-2) \otimes M_{s_{i+2}} \to \mathscr{O}(-i-1) \otimes M_{s_{i+1}}\Big]$$

and $\mathscr{U}_i := \mathscr{U}_i(C(q))$. Swan proves $\operatorname{End}(\mathscr{U}_i) \cong C_0(q)$, cf. [80, Corollary 8.8].

Let $\Lambda := \bigoplus_n \Lambda^n$, $\Lambda^{(i)} := \bigoplus_n \Lambda^{2n+i}$ for $i \in \mathbb{Z}/2\mathbb{Z}$. Then, $\Lambda = \Lambda^{(0)} \oplus \Lambda^{(1)}$ can be viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded left $C(q)$-module, cf. [80, Corollary 8.8]. Taking $\Lambda = \Lambda^{(0)} \oplus \Lambda^{(1)}$ to the sequence (4.2), we get

$$\cdots \to \mathscr{O}(-i-1) \otimes \Lambda^{(s_{i+1})} \to \mathscr{O}(-i) \otimes \Lambda^{(s_i)} \to \mathscr{O}(-i+1) \otimes \Lambda^{(s_{i-1})} \to \cdots \tag{4.3}$$

This sequence is exact, since Swan shows that, when $i \geq d$, the sequence (4.3) and the Tate resolution (4.1) coincide, cf. [80, Lemma 8.4].

Proposition 4.1.1. *The functors*

$$\mathscr{O}(i) \otimes ? : \mathscr{D}^b k \to \mathscr{D}^b Q \text{ and } \mathscr{U}_i \otimes_{C_0(q)} ? : \mathscr{D}^b C_0(q) \to \mathscr{D}^b Q$$

are fully faithful, where $\mathscr{D}^b C_0(q)$ is the derived category of bounded complexes of finitely generated projective left $C_0(q)$-modules.

Proof. Note that

$$\operatorname{Hom}(\mathscr{O}(i), \mathscr{O}(i)[p]) = H^p(Q, \mathscr{O}) = \begin{cases} 0 & \text{if } p > 0 \\ k & \text{if } p = 0 \end{cases}$$

by [80, Lemma 5.2] and that

$$\operatorname{Hom}(\mathscr{U}_i, \mathscr{U}_i[p]) = \operatorname{Ext}^p(\mathscr{U}_{d-1}, \mathscr{U}_{d-1}) = \begin{cases} 0 & \text{if } p > 0 \\ C_0(q) & \text{if } p = 0 \end{cases}$$

by [80, Corollary 8.8] and [80, Lemma 6.1]. The result follows. □

The inclusion $\det(P^*) \subset \Lambda$ induces a $\mathbb{Z}/2\mathbb{Z}$-graded $C(q)$-module isomorphism $C(q) \otimes \det P^* \to \Lambda$, cf. [80, Lemma 8.3]. Then, one sees that $\mathscr{U}_n(\Lambda) = \mathscr{U}_n \otimes \det P^*$ and that $\operatorname{End}(\mathscr{U}_n(\Lambda)) = C_0(q)$, cf. [80, Corollary 8.8].

Let $\mathcal{U}_i$ (resp. $\mathcal{A}_i$) be the smallest full idempotent complete triangulated subcategory of $\mathscr{D}^b Q$ containing the bundle $\mathscr{U}_i$ (resp. line bundle $\mathscr{O}(i)$), that is $\mathscr{U}_i \otimes_{C_0} \mathscr{D}^b C_0(q)$ (resp. $\mathscr{O}(i) \otimes \mathscr{D}^b k$).

Theorem 4.1.1. *There is a semi-orthogonal decomposition*

$$\mathscr{D}^bQ = \langle \mathcal{U}_{d-1}, \mathcal{A}_{1-d}, \cdots, \mathcal{A}_{-1}, \mathcal{A}_0 \rangle.$$

Proof. Firstly, we show that the set

$$\Sigma = \{\mathscr{U}_{d-1}, \mathscr{O}(1-d), \dots, \mathscr{O}\}$$

generates $\mathscr{D}^bQ$ as an idempotent complete triangulated category. Thus, $\mathscr{D}^bQ$ is generated by the idempotent complete full triangulated subcategories $\mathcal{U}_{d-1}, \mathcal{A}_{1-d}, \dots, \mathcal{A}_0$. Let $\langle \Sigma \rangle \subset \mathscr{D}^bQ$ denote the full triangulated subcategory generated by Σ. Note that the quadric Q is a scheme with an ample line bundle $\mathscr{O}(1)$. So, the triangulated subcategory of compact objects $\mathscr{D}^bQ$ is generated (as an idempotent complete triangulated category) by line bundles $\mathscr{O}(i)$ for $i \le 0$, cf. [72, Lemma 3.5.2]. Taking duals, we see $\mathscr{D}^bQ$ is generated (as an idempotent complete triangulated category) by $\mathscr{O}(i)$ for $i \ge 0$. The resolution in Section 3.4 implies that $\mathscr{O}(1)$ is in $\langle \Sigma \rangle$. The canonical resolution (cf. [80, p. 126 Section 6]) gives $\mathscr{O}(i)$ for $i \ge 2$ is in $\langle \Sigma \rangle$.

It is clear that

$$\operatorname{Hom}(\mathscr{O}(i), \mathscr{O}(j)[p]) = H^p(Q, \mathscr{O}(j-i)) = 0$$

for $1-d \le j < i \le 0$, cf. [80, Lemma 5.2]. Moreover, we have

$$\operatorname{Hom}(\mathscr{O}(i), \mathscr{U}_{d-1}[p]) = H^p(Q, \mathscr{U}_{d-1}(-i)) = 0$$

for $1-d \le i \le 0$, cf. [80, Proof of Lemma 9.3] for $p > 0$ and [80, Lemma 9.5] for $p = 0$. Thus, we conclude $\operatorname{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $j < i$ and $\operatorname{Hom}(\mathcal{A}_i, \mathcal{U}_{d-1}) = 0$. $\square$

Corollary 4.1.1. *There is a semi-orthogonal decomposition*

$$\mathscr{D}^bQ_d = \begin{cases} \langle \mathcal{A}_{-m}, \dots, \mathcal{A}_{-1}, \mathcal{U}_0, \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_m \rangle & \text{if } d = 2m+1; \\ \langle \mathcal{A}_{1-m}, \dots, \mathcal{A}_{-1}, \mathcal{U}_0, \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{m-1}, \mathcal{A}_m \rangle & \text{if } d = 2m. \end{cases}$$

Proof. Let $d = 2m+1$. By taking the tensor product $\mathscr{O}(m) \otimes E$ for every element $E \in \Sigma$, we get another set

$$\{\mathscr{U}_m, \mathscr{O}(-m), \dots, \mathscr{O}, \dots, \mathscr{O}(m)\}.$$

Clearly, this set generates $\mathscr{D}^bQ$ as an idempotent complete triangulated category. Note that

$$\Sigma' = \{\mathscr{O}(-m), \dots, \mathscr{O}(-1), \mathscr{U}_0, \mathscr{O}, \mathscr{O}(1), \dots, \mathscr{O}(m)\}$$

also generates $\mathscr{D}^bQ$ as an idempotent complete triangulated category. It is enough to show $\mathscr{U}_m$ is in Σ'. Applying the exact sequence

$$0 \longrightarrow \mathscr{U}_m \longrightarrow \mathscr{O}(-m) \otimes C_{s_m} \longrightarrow \cdots\cdots \longrightarrow \mathscr{O}(-1) \otimes C_{s_{-1}} \longrightarrow \mathscr{U}_0 \longrightarrow 0,$$

we see the claim. Besides, we have $H^p(Q, \mathscr{F} \otimes \mathscr{U}_0^\vee) = \mathrm{Ext}^p(\mathscr{U}_0, \mathscr{F})$. Thus,

$$\mathrm{Hom}(\mathscr{U}_0, \mathscr{O}(i)[p]) = H^p(Q, \mathscr{U}_0^\vee(i)) = H^p(Q, \mathscr{U}_{-1}(i)) = H^p(Q, \mathscr{U}_{d-1}(d+i)) = 0$$

for $-m \leq i \leq -1$ by [80, Proof of Lemma 9.3] for $p > 0$ and [80, Lemma 9.5] for $p = 0$. The rest of the proof is similar to the proof of Theorem 4.1.1. For the case of $d = 2m$, one could use a similar procedure. $\square$

Let $\mathcal{A}_{[i,j]} \subset \mathscr{D}^bQ$ (resp. $\mathcal{A} \subset \mathscr{D}^bQ$) denote the full triangulated subcategory

$$\langle \mathcal{A}_i, \ldots, \mathcal{A}_j \rangle \text{ (resp. } \langle \mathcal{U}_0, \mathcal{A}_0 \rangle),$$

that is the smallest triangulated subcategory of $\mathscr{D}^bQ$ containing $\mathcal{A}_i, \ldots, \mathcal{A}_j$ (resp. $\mathcal{U}_0, \mathcal{A}_0$).

Corollary 4.1.2. *There is a semi-orthogonal decomposition*

$$\mathscr{D}^bQ_d = \begin{cases} \langle \mathcal{A}_{[-m,-1]}, \mathcal{A}, \mathcal{A}_{[1,m]} \rangle & \text{if } d = 2m+1; \\ \langle \mathcal{A}_{[1-m,-1]}, \mathcal{A}, \mathcal{A}_{[1,m-1]}, \mathcal{A}_m \rangle & \text{if } d = 2m. \end{cases}$$

4.2 Clifford algebras and canonical involution

Let k be a commutative ring (possibly $\frac{1}{2} \notin k$ in this section). Let A be a k-algebra. Recall that an involution $\tau : A \to A^{\mathrm{op}}$ is a k-algebra homomorphism such that $\tau^2 = \mathrm{id}$. The inclusion $\varphi : P \hookrightarrow C(q)^{\mathrm{op}}$ satisfies $\varphi(v)^2 = q(v)$, hence it provides a k-algebra homomorphism $\sigma : C(q) \to C(q)^{\mathrm{op}}, x \mapsto \bar{x}$ which is an involution (called *canonical involution*). Certainly, the canonical involution preserves the $\mathbb{Z}/2\mathbb{Z}$-grading on $C(q)$, so it restricts to the even part $C_0(q)$.

For any Clifford algebra $C(q)$, one could define a 'reduced' trace form $\mathrm{tr} : C(q) \to k$, cf. [78, Exercise 3.14]. The trace tr restricts to the even part $C_0(q)$. The trace form in $C(q)$ together with the canonical involution gives a symmetric bilinear form

$$B : C(q) \times C(q) \to k : (x, y) \mapsto \mathrm{tr}(x\bar{y})$$

Lemma 4.2.1. *Assume (P, q) is non-degenerate. Then B is non-degenerate if $\frac{1}{2} \in k$.*

Proof. Since $\frac{1}{2} \in k$, we can use [80, Corollary 1.2] to identify non-singularity and non-degeneracy. Then, we apply [80, Proposition 1.1. (a)] to reduce the problem to the case when the base k is a field. The form B is non-degenerate over a field of characteristic $\neq 2$, cf. [78, Exercise 3.14]. $\square$

Let A be any k-algebra, and let $A^* := \mathrm{Hom}_k(A, k)$. If A is viewed as a left A-module, then A^* may be considered as a right A-module via the left multiplication. Let M be a left A-module. Its opposite module M^{op} equals M as a k-module and can be viewed as a right A-module: $m^{\mathrm{op}} \cdot a = (\bar{a}m)^{\mathrm{op}}$ for any $a \in A$. It follows that $\mathrm{Hom}_k(M^{\mathrm{op}}, k)$ can be considered as a left A-module.

Lemma 4.2.2. *Consider $C(q)$ as a right $C(q)$-module. The map*

$$\vartheta : C(q) \to \mathrm{Hom}_k(C(q)^{\mathrm{op}}, k) : x \mapsto \big(\vartheta(x) : y^{\mathrm{op}} \mapsto B(x, y)\big)$$

is a right $C(q)$-module isomorphism such that $\vartheta^\vee \circ \mathrm{can} = \vartheta$ where $\mathrm{can} : C(q) \to C(q)^{**}$ is the double dual identification.

Remark 4.2.1. The map ϑ preserves the $\mathbb{Z}/2\mathbb{Z}$-grading of $C(q)$.

If $\mathscr{F}$ is a vector bundle, we denote the sheaf hom $\mathscr{H}om(\mathscr{F}, \mathscr{O})$ by $\mathscr{F}^\vee$. Let $\mathrm{can}_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^{\vee\vee}$ be the canonical double dual identification. Recall the notation in the previous section.

Lemma 4.2.3. *Let (P, q) be a non-degenerate quadratic form over k. Then, there are isomorphisms*

$$h_i : \mathscr{U}_i \longrightarrow (\mathscr{U}^{\mathrm{op}}_{-1-i})^\vee$$

of $\mathscr{O}$-modules and right $C_0(q)$-modules. Moreover, we have the following equalities

$$h_{-1-i} = (h_i)^\vee \circ \mathrm{can}_{\mathscr{U}_{-1-i}}$$

Proof. Let C_s denote $C_s(q)$ for $s \in \mathbb{Z}/2\mathbb{Z}$ for simplicity. Define a map

$$\eta_i : \mathscr{O}(-i) \otimes C_{s_i} \longrightarrow (\mathscr{O}(i) \otimes C^{\mathrm{op}}_{s_{-i}})^\vee$$

by $\eta_i(f \otimes x)(g \otimes y^{\mathrm{op}}) = fg \cdot \mathrm{tr}(x\bar{y})$. It is evident that η_i is an isomorphism, because η_i is the following composition

$$\mathscr{O}(-i) \otimes C_{s_i} \xrightarrow{\psi \otimes \vartheta} \mathscr{O}(i)^\vee \otimes (C^{\mathrm{op}}_{s_{-i}})^* \longrightarrow (\mathscr{O}(i) \otimes C^{\mathrm{op}}_{s_{-i}})^\vee$$

where $\psi : \mathscr{O}(-i) \to \mathscr{O}(i)^\vee$ is the natural isomorphism, where ϑ is defined in Lemma 4.2.2 and where the last map is just the canonical isomorphism. It is a pleasure to note that $C_s = C_{-s}$. By Lemma 4.2.2 and Remark 4.2.1, we see η_i are well-defined right $C_0(q)$-homomorphisms.

The diagram labeled by the symbol $\square$

$$\begin{tikzcd}
\mathscr{O}(-i-2) \otimes C_{s_{i+2}} \arrow[r, "\ell"] \arrow[d, "\eta"] \arrow[dr, phantom, "\square"] & \mathscr{O}(-i-1) \otimes C_{s_{i+1}} \arrow[r, two heads] \arrow[d, "\eta"] & \mathscr{U}_i \arrow[d, dashed, "h_i"] \\
(\mathscr{O}(i+2) \otimes C^{\mathrm{op}}_{s-i-2})^\vee \arrow[r, "\ell^\vee"] & (\mathscr{O}(i+1) \otimes C^{\mathrm{op}}_{s-i-1})^\vee \arrow[r, two heads] & (\mathscr{U}^{\mathrm{op}}_{-1-i})^\vee
\end{tikzcd}$$

is commutative. This may be checked locally. Let $f \otimes m \in \mathscr{O}(-i-2) \otimes C_{s_{i+2}}$. Assume k is a local ring (by localizing at a prime ideal), we see the map ℓ may be interpreted as $\ell(f \otimes m) = \sum \beta_i f \otimes \xi_i m$ for some $\beta_i \in P^*, \xi_i \in P$. Let $g \otimes n^{\mathrm{op}} \in \mathscr{O}(i+1) \otimes C^{\mathrm{op}}_{s-i-1}$. It reduces to check

$$\eta(\sum \beta_i f \otimes \xi_i m)(g \otimes n^{\mathrm{op}}) = \eta(f \otimes m)(\sum \beta_i g \otimes (\xi_i n)^{\mathrm{op}})$$

The left-hand side equals $\sum \beta_i f g \cdot \mathrm{tr}(\xi_i m \bar{n})$, while the right-hand side equals $\sum \beta_i f g \cdot \mathrm{tr}(m \overline{\xi_i n}) = \sum \beta_i f g \cdot \mathrm{tr}(m \bar{n} \bar{\xi_i})$. Note that $m\bar{n}\bar{\xi_i} = m\bar{n}\xi_i$ by $\bar{\xi} = \xi \in P$, and that $\mathrm{tr}(m\bar{n}\xi_i) = \mathrm{tr}(\xi_i m \bar{n})$. This provides the equality.

Then, the dotted map h_i is just given by the universal property of cokernels. Since η_i are isomorphisms, so are the maps h_i. The last assertion is obtained by the symmetry of ϑ in Lemma 4.2.2. $\square$

4.3 Grothendieck-Witt spectra of Q_{odd}

Let $\mathrm{sPerf}(Q_d)$ be the dg category of strictly perfect complexes over Q_d and let $\mathscr{L}$ be any line bundle over Q_d. Let $\#_{\mathscr{L}}$ denote the dg functor $[-, \mathscr{L}]$ and $\mathrm{can}^{\mathscr{L}}$ be the canonical double dual identification. Take quasi-isomorphisms as weak equivalences. The quadruple

$$(\mathrm{sPerf}(Q_d), \mathrm{quis}, \#_{\mathscr{L}}, \mathrm{can}^{\mathscr{L}})$$

is a dg category with weak equivalences and duality, cf. [73, Section 9]. Then, the triangulated category $\mathcal{T}(\mathrm{sPerf}(Q_d), \mathrm{quis})$ is just $\mathscr{D}^b Q_d$.

Set

$$GW^{[i]}(Q_d) := GW(\mathrm{sPerf}(Q_d), \mathrm{quis}, \#_{\mathscr{O}[i]}, \mathrm{can}^{\mathscr{O}[i]}).$$

Note that

$$GW^{[i]}(Q_d) \approx GW^{[i]}(\mathrm{sPerf}(Q_d), \mathrm{quis}, \#_{\mathscr{O}}, \mathrm{can}^{\mathscr{O}}).$$

One could write a similar notation for the case of $\mathbb{G}W$-spectra.

For convenience, we introduce notations $\mathscr{A}_i, \mathscr{A}_{[k,l]}$ and $\mathscr{A}$ which are defined to be the full dg categories of $\mathrm{sPerf}(Q_d)$ corresponding to the full triangulated categories $\mathcal{A}_i, \mathcal{A}_{[k,l]}$ and $\mathcal{A}$ of $\mathscr{D}^b Q_d$ respectively. Explicitly, $\mathscr{A}_i, \mathscr{A}_{[k,l]}, \mathscr{A}$ have objects which lie in $\mathcal{A}_i$, $\mathcal{A}_{[k,l]}, \mathcal{A}$ respectively. Moreover, $\mathscr{A}_i, \mathscr{A}_{[k,l]}$ and $\mathscr{A}$ are pretriangulated.

Lemma 4.3.1. *$\mathscr{A} \subset \mathrm{sPerf}(Q_d)$ is fixed by the duality $\#_{\mathscr{O}}$.*

Proof. It is enough to show $\mathcal{A}$ is fixed by the duality $\#_{\mathscr{O}}$. By the definition of $\mathscr{U}_i$, we have an exact sequence

$$\mathscr{U}_0 \rightarrowtail \mathscr{O} \otimes C_0(q) \twoheadrightarrow \mathscr{U}_{-1}$$

It follows that $\mathscr{U}_{-1} \in \mathcal{A}$. By Lemma 4.2.3, we see $\mathscr{U}_0^\vee \approx \mathscr{U}_{-1}$ in $\mathcal{A}$. $\square$

Thus we have a pretriangulated dg category with weak equivalences and duality

$$(\mathscr{A}, \mathrm{quis}, \#_{\mathscr{O}}, \mathrm{can}^{\mathscr{O}}).$$

Next, recall the hyperbolic dg category $\mathcal{H}\mathscr{A}_{[1,m]}$ defined in [73, Section 4].

Theorem 4.3.1. *There is a stable equivalence of Grothendieck-Witt spectra*

$$GW(\mathcal{H}\mathscr{A}_{[1,m]}) \oplus GW^{[i]}(\mathscr{A}) \xrightarrow{\sim} GW^{[i]}(Q_d)$$

and a stable equivalence of Karoubi-Grothendieck-Witt spectra

$$\mathbb{G}W(\mathcal{H}\mathscr{A}_{[1,m]}) \oplus \mathbb{G}W^{[i]}(\mathscr{A}) \xrightarrow{\sim} \mathbb{G}W^{[i]}(Q_d).$$

Proof. This result is a consequence of the additivity theorem (cf. Theorem 3.5.6) and Corollary 4.1.2. $\square$

Recall from [73] that we have

$$K(\mathscr{A}_{[1,m]}) = GW(\mathcal{H}\mathscr{A}_{[1,m]}) \quad \text{and} \quad \mathbb{K}(\mathscr{A}_{[1,m]}) = \mathbb{G}W(\mathcal{H}\mathscr{A}_{[1,m]}).$$

By additivity in K-theory, we conclude

$$K(\mathscr{A}_{[1,m]}) = \oplus_{i=1}^m K(\mathscr{A}_i) \quad \text{and} \quad \mathbb{K}(\mathscr{A}_{[1,m]}) = \oplus_{i=1}^m \mathbb{K}(\mathscr{A}_i).$$

Moreover, note that the exact dg functor

$$\mathscr{O}(i) \otimes ? : (\mathrm{sPerf}(k), \mathrm{quis}) \longrightarrow (\mathscr{A}_i, \mathrm{quis})$$

induces an equivalence of associated triangulated categories

$$\mathscr{O}(i)\otimes ? : \mathscr{D}^b(k) \xrightarrow{\sim} \mathcal{A}_i.$$

Applying [72, Theorem 3.2.24] and [72, Theorem 3.2.29], we see

$$\mathscr{O}(i)\otimes ? : K(k) \xrightarrow{\sim} K(\mathscr{A}_i) \quad \text{and} \quad \mathscr{O}(i)\otimes ? : \mathbb{K}(k) \xrightarrow{\sim} \mathbb{K}(\mathscr{A}_i).$$

Corollary 4.3.1. *There is a stable equivalence of Grothendieck-Witt spectra*

$$(H\mathscr{O}(1)\otimes ?, \cdots, H\mathscr{O}(m)\otimes ?, ?) : \bigoplus_{i=1}^{m} K(k) \oplus GW^{[i]}(\mathscr{A}) \xrightarrow{\sim} GW^{[i]}(Q_d)$$

and a stable equivalence of Karoubi-Grothendieck-Witt spectra

$$(H\mathscr{O}(1)\otimes ?, \cdots, H\mathscr{O}(m)\otimes ?, ?) : \bigoplus_{i=1}^{m} \mathbb{K}(k) \oplus \mathbb{G}W^{[i]}(\mathscr{A}) \xrightarrow{\sim} \mathbb{G}W^{[i]}(Q_d).$$

It remains to understand $GW^{[i]}(\mathscr{A})$ and $\mathbb{G}W^{[i]}(\mathscr{A})$. Let w be the set of morphisms in $\mathscr{A}$ that become isomorphisms in the Verdier quotient

$$\mathcal{T}(\mathscr{A}, \text{quis})/\mathcal{T}(\mathscr{A}_0, \text{quis}),$$

which is equivalent to $\mathcal{T}(\mathscr{A}, w)$. Then, there is a quasi-exact sequence (hence Morita exact)

$$(\mathscr{A}_0, \text{quis}) \longrightarrow (\mathscr{A}, \text{quis}) \longrightarrow (\mathscr{A}, w)$$

in the sense of [73, Section 6] which provides localization sequences of $GW^{[i]}$-spectra

$$GW^{[i]}(\mathscr{A}_0) \longrightarrow GW^{[i]}(\mathscr{A}) \longrightarrow GW^{[i]}(\mathscr{A}, w)$$

by [73, Theorem 6.6], and localization sequences of Karoubi $\mathbb{G}W^{[i]}$-spectra

$$\mathbb{G}W^{[i]}(\mathscr{A}_0) \longrightarrow \mathbb{G}W^{[i]}(\mathscr{A}) \longrightarrow \mathbb{G}W^{[i]}(\mathscr{A}, w)$$

by [73, Theorem 8.9]. The exact dg form functor

$$(\mathscr{O}, \text{id})\otimes ? : (\text{sPerf}(k), \text{quis}, \#_k) \longrightarrow (\mathscr{A}_0, \text{quis}, \#_{\mathscr{O}})$$

gives an equivalence $\mathscr{O}\otimes ? : \mathscr{D}^b(k) \xrightarrow{\sim} \mathcal{A}_0$ of associated triangulated categories. By the invariance for GW (cf. [73, Theorem 6.5]) and the invariance for $\mathbb{G}W$ (cf. [73, Theorem 8.10]), one has equivalences of spectra

$$GW^{[i]}(k) \xrightarrow{\sim} GW^{[i]}(\mathscr{A}_0) \quad \text{and} \quad \mathbb{G}W^{[i]}(k) \xrightarrow{\sim} \mathbb{G}W^{[i]}(\mathscr{A}_0).$$

Thus, we deduce homotopy fibrations of GW-spectra

$$GW^{[i]}(k) \longrightarrow GW^{[i]}(\mathscr{A}) \longrightarrow GW^{[i]}(\mathscr{A}, w)$$

and of $\mathbb{G}W$-spectra

$$\mathbb{G}W^{[i]}(k) \longrightarrow \mathbb{G}W^{[i]}(\mathscr{A}) \longrightarrow \mathbb{G}W^{[i]}(\mathscr{A}, w).$$

Next, we study the implicit part $GW^{[i]}(\mathscr{A}, w)$ and $\mathbb{G}W^{[i]}(\mathscr{A}, w)$.

Recall the exact sequence $\mathscr{U}_0 \rightarrowtail \mathscr{O} \otimes C_0(q) \twoheadrightarrow \mathscr{U}_{-1}$ obtained by the definition of $\mathscr{U}_i$, and delete the component $\mathscr{U}_{-1}$. We obtain a (cochain) complex concentrated in degree $[0,1]$

$$\cdots \to 0 \to \mathscr{U}_0 \to \mathscr{O} \otimes C_0(q) \to 0 \to \cdots$$

which is denoted by $Cl_{[0,1]}$.

Lemma 4.3.2. *There is a symmetric space $(\mathscr{U}_0, \mu)$ in the category with duality*

$$(\mathcal{T}(\mathscr{A}, w), \#_{\mathscr{O}[-1]}, can^{\mathscr{O}})$$

where the form μ is represented by the following left roof

$$\begin{tikzcd} & Cl_{[0,1]} \arrow[dl, "t"'] \arrow[dr, "s"] & \\ \mu: \quad \mathscr{U}_0 \arrow[rr, dashed] & & (\mathscr{U}_0)^\vee[-1]. \end{tikzcd}$$

The only non-trivial component in the morphism $t : Cl_{[0,1]} \to \mathscr{U}_0$ is the map $id : \mathscr{U}_0 \to \mathscr{U}_0$ in the degree 0, and the only non-trivial component of the map $s : Cl_{[0,1]} \to (\mathscr{U}_0)^\vee[-1]$ is the composition $\mathscr{O} \otimes C_0(q) \longrightarrow \mathscr{U}_{-1} \xrightarrow{h_{-1}} (\mathscr{U}_0)^\vee$ in the degree 1.

Proof. It is clear that s is a quasi-isomorphism so that $s \in w$. Moreover, we observe that cone(t) is in $\mathcal{T}(\mathscr{A}_0, \text{quis})$ so that $t \in w$. Thus, μ is an isomorphism in $\mathcal{T}(\mathscr{A}, w)$. We show μ is symmetric, i.e. $\mu^\vee \circ \text{can}_{\mathscr{U}_0} = \mu$. Observe $\mu^\vee \circ \text{can}_{\mathscr{U}_0}$ is represented by a right roof. The result can be obtained by noting the following morphism of complexes is null-homotopic.

$$\begin{tikzcd} \mathscr{U}_0 \arrow[r, "l"] \arrow[d, "l^\vee \circ h_0"'] & \mathscr{O} \otimes C_0(q) \arrow[d, "-h_{-1} \circ \ell"] \arrow[dl, dashed, "\eta_0"'] \\ (\mathscr{O} \otimes C_0(q))^\vee \arrow[r, "-\ell^\vee"] & (\mathscr{U}_0)^\vee \end{tikzcd}$$

Here, all the maps are defined (recall Lemma 4.2.3 and its proof). $\square$

The proof of [73, Lemma 3.9] tells us that the form $(\mathscr{U}_0, \mu)$ (in Lemma 4.3.2) can be lifted to a symmetric form $(B_{\mathscr{U}_0}, B_\mu)$ in the dg category

$$(\mathscr{A}, w, \#_{\mathscr{O}[-1]}, \mathrm{can})$$

with weak equivalences and duality, such that the morphism B_μ is in w and that $(B_{\mathscr{U}_0}, B_\mu)$ is isometric to $(\mathscr{U}_0, \mu)$ in $(\mathcal{T}(\mathscr{A}, w), \#_{\mathscr{O}[-1]}, \mathrm{can}, \lambda)$. The form $(B_{\mathscr{U}_0}, B_\mu)$ is displayed as follows.

$$\begin{tikzcd} Cl_{[0,1]} \arrow[d, "B_\mu"] \\ {[Cl_{[0,1]}, \mathscr{O}[-1]]} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{U}_0 \arrow[r, "l"] \arrow[d, "l^\vee \circ h_0"'] & \mathscr{O} \otimes C_0(q) \arrow[d, "h_{-1} \circ \ell"] \\ (\mathscr{O} \otimes C_0(q))^\vee \arrow[r, "-\ell^\vee"] & (\mathscr{U}_0)^\vee \end{tikzcd}$$

Let A, B be dg k-algebras. Recall from [71, Section 7.2] the definition of dg A-modules. We denote by A-dgMod $:= A$-dgMod-k, dgMod-$B := k$-dgMod-B and A-dgMod-B the dg category of dg left A-modules, dg right B-modules, and of dg left A-modules and right B-modules. If A is a dg algebra with involution, then for any dg left A-module M we have a dg right A-module M^{op}, cf. [71, Section 7.3].

Let (I, i) denote an A-bimodule I together with an A-bimodule isomorphism $i : I \to I^{\mathrm{op}}$ such that $i^{\mathrm{op}} \circ i = \mathrm{id}$. By abuse of the notation, we write I for (I, i) if the isomorphism i is understood. In fact, I is called a *duality coefficient* in [71, Section 7.3]. There is a dg category with duality

$$(A\text{-dgMod-}B, \#_I, \mathrm{can}_I)$$

with $\#_I : (A\text{-dgMod-}B)^{\mathrm{op}} \to A\text{-dgMod-}B$ by $M^{\#_I} = [M^{\mathrm{op}}, I]_A$ and $\mathrm{can}_I : M \to M^{\#\#}$ by

$$\mathrm{can}(x)(f^{\mathrm{op}}) = (-1)^{|f||x|} i(f(x^{\mathrm{op}})).$$

By abuse of notation, we write $(A\text{-dgMod-}B, \#_I)$ for this dg category with duality. Let A, B be dg algebras with involution. Then, there is a dg form functor

$$(A\text{-dgMod-}B, \#_I) \otimes (B\text{-dgMod}, \#_B) \to (A\text{-dgMod}, \#_I)$$

sending (M, N) to $M \otimes_B N$ with the duality compatibility map

$$\gamma : [M^{\mathrm{op}}, I]_A \otimes_B [N^{\mathrm{op}}, B]_B \to [(M \otimes_B N)^{\mathrm{op}}, I]_A$$

defined by

$$\gamma(f \otimes g)((m \otimes n)^{\mathrm{op}}) = (-1)^{|m||n|} f(g(n^{\mathrm{op}}) m^{\mathrm{op}}).$$

Applying $A = (\mathscr{O}, \mathrm{id})$, $I = (\mathscr{O}[-1], \mathrm{id})$ and $B = (C_0(q), \sigma)$, we deduce that

Lemma 4.3.3. *There is a dg form functor*

$$(\mathscr{O}\text{-dgMod-}C_0(q), \#_{\mathscr{O}[-1]}) \otimes (C_0(q)\text{-dgMod}, \#^{\sigma}_{C_0(q)}) \to (\mathscr{O}\text{-dgMod}, \#_{\mathscr{O}[-1]}).$$

We have already seen $B_\mu : Cl_{[0,1]} \to [Cl_{[0,1]}, \mathscr{O}[-1]]$ is a symmetric form in the dg category with duality $(\mathscr{O}\text{-dgMod}, \#_{\mathscr{O}[-1]})$. We further observe the following result.

Lemma 4.3.4. *The map $B_\mu : Cl_{[0,1]} \to [(Cl_{[0,1]})^{\mathrm{op}}, \mathscr{O}[-1]]$ is a symmetric form in $(\mathscr{O}\text{-dgMod-}C_0(q), \#_{\mathscr{O}[-1]})$.*

Proof. We only need to show B_μ is a right $C_0(q)$-module map. This can be seen directly by Lemma 4.2.3 and its proof. □

By Lemma 3.2.4, we obtain

Corollary 4.3.2. *There is a dg form functor*

$$(B_{\mathscr{U}_0}, B_\mu) \otimes ? : (C_0(q)\text{-dgMod}, \#^{\sigma}_{C_0(q)}) \to (\mathscr{O}\text{-dgMod}, \#_{\mathscr{O}[-1]}).$$

Let $\mathrm{sPerf}(C_0(q))$ be the dg category of strictly perfect complexes of finitely generated left projective $C_0(q)$-modules. Since $\mathrm{sPerf}(C_0(q)) \subset C_0(q)\text{-dgMod}$ and $\mathscr{A} \subset \mathscr{O}\text{-dgMod}$ are full dg subcategories, we conclude

Corollary 4.3.3. *There is a dg form functor*

$$(B_{\mathscr{U}_0}, B_\mu) \otimes ? : (\mathrm{sPerf}(C_0(q)), \#^{\sigma}_{C_0(q)}) \to (\mathscr{A}, \#_{\mathscr{O}[-1]}).$$

Taking weak equivalences into account, we get an exact dg form functor

$$(B_{\mathscr{U}_0}, B_\mu) \otimes ? : (\mathrm{sPerf}(C_0(q)), \mathrm{quis}, \#^{\sigma}_{C_0(q)}) \longrightarrow (\mathscr{A}, w, \#_{\mathscr{O}}[-1])$$

which induces an equivalences of associated triangulated categories. Let

$$GW^{[i]}(C_0(q), \sigma) := GW^{[i]}(\mathrm{sPerf}(C_0(q)), \mathrm{quis}, \#^{\sigma}_{C_0(q)}).$$

By invariance for GW and $\mathbb{G}W$, we find stable equivalences

$$(B_{\mathscr{U}_0}, B_\mu) \cup ? : GW^{[i+1]}(C_0(q), \sigma) \xrightarrow{\sim} GW^{[i]}(\mathscr{A}, w)$$

and

$$(B_{\mathscr{U}_0}, B_\mu) \cup ? : \mathbb{G}W^{[i+1]}(C_0(q), \sigma) \xrightarrow{\sim} \mathbb{G}W^{[i]}(\mathscr{A}, w)$$

Theorem 4.3.2. *There is a stable equivalence of spectra*

$$(H\mathscr{O}(1) \otimes ?, \cdots, H\mathscr{O}(m) \otimes ?, ?) : \bigoplus_{i=1}^{m} K(k) \oplus GW^{[i]}(\mathscr{A}) \xrightarrow{\sim} GW^{[i]}(Q_d).$$

where $GW^{[i]}(\mathscr{A})$ fits into another homotopy fibration sequence

$$GW^{[i]}(k) \longrightarrow GW^{[i]}(\mathscr{A}) \longrightarrow GW^{[i+1]}(C_0(q), \sigma).$$

Theorem 4.3.3. *There is a stable equivalence of spectra*

$$(H\mathscr{O}(1)\otimes ?, \cdots, H\mathscr{O}(m)\otimes ?, ?) : \bigoplus_{i=1}^{m} \mathbb{K}(k) \oplus \mathbb{G}W^{[i]}(\mathscr{A}) \xrightarrow{\sim} \mathbb{G}W^{[i]}(Q_d).$$

where $\mathbb{G}W^{[i]}(\mathscr{A})$ *fits into another homotopy fibration sequence*

$$\mathbb{G}W^{[i]}(k) \longrightarrow \mathbb{G}W^{[i]}(\mathscr{A}) \longrightarrow \mathbb{G}W^{[i+1]}(C_0(q)).$$

4.4 Application: Witt groups of Q_{odd}

In this section, we prove Theorem 1.2.3. Let $d > 0$ be an odd integer. Note that Balmer's Witt groups $W^i(Q_d)$ are just

$$W^{[i]}(Q_d) := W^{[i]}(\mathrm{sPerf}(Q_d), \mathrm{quis}, \#_{\mathscr{O}}, \mathrm{can})$$

in Schlichting's framework. In fact, $W^0(Q_d)$ is isomorphic to the classical Witt group $W(Q_d)$, cf. [7].

Lemma 4.4.1. *Let k be a regular ring with $\frac{1}{2} \in k$. There is an isomorphism $W^{[i]}(\mathscr{A}) \approx W^{[i]}(Q_d)$.*

Proof. Note that the negative homotopy groups of $K(k)$ vanish if k is regular, cf. [69, Remark 7]. Taking negative homotopy groups over both sides of the equivalence

$$\bigoplus_{i=1}^{m} K(k) \oplus GW^{[i]}(\mathscr{A}) \xrightarrow{\sim} GW^{[i]}(Q_d),$$

we see the result. □

Let $W^i(C_0(q), \sigma)$ be the Balmer's Witt groups of the even part Clifford algebra $C_0(q)$ with the canonical involution σ, which are just

$$W^{[i]}(C_0(q), \sigma) := W^{[i]}(\mathrm{sPerf}(C_0(q)), \mathrm{quis}, \#^{\sigma}_{C_0(q)}, \mathrm{can}).$$

Note that $W^0(C_0(q), \sigma)$ is isomorphic to the classical Witt group of the non-commutative algebra $C_0(q)$ with the involution σ.

Then, we have a dg functor

$$F : C_0(q)\text{-dgMod-}k \longrightarrow \text{dgMod-}k$$

by forgetting the left $C_0(q)$-module structure. Observe that there is a dg form functor

$$(F, \mathrm{tr}) : (C_0(q)\text{-dgMod-}k, \#^{\sigma}_{C_0(q)}) \longrightarrow (\text{dgMod-}k, \#_k)$$

with the duality compatibility map defined by the composition of k-module maps

$$[M, C_0(q)]_{C_0(q)} \xrightarrow{i} [M, C_0(q)]_k \xrightarrow{[1,\mathrm{tr}]} [M, k]_k$$

where the map i is the inclusion and where the map tr is the trace map. This dg form functor gives an exact dg form functor

$$(F, \mathrm{tr}) : (\mathrm{sPerf}(C_0(q)), \mathrm{quis}, \#^{\sigma}_{C_0(q)}) \longrightarrow (\mathrm{sPerf}(k), \mathrm{quis}, \#_k)$$

which provides a map $(F, \mathrm{tr}) : W^i(C_0(q), \sigma) \to W^i(k)$.

Theorem 4.4.1. *Let k be a regular local ring with $\frac{1}{2} \in k$. If d is odd, then*

$$\begin{aligned} W^0(Q_d) &\approx \mathrm{coker}(F, \mathrm{tr}) \\ W^1(Q_d) &\approx W^2(C_0(q), \sigma) \\ W^2(Q_d) &\approx 0 \\ W^3(Q_d) &\approx \ker(F, \mathrm{tr}) \end{aligned}$$

Recall the exact sequence $\mathcal{A}_0 \to \mathcal{A} \to \mathcal{A}/\mathcal{A}_0$ of triangulated categories. There is a map $\partial^i : W^i(\mathcal{A}/\mathcal{A}_0) \to W^{i+1}(\mathcal{A}_0)$ called the "connecting homomorphism" in [9]. The map ∂^3 may be interpreted by the map (F, tr) in the following sense.

Proposition 4.4.1. *The diagram*

$$\begin{tikzcd} W^0(C_0(q), \sigma) \arrow[r, "(F,\mathrm{tr})"] \arrow[d, "U"', "\cong"] & W^0(k) \arrow[d, "G"', "\cong"] \\ W^{-1}(\mathcal{A}/\mathcal{A}_0) \arrow[r, "\partial^3"] & W^0(\mathcal{A}_0) \end{tikzcd}$$

is commutative, where U (resp. G) denotes the map $(\mathscr{U}_0, \mu)\otimes?$ (resp. $(\mathscr{O}, \mathrm{id})\otimes?$).

Proof. Let b be an element in $W^0(C_0(q), \sigma)$ corresponding to a symmetric space

$$b : M \to [M^{\mathrm{op}}, C_0(q)]_{C_0(q)},$$

where M is a finitely generated left projective $C_0(q)$-module. It is enough to check

$$G \circ (F, \mathrm{tr})(b) = \partial^3 \circ U(b).$$

Note that $G \circ (F, \mathrm{tr})(b)$ is the symmetric space

$$\psi : \mathscr{O} \otimes M \to [\mathscr{O} \otimes M, \mathscr{O}]_{\mathscr{O}}$$

defined by $\psi(f\otimes x)(g\otimes y) = fg\cdot \mathrm{tr}(b(x)(y^{\mathrm{op}}))$. Moreover, I claim $\partial^3\circ U$ sends b to the symmetric space ψ. Firstly, observe that $U(b)$ is the symmetric space $\mathscr{U}_0\otimes_{C_0(q)} M \to (\mathscr{U}_0\otimes_{C_0(q)} M)^\vee[-1]$ represented by the left roof $t^{-1}s$

$$\begin{tikzcd}
& Cl_{[0,1]}\otimes_{C_0(q)} M \arrow[dl, "t"'] \arrow[dr, "s"] & \\
\mathscr{U}_0\otimes_{C_0(q)} M \arrow[rr, dashed] & & (\mathscr{U}_0\otimes_{C_0(q)} M)^\vee[-1].
\end{tikzcd}$$

where t is the projection and where s consists of the composition

$$e : (\mathscr{O}\otimes C_0(q))\otimes_{C_0} M \xrightarrow{\ell\otimes 1} \mathscr{U}_{-1}\otimes_{C_0} M \xrightarrow{h\otimes b} (\mathscr{U}_0^{\mathrm{op}})^\vee\otimes_{C_0}[M^{\mathrm{op}}, C_0(q)] \xrightarrow{\gamma} (\mathscr{U}_0\otimes_{C_0} M)^\vee$$

in the degree 1. Note that s is a quasi-isomorphism. Now, we use the procedure introduced in [9] to investigate $\partial^3(U(b))$. Observe that there is an exact triangle

$$(\mathscr{U}_0\otimes_{C_0} M)^\vee[-1]\xrightarrow{s^{-1}t}\mathscr{U}_0\otimes_{C_0} M\xrightarrow{\ell\otimes 1}\mathscr{O}\otimes M\xrightarrow{e}(\mathscr{U}_0\otimes_{C_0} M)^\vee$$

in the triangulated category $\mathcal{A}$, and a symmetric form of exact triangles

$$\begin{tikzcd}
(\mathscr{U}_0\otimes_{C_0} M)^\vee[-1] \arrow[r, "s^{-1}t"] \arrow[d, "="] & \mathscr{U}_0\otimes_{C_0} M \arrow[r, "\ell\otimes 1"] \arrow[d, "\mathrm{can}"] & \mathscr{O}\otimes M \arrow[r, "e"] \arrow[d, "\psi"] & (\mathscr{U}_0\otimes_{C_0} M)^\vee \arrow[d, "="] \\
(\mathscr{U}_0\otimes_{C_0} M)^\vee[-1] \arrow[r] & (\mathscr{U}_0\otimes_{C_0} M)^{\vee\vee} \arrow[r, "e^\vee"] & (\mathscr{O}\otimes M)^\vee \arrow[r, "(\ell\otimes 1)^\vee"] & (\mathscr{U}_0\otimes_{C_0} M)^\vee
\end{tikzcd}$$

in the triangulated category $(\mathcal{A}, \#_{\mathscr{O}[-1]})$ with duality. Note that the map

$$s^{-1}t : (\mathscr{U}_0\otimes_{C_0} M)^\vee[-1]\longrightarrow \mathscr{U}_0\otimes_{C_0} M$$

is a symmetric form in the triangulated category $(\mathcal{A}, \#_{\mathscr{O}[-1]})$ with duality. Moreover, it is isometric to the symmetric space

$$U(b) = t^{-1}s : \mathscr{U}_0\otimes_{C_0} M\longrightarrow (\mathscr{U}_0\otimes_{C_0} M)^\vee[-1]$$

in the triangulated category $(\mathcal{A}/\mathcal{A}_0, \#_{\mathscr{O}[-1]})$ with duality. $\square$

Proof of Theorem 1.2.3. Note that if k is local, then $W^i(k) = 0$ for $i \not\equiv 0 \mod 4$ and $W^i(C_0(q),\sigma) = 0$ for i odd (cf. [12]). By four-periodicity and localization, we find a

12-term long exact sequence

$$\begin{tikzcd}
 & W^0(k) \arrow[r] & W^0(\mathcal{A}) \arrow[r] & 0 \arrow[dr] & \\
W^4(C_0(q),\sigma) \arrow[ur, "{(F,\mathrm{tr})}"] & & & & 0 \arrow[d] \\
W^3(\mathcal{A}) \arrow[u] & & & & W^1(\mathcal{A}) \arrow[d] \\
0 \arrow[u] & & & & W^2(C_0(q),\sigma) \arrow[dl] \\
 & 0 \arrow[ul] & W^2(\mathcal{A}) \arrow[l] & 0 \arrow[l] &
\end{tikzcd} \tag{4.4}$$

Then, we apply Lemma 4.4.1 to finish the proof. □

Let q denote the quadratic form $\langle a_1, \dots, a_n \rangle$ with $a_i \in k^\times$.

Theorem 4.4.2. *Let k be a field with $\frac{1}{2} \in k$. Then, we have*

$$\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle W^0(k) \subset \ker(p^* : W^0(k) \to W^0(Q_d))$$

where $\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle$ is the Pfister form. Moreover, if $C_0(q)$ is a division algebra, then

$$W^0(Q_d) \approx W^0(k)/\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle W^0(k).$$

Proof. There is always an element β in $W^0(C_0(q), \sigma)$ represented by the form

$$\beta : C_0(q) \times C_0(q) \to C_0(q) : (x, y) \mapsto x\bar{y}$$

Let $\{e_1, \dots, e_n\}$ be an orthogonal basis of the quadratic form q. The Clifford algebra $C(q)$ has a basis $\{e^\Delta : \Delta \in \mathbb{F}_2^n\}$ where $e^\Delta := e_1^{b_1} \cdots\cdot e_n^{b_n}$ with $\Delta = (b_1, \dots, b_n)$. Let $|\Delta| = \sum b_i$. Then, $C_0(q)$ has a basis $\{e^\Delta : \Delta \in \Omega\}$ where Ω is the set $\{\Delta \in \mathbb{F}_2^n : |\Delta| \text{ is even}\}$. Thus, we see $(F, \mathrm{tr})(\beta) = \perp_{\Delta\in\Omega} q(e^\Delta) \in W^0(k)$ where $q(e^\Delta) := q(e_1)^{b_1} \cdots\cdot q(e_n)^{b_n} = a_1^{b_1} \cdots\cdot a_n^{b_n}$. Observe that $\perp_{\Delta\in\Omega} q(e^\Delta) \approx \langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle$ in $W^0(k)$ (one may find that [78, Exercise 3.14] is helpful to see this).

Let (D, σ) be a division algebra with involution, and let $S(D)$ denote the involution center, i.e. $\{x \in D : x = \bar{x}\}$. It is well-known that $W^0(D, \sigma)$ is generated by $\langle d \rangle$ with $d \in S(D)^\times$. If $C_0(q)$ is a division algebra, I claim this element is exactly the generator of $\ker(p^*) \approx \mathrm{Im}(F, \mathrm{tr})$. Taking $\langle d \rangle \in W^0(C_0(q), \sigma)$ with $d \in S(C_0(q)) \cap \{e^\Delta : \Delta \in \Omega\}$, it is enough to show $(F,\mathrm{tr})(\langle d\rangle) \in \langle\langle a_1a_2,\dots,a_1a_n\rangle\rangle W^0(k)$. A computation shows that

$$(F,\mathrm{tr})(\langle d\rangle) = \pm q(d)\langle\langle a_1a_2,\dots,a_1a_n\rangle\rangle,$$

but $\pm q(d)$ disappears in $W^0(k)$. □

Suppose the field k is quadratically closed, i.e. every unit is a square. It is clear that $W^0(k) \cong \mathbb{Z}/2\mathbb{Z}$. Observe that the map (F,tr) vanishes. Thus, we have

Proposition 4.4.2. *$W^0(Q) \approx W^0(k)$ and $W^3(Q) \approx W^0(C_0(q),\sigma)$.*

Corollary 4.4.1. *If $d \equiv 3,5 \mod 8$, then $W^0(Q) = \mathbb{Z}/2\mathbb{Z}$, $W^1(Q) = W^2(Q) = 0$ and $W^3(Q) = \mathbb{Z}/2\mathbb{Z}$. If $d \equiv 1,7 \mod 8$, then $W^0(Q) = \mathbb{Z}/2\mathbb{Z}$, $W^2(Q) = W^3(Q) = 0$ and $W^1(Q) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Combining Proposition 4.4.1 and Theorem 1.2.3, we conclude the results by [12, Lemma 4.13] and [48, Proposition 8.4]. □

This result is well-known in the literature. Over a quadratically closed field k, every quadric is isomorphic to the split one. The Witt groups of split quadrics are computed in [60] and [93]. Now, we are interested in the case when the field k is not quadratically closed, e.g. the real numbers $\mathbb{R}$.

Let Q be the quadric defined by the quadratic form $q_n = n\langle 1\rangle$. The Clifford algebra of the quadratic form q_n is well-known. I will use the exact sequence

$$0 \longrightarrow W^{-1}(Q) \longrightarrow W^0(C_0(q),\sigma) \xrightarrow{(F,\mathrm{tr})} W^0(k) \longrightarrow W^0(Q) \longrightarrow 0$$

to compute $W^0(Q)$. Note that $C_0(q) \approx C(q')$ where $q' = (n-1)\langle -1\rangle$. Let Y denote the quaternion algebra $(\frac{-1,-1}{k})$.

Assume -1 is not a sum of two squares. The algebra Y is a division algebra. We copy parts of the table on [56, p. 125] for this case.

Table 1	n=3	n=5	n=7	n=9
$C_0(q_n)$	Y	$M_2(Y)$	$M_8(k)$	$M_{16}(k)$
Type of σ	symplectic		orthogonal	

The type of the canonical involution is determined in [48, Proposition 8.4]. The Clifford algebra is 8-periodic in the sense that $C(q_{n+8}) \approx M_{16}(C(q_n))$. By Morita equivalence, we see that

$$W^0(C_0(q),\sigma) = \begin{cases} W^0(Y,\sigma_s) & \text{if } n \equiv 3,5 \mod 8 \\ W^0(k) & \text{if } n \equiv 1,7 \mod 8 \end{cases}$$

where σ_s is the involution which is the 1 on the center and -1 outside the center.

Theorem 4.4.3. *Let Q be the quadric defined by the quadratic form $n\langle 1\rangle$. Assume -1 is not a sum of two squares in k, e.g. $\mathbb{R}$. Then,*

$$W^0(Q) \approx W^0(k)/2^{\delta(n)}W^0(k)$$

where $\delta(n)$ is the cardinality of the set $\{l \in \mathbb{Z} : 0 < l < n, l \equiv 0, 1, 2 \text{ or } 4 \mod 8\}$.

Proof. It is enough to show $\operatorname{Im}(F, \operatorname{tr}) \approx \ker(p^*) \approx 2^{\delta(n)}W^0(k)$. Recall that the map (F, tr) takes a symmetric space $M \times M \to C_0(q)$ in $W^0(C_0(q), \sigma)$ to the symmetric space $M \times M \to C_0(q) \xrightarrow{\operatorname{tr}} k$ in $W^0(k)$ by forgetting the left $C_0(q)$ structure. Since there is a surjective group homomorphism $\mathbb{Z}[k^\times/k^{2\times}] \to W^0(C_0(q), \sigma)$ (via the Morita equivalence), it is enough to investigate where a symmetric space represented by the irreducible $C_0(q)$-module goes under the map (F, tr). Note that in this case $C_0(q)$ is simple. Let $\Sigma(q)$ be the unique (up to isomorphism) irreducible left $C_0(q)$-module. By Table 1, $\Sigma(q)$ may be understood as a left ideal of $C_0(q)$. As a k-vector space, $\Sigma(q)$ is of rank $2^{\delta(n)}$. Consider a space represented by the form $b : \Sigma(q) \times \Sigma(q) \hookrightarrow C_0(q) \times C_0(q) \xrightarrow{\beta} C_0(q)$. Note that $\beta(e, e) = e\bar{e} = 1$ for any e in the set of basis $\{e^\Delta : \Delta \in \mathbb{F}_2^n\}$, and that $\operatorname{tr}(e^\Delta) = 0$ if $\Delta \neq 0$. By a proper choice of k-vector space basis of $\Sigma(q)$ with respect to the k-vector space basis of $C_0(q)$, we see (F, tr) sends b to the diagonal form $2^{\delta(n)}\langle 1\rangle$ in $W^0(k)$. $\square$

4.5 An exact sequence

Let (P, q) be an n-dimensional non-degenerate quadratic form over k. Recall that Q_d is the variety $\operatorname{Proj}(S(P^*)/q)$ of dimension $d = n-2$. Consider the following maps of exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker(l) \arrow[r, "\cong"] \arrow[d] & \ker(v) \arrow[r] \arrow[d] & 0 \\
Cl_{d+1}(\Lambda) \arrow[r] \arrow[d, "="] & Cl_d(\Lambda) \arrow[r] \arrow[d, "="] & Cl_{d-1}(\Lambda) \arrow[r] \arrow[d, "l"] & \mathscr{U}_{d-2}(\Lambda) \arrow[r] \arrow[d, dashed, "\exists v"] & 0 \\
L_{d+1} \arrow[r] & L_d \arrow[r, "\varphi_d"] & L_{d-1} \arrow[r] & \operatorname{coker}(\varphi_d) \arrow[r] & 0
\end{tikzcd} \tag{4.5}$$

where the middle row is the exact sequence (4.3) with $Cl_i(\Lambda) := \mathscr{O}(-i) \otimes \Lambda^{(s_i)}$, the bottom row is the Tate resolution, and where the map l is the projection by taking the component $\mathscr{O}(1-d) \otimes \Lambda^{d+2}(P^*)$ to 0. By the universal property of cokernels, we observe that there is a surjective map $v : \mathscr{U}_{d-2}(\Lambda) \to \mathrm{coker}(\varphi_d)$ such that the lower-third square in the diagram (4.5) is commutative. By taking kernels of maps between the bottom and the middle exact sequences of the diagram (4.5), we obtain the exact sequence in the top row. In particular, we see that

$$\mathscr{O}(1-d) \otimes \det(P^*) \cong \ker(l) \cong \ker(v).$$

Let s denote the composition

$$\mathscr{O}(1-d) \otimes \det(P^*) \to \ker(l) \to \ker(v) \to \mathscr{U}_{d-2}(\Lambda).$$

Proposition 4.5.1. *There is an exact sequence*

$$0 \to \mathscr{O}(1-d) \otimes \det(P^*) \xrightarrow{s} \mathscr{U}_{d-2}(\Lambda) \to L_{-d+2} \to \cdots \to L_1 \to L_0 \to \mathscr{O}(1) \to 0$$

where the part

$$L_{-d+2} \to \cdots \to L_1 \to L_0 \to \mathscr{O}(1) \to 0$$

is the Tate resolution truncated from L_{-d+2}.

Corollary 4.5.1. *Let $d = 2m$ be an even number. Tensoring the exact sequence of Proposition 4.5.1 with the line bundle $\mathscr{O}(m-1)$, we obtain another exact sequence*

$$0 \to \mathscr{O}(-m) \otimes \det(P^*) \xrightarrow{s} \mathscr{U}_{d-2}(\Lambda)(m-1) \to L_{-d+2}(m-1) \to \cdots \to L_0(m-1) \to \mathscr{O}(m) \to 0. \tag{4.6}$$

This exact sequence is of length $n = d+2$. We denote it by $T_{[-m,m]}$.

4.6 Grothendieck-Witt spectra of Q_{even}

In this section, we let Q_d be a smooth quadric hypersurface of dimension $d = 2m$ corresponding to a non-degenerate quadratic form (P, q) of rank $n = d+2$. Recall from Corollary 4.1.2 the semi-decomposition

$$\mathscr{D}^b Q_d = \langle \mathcal{A}_{[1-m,-1]}, \mathcal{A}, \mathcal{A}_{[1,m-1]}, \mathcal{A}_m \rangle.$$

Define $\mathcal{A}' := \langle \mathcal{A}_{[1-m,-1]}, \mathcal{A}, \mathcal{A}_{[1,m-1]} \rangle$. Note that $\mathcal{A}'$ is fixed by the duality $\vee := \mathscr{H}om(-, \mathscr{O})$.

Recall that the quadruple

$$(\mathrm{sPerf}(Q_d), \mathrm{quis}, \#_{\mathscr{O}}, \mathrm{can}^{\mathscr{O}})$$

is a dg category with weak equivalences and duality. Let $\mathscr{C}'$ be the full dg subcategory of $\mathrm{sPerf}(Q_d)$ associated to the triangulated category $\mathcal{A}'$. Let v be the set of morphisms in $\mathrm{sPerf}(Q_d)$ which become isomorphisms in the Verdier quotient $\mathcal{T}(\mathrm{sPerf}(Q_d), \mathrm{quis})/\mathcal{T}(\mathscr{C}', \mathrm{quis})$. Note that $\mathscr{C}'$ is fixed by the duality $\#_{\mathscr{O}} = [-, \mathscr{O}]$.

Note that the sequence of dg categories with weak equivalences and duality

$$(\mathscr{C}', \mathrm{quis}) \longrightarrow (\mathrm{sPerf}(Q_d), \mathrm{quis}) \longrightarrow (\mathrm{sPerf}(Q_d), v)$$

is quasi-exact (hence Morita exact). So, it induces localization sequences of GW-spectra

$$GW^{[i]}(\mathscr{C}') \longrightarrow GW^{[i]}(Q_d) \longrightarrow GW^{[i]}(Q_d, v)$$

and localization sequences of $\mathbb{G}W$-spectra

$$\mathbb{G}W^{[i]}(\mathscr{C}') \longrightarrow \mathbb{G}W^{[i]}(Q_d) \longrightarrow \mathbb{G}W^{[i]}(Q_d, v).$$

Observe that $GW^{[i]}(\mathscr{C}')$ and $\mathbb{G}W^{[i]}(\mathscr{C}')$ may be manipulated similarly as in the case of odd dimensional quadrics. Thus, we conclude

Theorem 4.6.1. *There is a stable equivalence of spectra*

$$(H\mathscr{O}(1)\otimes ?, \cdots, H\mathscr{O}(m)\otimes ?, ?) : \bigoplus_{i=1}^{m} K(k) \oplus GW^{[i]}(\mathscr{A}) \xrightarrow{\sim} GW^{[i]}(\mathscr{C}')$$

where $GW^{[i]}(\mathscr{A})$ fits into another homotopy fibration sequence

$$GW^{[i]}(k) \longrightarrow GW^{[i]}(\mathscr{A}) \longrightarrow GW^{[i+1]}(C_0(q), \sigma).$$

Moreover, a similar result holds for $\mathbb{G}W$.

It turns out we really need to study $GW^{[i]}(Q_d, v)$ and $\mathbb{G}W^{[i]}(Q_d, v)$.

Consider the exact sequence $T_{[-m,m]}$ in Corollary 4.5.1, and delete the component $\mathscr{O}(-m) \otimes \det(P^*)$ in $T_{[-m,m]}$. We obtain a complex concentrated in degree $[-d, 0]$

$$0 \to \mathscr{U}_{d-2}(\Lambda)(m-1) \to L_{-d+2}(m-1) \to \cdots \to L_0(m-1) \to \mathscr{O}(m) \to 0 \qquad (4.7)$$

Denote this new complex by $T_{[1-m,m]}$. Note that

$$L_{-d+2}(m-1) = \mathscr{O}(1-m) \otimes \big(\bigoplus_{j\geq 0} \Lambda^{(-d+2)+1-2j}(P^*)\big).$$

That is why I write $1-m$ in the brackets of $T_{[1-m,m]}$.

Lemma 4.6.1. *Assume P is free. Then, there exists a symmetric space $(\mathscr{O}(m), \psi)$ in the category with duality*

$$(\mathcal{T}(\mathrm{sPerf}(Q), v), \#_{\mathscr{O}[d]}, \epsilon \cdot \mathrm{can})$$

where $\epsilon = \pm 1$ and where the form ψ is represented by the following right roof

$$\begin{tikzcd} & T_{[1-m,m]} & \\ \psi: \quad \mathscr{O}(m) \arrow[ur, "\iota"] \arrow[rr, dashed] & & {[\mathscr{O}(m), \mathscr{O}[d]]_{\mathscr{O}}} \arrow[ul, "s"'] \end{tikzcd}$$

The only non-trivial morphism of $\iota : \mathscr{O}(m) \to T_{[1-m,m]}$ is the map $\mathrm{id} : \mathscr{O}(m) \to \mathscr{O}(m)$ *in the degree* 0. *The only non-trivial morphism of*

$$s : [\mathscr{O}(m), \mathscr{O}[d]] \to T_{[1-m,m]}$$

is the composition

$$\mathscr{O}(m)^\vee \xrightarrow{\sim} \mathscr{O}(-m) \xrightarrow{\sim} \mathscr{O}(-m) \otimes \det(P^*) \xrightarrow{v} \mathscr{U}_{d-2}(\Lambda)$$

in the degree $-d$.

Proof. It is enough to show that ι and s are both weak equivalences and that ψ is symmetric. Indeed, s is a quasi-isomorphism, hence also a weak equivalence. Furthermore, the cone of the morphism ι in $\mathscr{D}^b Q$ is already in the triangulated category $\mathcal{A}_{[1-m,m-1]}$. Thus, ι is a weak equivalence and we conclude ψ is an isomorphism in

$$\mathrm{Hom}_{\mathcal{T}(\mathrm{sPerf}(Q),v)}(\mathscr{O}(m), \mathscr{O}(m)^\vee[d]).$$

Define $\psi^t := \psi^\# \circ \mathrm{can}_{\mathscr{O}(m)}$. Note that $\psi^{tt} = \psi$. Next, we show that $\psi = \epsilon\psi^t$ for some sign ϵ. In fact, the morphism ψ can also induce an isomorphism

$$\mathrm{Hom}_{\mathcal{T}(\mathrm{sPerf}(Q),v)}(\mathscr{O}(m), \mathscr{O}(m)^\vee[d]) \approx \mathrm{Hom}_{\mathcal{T}(\mathrm{sPerf}(Q),v)}(\mathscr{O}(m), \mathscr{O}(m))$$

The right-hand side is isomorphic to

$$\mathrm{Hom}_{\mathscr{D}^b Q}(\mathscr{O}(m), \mathscr{O}(m)) \approx k,$$

since $\mathcal{T}(\mathrm{sPerf}(Q), v)$ is just the Verdier quotient $\mathscr{D}^b Q/\mathcal{A}'$. Thus, we conclude $\psi^t = \epsilon\psi$ for some $\epsilon \in k^\times$. Observe that $\psi = \psi^{tt} = (\epsilon\psi)^t = \epsilon^2\psi$. This suggests $\epsilon^2 = 1$ (so $\epsilon = \pm 1$), because ψ is an isomorphism. $\square$

The sign ϵ will be determined later. Consider the dg form functor

$$(\mathscr{O}\text{-dgMod-}k, \#_{\mathscr{O}[d]}) \otimes (k\text{-dgMod}, \#_k) \to (\mathscr{O}\text{-dgMod}, \#_{\mathscr{O}[d]})$$

with the duality compatibility map

$$\gamma : [M, \mathscr{O}[d]]_{\mathscr{O}} \otimes [N, k]_k \to [M \otimes N, \mathscr{O}[d]]_{\mathscr{O}}$$

defined by $\gamma(f \otimes g)(m \otimes n) = (-1)^{|g||m|} f(m)g(n)$. The proof of [73, Lemma 3.9] tells us that the form $(\mathscr{O}(m), \psi)$ (in Lemma 4.6.1) can be lifted to a symmetric form $(B_{\mathscr{O}(m)}, B_\psi)$ in the dg category

$$(\mathrm{sPerf}(Q_d), v, \#_{\mathscr{O}[d]}, \epsilon \cdot \mathrm{can})$$

with weak equivalences and duality, such that the morphism B_ψ is in v and that $(B_{\mathscr{O}(m)}, B_\psi)$ is isometric to $(\mathscr{O}(m), \psi)$ in $(\mathcal{T}(\mathrm{sPerf}(Q), v), \#_{\mathscr{O}[d]}, \epsilon \cdot \mathrm{can})$.

Lemma 4.6.2. *There is a dg form functor*

$$(B_{\mathscr{O}(m)}, B_\psi) \otimes ? : (k\text{-dgMod}, \#_k) \longrightarrow (\mathscr{O}\text{-dgMod}, \#_{\mathscr{O}[d]}, \epsilon \cdot \mathrm{can}^{\mathscr{O}})$$

for some sign ϵ.

Proof. This is a consequence of Lemma 3.2.4. □

Restricting ourselves to the full dg subcategory $\mathrm{sPerf}(k) \subset k\text{-dgMod}$ and taking the set v of weak equivalences into account, we obtain

Lemma 4.6.3. *There is an exact dg form functor*

$$(B_{\mathscr{O}(m)}, B_\psi) \otimes ? : (\mathrm{sPerf}(k), \mathrm{quis}, \#_k, \mathrm{can}) \longrightarrow (\mathrm{sPerf}(Q_d), v, \#_{\mathscr{O}[d]}, \epsilon \cdot \mathrm{can})$$

of dg categories with weak equivalences and duality for $\epsilon = \pm 1$.

It is well-known that the composition of functors

$$i : \mathcal{A}_m \longrightarrow \langle \mathcal{A}', \mathcal{A}_m \rangle \longrightarrow \langle \mathcal{A}', \mathcal{A}_m \rangle / \mathcal{A}' \approx \mathcal{T}(\mathrm{sPerf}(Q), v)$$

is an equivalence, where the first functor is the inclusion and the second one is the quotient. Furthermore, this equivalence induces an equivalence of categories

$$\mathscr{D}^b k \longrightarrow \mathcal{T}(\mathrm{sPerf}(Q), v), E \mapsto \mathscr{O}(m) \otimes E.$$

Thus, the functor $B_{\mathscr{O}(m)} \otimes ? : \mathrm{sPerf}(k) \to \mathrm{sPerf}(Q)$ induces an equivalence of associated triangulated categories. By invariance of GW and $\mathbb{G}W$, we get

Lemma 4.6.4. *There is an equivalence*

$$(B_{\mathscr{O}(m)}, B_\psi)\otimes ? : GW^{[i]}(k) \xrightarrow{\sim} {}_\epsilon GW^{[i+d]}(Q,v)$$

of spectra for $\epsilon = \pm 1$, *where* ${}_\epsilon GW^{[i+d]}(Q,v) := GW^{[i+d]}(\mathrm{sPerf}(Q_d), v, \#_{\mathscr{O}[d]}, \epsilon\cdot \mathrm{can})$. *A similar result holds for the case of* $\mathbb{G}W$.

Recall $d = 2m$. The sign ϵ is determined by the following:

Lemma 4.6.5. *There is an equivalence*

$$(B_{\mathscr{O}(m)}, B_\psi)\otimes ? : GW^{[i]}(k) \xrightarrow{\sim} {}_\epsilon GW^{[i+d]}(Q_d,v)$$

of spectra where $\epsilon = (-1)^{\frac{d(d-1)}{2}}$. *A similar result holds for the case of* $\mathbb{G}W$*-spectra.*

Proof. Assume first that k is a field. If m is odd, we argue $\epsilon = 1$ is impossible, hence ϵ must equal -1. Note that by our notation ${}_1GW^{[i]}(Q,v) = GW^{[i]}(Q,v)$. Then, there is a homotopy fibration

$$GW^{[2]}(\mathscr{C}') \longrightarrow GW^{[2]}(Q) \longrightarrow {}_1GW^{[2]}(Q,v)$$

of spectra by the localization. If $d = 1$, we have ${}_1GW^{[2]}(Q,v) \approx GW^{[2]}(k)$. Taking the negative homotopy group π_{-4} to the fibration, one has an exact sequence

$$W^2(\mathcal{A}') \longrightarrow W^2(Q) \longrightarrow W^2(k)$$

By the base change $k \to \bar{k}$, we get an exact sequence

$$W^2(\mathcal{A}'_{\bar{k}}) \longrightarrow W^2(Q_{\bar{k}}) \longrightarrow W^2(\bar{k}).$$

Note that $W^2(\mathcal{A}'_{\bar{k}}) = 0$ by a similar argument to the proof of Theorem 1.2.3. Besides, $W^2(\bar{k}) = 0$ is well-known. This implies $W^2(Q_{\bar{k}}) = 0$. This is a contradiction because $W^2(Q_{\bar{k}}) \approx W^2(Q_{\mathbb{C}})$ by [86] and $W^2(Q_{\mathbb{C}}) \approx \mathbb{Z}/2\mathbb{Z}$ by [93].

If m is even, we prove that $\epsilon = -1$ is impossible, hence ϵ must equal 1. Under our notation, we have ${}_{-1}GW^{[i+2]}(Q,v) = GW^{[i]}(Q,v)$. Applying the localization theorem, we deduce a homotopy fibration sequence

$$GW^{[0]}(\mathscr{C}') \longrightarrow GW^{[0]}(Q) \longrightarrow {}_{-1}GW^{[2]}(Q,v)$$

Assume $\epsilon = -1$. By Lemma 4.6.4 and the 4-periodicity, we have ${}_{-1}GW^{[2]}(Q,v) \approx GW^{[2]}(k)$. Similar to the procedure above, we get an exact sequence

$$W^0(\mathcal{A}'_{\bar{k}}) \longrightarrow W^0(Q_{\bar{k}}) \longrightarrow W^2(\bar{k}).$$

By the proof of Theorem 4.4.2, we deduce $W^0(\mathcal{A}'_{\bar{k}}) \approx W^0(\bar{k}) \approx \mathbb{Z}/2\mathbb{Z}$. Note that $W^2(\bar{k}) = 0$ gives a surjective map $W^0(\bar{k}) \to W^0(Q_{\bar{k}})$, so $W^0(Q_{\bar{k}}) = 0$ or $\mathbb{Z}/2\mathbb{Z}$. This is a contradiction because $W^0(Q_{\bar{k}}) \approx W^0(Q_{\mathbb{C}}) = (\mathbb{Z}/2\mathbb{Z})^2$ by [60] or [93].

For the general case (k is any commutative ring with $\frac{1}{2} \in k$), we consider the base change $k \to k_P$ (with P any prime ideal of k) and take the field of fractions of k_P. □

Note that by our sign convention, ${}_{\epsilon}GW^{[i]}(Q)$ is as the same as $GW^{[i]}(Q)$ if $\epsilon = (-1)^{\frac{d(d-1)}{2}}$ and $d = 2m$ is even.

Theorem 4.6.2. *Let $d = 2m$. There exists a homotopy fibration sequence of spectra*

$$\bigoplus_{j=1}^{m-1} K(k) \oplus GW^{[i]}(\mathscr{A}) \longrightarrow GW^{[i]}(Q_d) \longrightarrow GW^{[i-d]}(k)$$

where $GW^{[i]}(\mathscr{A})$ fits into another homotopy fibration sequence

$$GW^{[i]}(k) \longrightarrow GW^{[i]}(\mathscr{A}) \longrightarrow GW^{[i+1]}(C_0(q), \sigma).$$

Theorem 4.6.3. *Let $d = 2m$. There exists a homotopy fibration sequence of spectra*

$$\bigoplus_{j=1}^{m-1} \mathbb{K}(k) \oplus \mathbb{G}W^{[i]}(\mathscr{A}) \longrightarrow \mathbb{G}W^{[i]}(Q_d) \longrightarrow \mathbb{G}W^{[i-d]}(k)$$

where $\mathbb{G}W^{[i]}(\mathscr{A})$ fits into another homotopy fibration sequence

$$\mathbb{G}W^{[i]}(k) \longrightarrow \mathbb{G}W^{[i]}(\mathscr{A}) \longrightarrow \mathbb{G}W^{[i+1]}(C_0(q), \sigma).$$

4.7 Application: Witt groups of Q_{even}

Let $d = 2m$ be an even integer. In this section, we continue our study of Balmer's Witt groups $W^i(Q_d)$ in Section 4.4 by considering $d = 2m$.

Lemma 4.7.1. *Let k be a regular ring with $\frac{1}{2}$. Let $d = 2m$. Then, there is an exact sequence*

$$\cdots \longrightarrow W^{[i]}(\mathscr{A}) \longrightarrow W^{[i]}(Q) \longrightarrow W^{[i-d]}(k) \longrightarrow W^{[i+1]}(\mathscr{A}) \longrightarrow \cdots$$

Proof. Taking the negative homotopy groups of the fibration sequence

$$\bigoplus_{j=1}^{m-1} K(k) \oplus GW^{[i]}(\mathscr{A}) \longrightarrow GW^{[i]}(Q_d) \longrightarrow GW^{[i-d]}(k)\ ,$$

we see the results. □

Having this lemma in hand, we are able to prove

Theorem 4.7.1. *Let k be a regular local ring with $\frac{1}{2} \in k$. If d is even, then there is a 12-term long exact sequence*

$$\begin{tikzcd}
& \operatorname{coker}(F,\mathrm{tr}) \arrow[r] & W^0(Q) \arrow[r] & W^{-d}(k) \arrow[dr] & \\
W^{3-d}(k) \arrow[ur] & & & & W^2(C_0(q),\sigma) \arrow[d] \\
W^3(Q) \arrow[u] & & & & W^1(Q) \arrow[d] \\
\ker(F,\mathrm{tr}) \arrow[u] & & & & W^{1-d}(k) \arrow[dl] \\
& W^{2-d}(k) \arrow[ul] & W^2(Q) \arrow[l] & 0 \arrow[l] &
\end{tikzcd}$$

Proof. By Lemma 4.7.1, we are reduced to study $W^{[i]}(\mathscr{A})$. The computation of $W^{[i]}(\mathscr{A}) \approx W^i(\mathcal{A})$ is given in Section 4.4. □

Let q be the quadratic form $\langle a_1, \dots, a_n \rangle$ with $a_i \in k^\times$.

Theorem 4.7.2. *Let k be a field of characteristic $\neq 2$. Then, we have*

$$\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle W^0(k) \subset \ker(p^* : W^0(k) \to W^0(Q_d))$$

where $\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle$ is the Pfister form. Moreover, if $C_0(q)$ is a division algebra, then

$$W^0(Q_d) \approx W^0(k)/\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle W^0(k)$$

when $d \equiv 2 \mod 4$, and there is an exact sequence

$$0 \longrightarrow W^0(k)/\langle\langle a_1a_2, \dots, a_1a_n \rangle\rangle W^0(k) \longrightarrow W^0(Q_d) \longrightarrow W^0(k)$$

when $d \equiv 0 \mod 4$.

Proof. Apply Lemma 4.7.1. The rest is similar to the proof of Theorem 4.4.2. □

Let Q be the quadric defined by the quadratic form $q_n = n\langle 1 \rangle$ with n even. Let X denote $k(\sqrt{-1})$, and let Y denote the quaternion algebra $(\frac{-1,-1}{k})$.

Assume -1 is not a sum of two squares. Then, X is a field and Y is a division algebra. We copy parts of the table on [56, p. 125] for this case.

Table 2	n=2	n=4	n=6	n= 8
$C_0(q_n)$	X	$Y \times Y$	$M_4(X)$	$M_8(k) \times M_8(k)$
Type of σ	unitary	symplectic	unitary	orthogonal

The type of the canonical involution is determined in [48, Proposition 8.4]. The Clifford algebra is 8-periodic in the sense that $C(q_{n+8}) \approx M_{16}(C(q_n))$. By Morita equivalence, we see that

$$W^0(C_0(q), \sigma) = \begin{cases} W^0(X, \sigma_u) & \text{if } n \equiv 2, 6 \mod 8 \\ W^0(Y, \sigma_s) \oplus W^0(Y, \sigma_s) & \text{if } n \equiv 4 \mod 8 \\ W^0(k) \oplus W^0(k) & \text{if } n \equiv 8 \mod 8 \end{cases}$$

where σ_u is the unitary involution and σ_s is the involution which is identity on the center and -1 outside the center.

Theorem 4.7.3. *Assume -1 is not a sum of two squares in k. If $d \equiv 2 \mod 4$, then $W^0(Q_d) \approx W^0(k)/2^{\delta(n)}W^0(k)$ and if $d \equiv 0 \mod 4$, then there is an exact sequence*

$$0 \longrightarrow W^0(k)/2^{\delta(n)}W^0(k) \longrightarrow W^0(Q_d) \longrightarrow W^0(k) .$$

Moreover, if k is Euclidean, then $W^0(Q_d) \approx \mathbb{Z}/2^{\delta(n)}\mathbb{Z}$.

Proof. A proof similar to the proof of Theorem 4.4.3 can be applied to obtain the first part.

If k is Euclidean, then $W^0(k) \approx \mathbb{Z}$. By [21, Proposition 3.1], we conclude $W^0(Q_d)$ is a 2-primary torsion group. Thus, if d is even, $W^0(Q_d)$ is isomorphic to $\mathbb{Z}/2^{\delta(n)}\mathbb{Z}$ because $\mathrm{Hom}(\mathbb{Z}/2^c\mathbb{Z}, \mathbb{Z}) = 0$ for any integer $c > 0$. □

Chapter 5

Sums-of-squares formulas

A sums-of-squares formula of type $[r, s, n]$ over a field F of characteristic $\neq 2$ (with strictly positive integers r, s and n) is a formula

$$\Big(\sum_{i=1}^{r} x_i^2\Big) \cdot \Big(\sum_{i=1}^{s} y_i^2\Big) = \Big(\sum_{i=1}^{n} z_i^2\Big) \in F[x_1, \dots, x_r, y_1, \dots, y_s] \tag{5.1}$$

where $z_i = z_i(X, Y)$ for each $i \in \{1, \dots, n\}$ is a bilinear form in X and Y (with coefficients in F), i.e. $z_i \in F[x_1, \dots, x_r, y_1, \dots, y_s]$ is homogeneous of degree 2 and F-linear in X and Y. Here, $X = (x_1, \dots, x_r)$ and $Y = (y_1, \dots, y_s)$ are coordinate systems. To be specific, $z_i = \sum_{k,j} c_{kj}^{(i)} x_k y_j$ for $c_{kj}^{(i)} \in F$.

An old problem of Adolf Hurwitz concerns the existence of sums-of-squares formulas. This problem is valuable because it is related to several other branches of mathematics. Let m be a strictly positive integer, and let $\delta(m)$ denote the cardinality of the set $\{l \in \mathbb{Z} : 0 < l < m \text{ and } l \equiv 0, 1, 2 \text{ or } 4 \pmod 8\}$. The aim of this chapter is to prove the following result.

Theorem 5.0.4 (Theorem 1.1 [86]). *If a sums-of-squares formula of type $[r, s, n]$ exists over a field F of characteristic $\neq 2$, then $2^{\delta(s)-i+1}$ divides $\binom{n}{i}$ for all $i \in \mathbb{Z}$ such that $n - r < i \leq \delta(s)$.*

Corollary 5.0.1 (Hurwitz-Radon Theorem [39] and [67]). *If a sums-of-squares formula of type $[r, n, n]$ exists over F, then $r \leq \rho(n)$.*

Proof. Putting the value r, n, n in the numerical condition of Theorem 5.0.4, we obtain $2^{\delta(r)}$ divides $\binom{n}{1} = n$. It follows that $r \leq \rho(n)$ by [78, Exercise 6, Chapter 0]. □

Remark 5.0.1. This result implies one direction of the Hurwitz-Radon Theorem. There is a mistake in the published version [86]. The triple $[3,5,5]$ does not satisfy the condition of Theorem 5.0.4 (because $2^{\delta(3)}$ does not divide $\binom{5}{1}$).

Another consequence of Theorem 5.0.4 is the following result, which has been proved in [2] and [3] (see also [78, Theorem 14.10]).

Corollary 5.0.2 (Adem's Theorem)**.** *Suppose a sums-of-squares formula of type $[r, n-1, n]$ exists over F.*

- *If n is even, then $r \leq \rho(n)$;*
- *If n is odd, then $r \leq \rho(n-1)$.*

Proof. Putting the value $r, n-1, n$ in the numerical condition of Theorem 5.0.4, we obtain $2^{\delta(r)}$ divides $\binom{n}{2} = n(n-1)$. If n is even, $2^{\delta(s)}|n$. If n is odd, $2^{\delta(s)}|(n-1)$. The results follow by [78, Exercise 6, Chapter 0]. □

Example 5.0.1. Consider the triplet $[15,10,16]$ which does not exist over F by the above theorem. Neither Hopf's condition [23] nor the weaker condition in [22] can give the non-existence of $[15,10,16]$.

Remark 5.0.2. The algebraic K-theory analog (cf. [22, Theorem 1.1]) of our main theorem works even if the assumption 'if a sums-of-squares formula of type $[r,s,n]$ exists over F' is replaced by 'if a nonsingular bilinear map of size $[r,s,n]$ exists over F'. The statement with the latter assumption is 'stronger'. However, this is not the case under our proof, since we will use the sums-of-squares formula (5.1).

Remark 5.0.3. The triplet $[r,s,n]$ is independent of the base fields whenever $r \leq 4$ and whenever $s \geq n-2$ (cf. [78, Corollary 14.21]), so that the main theorem is true. There is a bold conjecture which states that the existence of $[r,s,n]$ is independent of the base field F (of characteristic $\neq 2$), cf. [76, Conjecture 3.8] or [78, Conjecture 14.22]. Our main theorem and Dugger-Isaksen's Hopf condition (cf. [23]) suggest this conjecture to some extent.

In [93], it was shown that the Grothendieck-Witt group of a complex cellular variety is isomorphic to the KO-theory of its set of $\mathbb{C}$-rational points with analytic topology. The set of $\mathbb{C}$-rational points of a deleted quadric is homotopy equivalent to the real projective space of the same dimension, cf. [57, Lemma 6.3]. Moreover, the computation of topological KO-theory of a real projective space is well-known, cf. [1, Theorem 7.4]. We therefore have motivations to work on the Grothendieck-Witt group of a deleted quadric and on the γ^i-operations. The proof of our main theorem requires the computation of Grothendieck-Witt group of a deleted quadric which will be explored in Section 5.2.

5.1 Terminologies, notations and remarks

Let $(\mathcal{E}, *, \eta)$ be a $\mathbb{Z}[\frac{1}{2}]$-linear exact category with duality. For $i \in \mathbb{Z}$, Walter's Grothendieck-Witt groups $GW^i(\mathcal{E}, *, \eta)$ are defined in [70, Section 4.3]. The triplet $(\mathrm{Vect}(X), \mathscr{H}om(\ ,\mathcal{L}), \mathrm{can})$ (the notation in [70, Example 2.3]) is an exact category with duality. If X is any $\mathbb{Z}[\frac{1}{2}]$-scheme, then we define

$$GW^i(X, \mathcal{L}) := GW^i(\mathrm{Vect}(X), \mathscr{H}om(\ ,\mathcal{L}), \mathrm{can}).$$

By the symbols $GW^i(X)$, we mean the groups $GW^i(X, \mathcal{O})$. Note that $GW^0(X)$ is just Knebusch's $L(X)$ which is defined in [46]. The notation in [7] is used for the Witt theory. For KO-theory and comparison maps, we refer to [93].

Definition 5.1.1. Let T be a scheme. For us, a smooth T-variety X is called T-*cellular* if it has a filtration by closed subvarieties

$$X = Z_0 \supset Z_1 \supset \cdots \supset Z_N = \varnothing$$

such that $Z_{k-1} - Z_k \cong \mathbb{A}_T^{n_k}$ for each k.

In this chapter, the following notations are introduced for convenience:

F	—	a field of characteristic $\neq 2$;
K	—	an algebraically closed field of characteristic $\neq 2$;
V	—	the ring of Witt vectors over K;
L	—	the field of fractions of V;
X_F	—	the base-change scheme $X \times_{\mathbb{Z}[\frac{1}{2}]} F$ for any $\mathbb{Z}[\frac{1}{2}]$-scheme X;
S	—	the polynomial ring $F[y_1, \ldots, y_s]$;
$\mathbb{P}^{s-1}$	—	the scheme Proj $\mathbb{Z}[\frac{1}{2}][y_1, \ldots, y_s]$;
q_s	—	the quadratic polynomial $q_s(y) = y_1^2 + \ldots + y_s^2$;
$V_+(q_s)$	—	the closed subscheme of $\mathbb{P}^{s-1}$ defined by q_s;
$D_+(q_s)$	—	the open subscheme $\mathbb{P}^{s-1} - V_+(q_s)$ of $\mathbb{P}^{s-1}$;
ξ	—	the line bundle $\mathcal{O}(-1)$ of $\mathbb{P}_F^{s-1}$ restricted to $D_+(q_s)_F$;
R	—	the ring of elements of total degrees 0 in S_{q_s};
P	—	the R-module of elements of total degrees -1 in S_{q_s};
Q_n	—	the $\mathbb{Z}[\frac{1}{2}]$-scheme defined by $\sum_{i=0}^{n/2} x_i y_i = 0$ in $\mathbb{P}^{n+1}$, if $n > 0$ is even; $\sum_{i=0}^{(n-1)/2} x_i y_i + c^2 = 0$ in $\mathbb{P}^{n+1}$, if $n > 0$ is odd;
DQ_{n+1}	—	the open subscheme $\mathbb{P}^{n+1} - Q_n$ of $\mathbb{P}^{n+1}$.

Remark 5.1.1. (i) Let E be a field containing $\sqrt{-1}$ and of characteristic $\neq 2$. Note that $(Q_{s-2})_E$ is isomorphic to the projective variety $V_+(q_s)_E$, cf. [23, Lemma 2.2]. This map induces an isomorphism $i_E : (DQ_{s-1})_E \to D_+(q_s)_E$.
(ii) Observe that V is a complete DVR with the quotient field K, cf. [75, Chapter II]. Also, note that the fraction field L of V has characteristic 0, cf. *loc. cit.*.
(iii) The scheme $D_+(q_s)_F$ is affine over the base field F, since $D_+(q_s)_F$ and Spec R are isomorphic, cf. [22, Proof of Proposition 2.2].

5.2 Proof of Theorem 5.0.4

Lemma 5.2.1. *If a sums-of-squares formula of type $[r, s, n]$ exists over F, then there exist a non-degenerate bilinear form $\sigma : \xi \times \xi \to \mathcal{O}$ on $D_+(q_s)_F$ and a bilinear space ζ on $D_+(q_s)_F$ of rank $n - r$ such that*

$$r[\xi, \sigma] + [\zeta] = n \in GW^0(D_+(q_s)_F)$$

where n is the trivial bilinear space of the rank n.

Proof. The K-theory analog has been proved, cf. [22, Proposition 2.2]. It is clear that the group $GW^0(D_+(q_s)_F)$ is isomorphic to $GW_0(R)$ by Remark 5.1.1 (iii). If the equation (5.1) exits, we are able to construct a graded S-module homomorphism $(S(-1))^r \to S^n$ by $f = (f_1, \dots, f_r) \mapsto (z_1(f, Y), \dots, z_n(f, Y))$ where $Y = (y_1, \dots, y_s)$ is the coordinate system. This map induces a homomorphism $\alpha : P^r \to R^n$ of R-modules by localizing it at q_s.

The isomorphism $P \otimes_R P \to R, f \otimes g \mapsto (fg) \cdot q_s$ gives a non-degenerate bilinear form $\sigma : P \times P \to R$. Let $\langle -, - \rangle_{R^n}$ be the unit bilinear form over R^n. Let $f = (f_1, \dots, f_r), g = (g_1, \dots, g_r) \in P^r$. We claim that $\langle \alpha(f), \alpha(g) \rangle_{R^n}$ equals $\sum_{i=1}^r \sigma(f_i, g_i)$. It is enough to show that $\langle \alpha(f), \alpha(f) \rangle_{R^n} = \sum_{i=1}^r \sigma(f_i, f_i)$. Note that $\langle \alpha(f), \alpha(f) \rangle_{R^n} = z_1(f, Y)^2 + ... + z_n(f, Y)^2$. By the existence of the triplet $[r, s, n]$, we obtain $z_1(f, Y)^2 + \dots + z_n(f, Y)^2 = (f_1^2 + \dots + f_r^2) q_s = \sum_{i=1}^r \sigma(f_i, f_i)$.

Note that $(P^r, \sum_{i=1}^r \sigma)$ is non-degenerate. It follows that α is injective and $(P^r, \sum_{i=1}^r \sigma)$ can be viewed as a non-degenerate subspace of $(R^n, \langle -, - \rangle_{R^n})$ via α. Define ζ to be its orthogonal complement $(P^r)^\perp$ with the unit form $\langle -, - \rangle_{R_n}$ restricting to $(P^r)^\perp$. By a basic fact of quadratic form theory, ζ is non-degenerate and $\zeta \perp (P^r, \sum_{i=1}^r \sigma) \cong (R^n, \langle -, - \rangle_{R^n})$. □

Theorem 5.2.1. *Let ν denote the element $[\xi,\sigma]-1$ in the ring $GW^0((D_+(q_s)_K)$. Then, the ring $GW^0((D_+(q_s)_K)$ is isomorphic to*

$$\mathbb{Z}[\nu]/(\nu^2+2\nu, 2^{\delta(s)}\nu)$$

where $\delta(s)$ is the number defined in the beginning of this chapter. Therefore, for any rational point $\varsigma : \operatorname{Spec} K \to (D_+(q_s)_K)$, the reduced Grothendieck-Witt ring

$$\widetilde{GW}^0(D_+(q_s)_K) := \ker\left(\varsigma^* : GW^0(D_+(q_s)_K) \to GW^0(\operatorname{Spec} K) \cong \mathbb{Z}\right)$$

is isomorphic to $\mathbb{Z}/2^{\delta(s)}\mathbb{Z}$.

Theorem 5.2.1 will be proved in the next section.

Proof of Theorem 5.0.4. It is enough to show this theorem over the algebraic closure $\bar{F}$ of F. Indeed, if $[r,s,n]$ exists over F, then it also exists over $\bar{F}$.

In order to apply the standard trick (cf. [22, Proof of Theorem 1.3]), we have to take care of γ^i-operations on $GW^0(D_+(q_s)_{\bar{F}})$. To be specific, this standard trick can not be applied without the list of three properties (cf. Properties (i)-(iii) in *loc. cit.*) of γ^i-operations and their generating power series $\gamma_t = 1+\sum_{i>0}\gamma^i t^i$ on $GW^0(D_+(q_s)_{\bar{F}})$. Due to the lack of reference, we will develop γ^i-operations on $K(\mathrm{Bil}(X))$ and prove these three properties (see Section 5.4). It is enough for our purpose because $GW^0(X)$ is just $K(\mathrm{Bil}(X))$ if X is affine (see Remark 5.4.1), and the scheme $D_+(q_s)_{\bar{F}}$ is affine by Remark 5.1.1 (iii).

By Lemma 5.2.1, we have $r\nu+(\zeta-(n-r)) = 0 \in \widetilde{GW}_0(DQ_{s-1})$. Applying γ_t, we obtain that $\gamma_t(\zeta-(n-r)) = \gamma_t(\nu)^{-r} = (1+t\nu)^{-r} = \sum_i \binom{n}{-i}\nu^i t^i = \sum_i(-1)^{i-1}\binom{n}{-i}2^{i-1}\nu t^i = -\sum_i\binom{r+i-1}{i}2^{i-1}\nu t^i$ by Theorem 5.2.1. By the property of γ-operation, one sees that $2^{\delta(s)}$ divides $2^{i-1}\binom{r+i-1}{i}$ for $i>n-r$. If $n-r<\delta(s)$, then $2^{\delta(s)-i+1}$ divides $\binom{r+i-1}{i}$ for $n-r<i\le\delta(s)$. Combining with a reformulation of powers of 2 dividing correspondent binomial coefficients (cf. [22, Section 1.2]), we are done. □

5.3 Proof of Theorem 5.2.1

5.3.1 Rigidity and Hermitian K-theory of cellular varieties

By Remark 5.1.1 (ii), there is always an inclusion map $\overline{\mathbb{Q}} \to \overline{L}$ where $\overline{\mathbb{Q}}$ (resp. $\overline{L}$) is the algebraic closure of $\mathbb{Q}$ (resp. $\overline{L}$). Consider the following diagram (5.2).

$$
\begin{tikzcd}
K & V \arrow[l] \arrow[r] & \overline{L} \\
\mathbb{C} & & \overline{\mathbb{Q}} \arrow[ll] \arrow[u]
\end{tikzcd}
\quad
\begin{tikzcd}
W^i(K) & W^i(V) \arrow[l, "\beta^i", "\cong"'] \arrow[r, "\alpha^i", "\cong"'] & W^i(\overline{L}) \\
& W^i(\mathbb{C}) & W^i(\overline{\mathbb{Q}}) \arrow[l, "\cong", "\eta^i"'] \arrow[u, "\cong", "\chi^i"']
\end{tikzcd}
\tag{5.2}
$$

On the right-hand side of the diagram (5.2), the maps of Witt groups are all induced by the correspondent ring maps of the left-hand side for a fixed $i \in \mathbb{Z}$. All these Witt groups are trivial if $i \not\equiv 0 \pmod 4$, cf. [10, Theorem 5.6]. Note that β^0 is an isomorphism by [45, Satz 3.3]. It is also clear that $W^0(K)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and that all the maps on the right-hand side of the diagram (5.2) preserve multiplicative identities for $i = 0$. Since Witt groups are four periodicity in shifting, we obtain

Lemma 5.3.1. *The map $\eta^i \circ (\chi^i)^{-1} \circ \alpha^i \circ (\beta^i)^{-1}$ yields an isomorphism from $W^i(K)$ to $W^i(\mathbb{C})$. Moreover, by Karoubi induction (cf. [15, Section 3]), the left-hand side of the diagram (5.2) gives an isomorphism $GW^i(K) \to GW^i(\mathbb{C})$ of Grothendieck-Witt groups.* □

Lemma 5.3.2. *Let X be a smooth $\mathbb{Z}[\frac{1}{2}]$-cellular variety. Let $f : A \to B$ be a map of regular local rings of finite Krull dimensions with $1/2$. Suppose that the map $W^i(A) \to W^i(B)$ induced by f is an isomorphism for each i, then f gives an isomorphism of Witt groups (resp. Grothendieck-Witt groups)*

$$W^i(X_A, \mathcal{L}_A) \to W^i(X_B, \mathcal{L}_B) \text{ (resp. } GW^i(X_A, \mathcal{L}_A) \to GW^i(X_B, \mathcal{L}_B))$$

for each i and any line bundle $\mathcal{L}$ over X.

Proof. We may use $W^i(X, \mathcal{L})_*$ to simplify the notation $W^i(X_*, \mathcal{L}_*)$. We wish to prove the Witt theory case by induction on cells. Firstly, note that the pullback maps $W^i(A) \to W^i(\mathbb{A}^n_A)$ and $W^i(B) \to W^i(\mathbb{A}^n_B)$ are isomorphisms by homotopy invariance, cf. [8, Theorem 3.1]. It follows that

$$W^i(\mathbb{A}^n_A) \cong W^i(\mathbb{A}^n_B).$$

Let $X = Z_0 \supset Z_1 \supset \cdots \supset Z_N = \varnothing$ be the filtration such that

$$Z_{k-1} - Z_k \cong \mathbb{A}^{n_k} =: C_k.$$

In general, the closed subvarieties Z_k may not be smooth. However, let U_k be the open subvariety $X - Z_k$ for each $0 \le k \le N$. Every U_k is smooth in X. There is another filtration $X = U_N \supset U_{N-1} \supset \cdots \supset U_0 = \varnothing$ with $U_k - U_{k-1} = Z_{k-1} - Z_k \cong C_k$ closed in

U_k of codimension d_k. Consider the following commutative diagram of localization sequences.

$$\begin{tikzcd} W^{i-1}(U_{k-1})_A \arrow[r] \arrow[d] & W^i_{C_k}(U_k,\mathcal{L})_A \arrow[r] \arrow[d] & W^i(U_k,\mathcal{L})_A \arrow[r] \arrow[d] & W^i(U_{k-1})_A \arrow[r] \arrow[d] & W^{i+1}_{C_k}(U_k,\mathcal{L})_A \arrow[d] \\ W^{i-1}(U_{k-1})_B \arrow[r] & W^i_{C_k}(U_k,\mathcal{L})_B \arrow[r] & W^i(U_k,\mathcal{L})_B \arrow[r] & W^i(U_{k-1})_B \arrow[r] & W^{i+1}_{C_k}(U_k,\mathcal{L})_B \end{tikzcd}$$

Here, $W^i_{C_k}(U_k,\mathcal{L})$ means the $\mathcal{L}$-twisted ith-Witt group of U_k with support on C_k. Note that any line bundle over $(C_k)_A$ is trivial, since

$$\mathrm{Pic}(\mathbb{A}^n_A) \cong \mathrm{Pic}(A) = 0 \text{ (A is regular local and so it is a UFD).}$$

By the dévissage theorem (cf. [32]), $W^i_{C_k}(U_k,\mathcal{L})$ is isomorphic to $W^{i-d_k}(C_k)$. Thus,

$$W^i_{C_k}(U_k,\mathcal{L})_A \cong W^i_{C_k}(U_k,\mathcal{L})_B \text{ for all } i.$$

Moreover, by induction hypothesis,

$$W^i(U_{k-1})_A \cong W^i(U_{k-1})_B \text{ for all } i.$$

Applying the 5-lemma, one sees that the middle vertical map is an isomorphism. Since the K-theory analog of this theorem is also true by induction on cells, the GW-theory cases follow by Karoubi induction, cf. [15, Section 3]. □

Corollary 5.3.1. *The Witt group (resp. the Grothendieck-Witt group)*

$$W^i(X,\mathcal{L})_K \text{ (resp. } GW^i(X,\mathcal{L})_K)$$

is isomorphic to

$$W^i(X,\mathcal{L})_{\mathbb{C}} \text{ (resp. } GW^i(X,\mathcal{L})_{\mathbb{C}})$$

for each i and any line bundle $\mathcal{L}$ over X.

5.3.2 Comparison maps and rank one bilinear spaces

If X is a smooth variety over $\mathbb{C}$, we let $X(\mathbb{C})$ be the set of $\mathbb{C}$-rational points of X with analytic topology. One can define comparison maps (cf. [93, Section 2])

$$\begin{array}{rccl} k^0: & K_0(X) & \to & K^0(X(\mathbb{C})) \\ gw^0: & GW^0(X) & \to & KO^0(X(\mathbb{C})) \\ w^0: & W^0(X) & \to & \frac{KO}{K}^0(X(\mathbb{C})) \end{array} \tag{5.3}$$

where $\frac{KO}{K}^0(X(\mathbb{C}))$ means the cokernel of the realification map from $K^{\mathrm{top}0}(X)$ to $KO^0(X(\mathbb{C}))$. Let $GW^0_{\mathrm{top}}(X(\mathbb{C}))$ be the Grothendieck-Witt group of complex bilinear spaces over $X(\mathbb{C})$. The map gw^0 consists of the composition of the following two maps

$$f : GW^0(X) \to GW^0_{\mathrm{top}}(X(\mathbb{C})) \qquad g : GW^0_{\mathrm{top}}(X(\mathbb{C})) \to KO^0(X(\mathbb{C}))$$

where the map f takes a class $[M, \phi]$ on X to the class $[M(\mathbb{C}), \phi(\mathbb{C})]$ on $X(\mathbb{C})$. The map g sends a class $[N, \epsilon]$ on $X(\mathbb{C})$ to the class represented by the underlying real vector bundle $\mathfrak{R}(N, \epsilon)$ such that $\mathfrak{R}(N, \epsilon) \otimes_{\mathbb{R}} \mathbb{C} = N$ and that $\epsilon|_{\mathfrak{R}(N,\epsilon)}$ is real and positive definite, cf. [93, Lemma 1.3]. Let $Q(X)$ (resp. $Q_{\mathrm{top}}(X)$) denote the group of isometry (resp. isomorphism) classes of rank one bilinear spaces (resp. rank one complex bilinear spaces) over X (resp. $X(\mathbb{C})$) with the group law defined by the tensor product. There are maps of sets

$$Q(X) \to GW^0(X), [\mathcal{L}, \phi] \mapsto [\mathcal{L}, \phi] \qquad Q_{\mathrm{top}}(X(\mathbb{C})) \to GW^0_{\mathrm{top}}(X(\mathbb{C})), [L, \epsilon] \mapsto [L, \epsilon].$$

Let $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$ be the group of isomorphism classes of rank one real vector bundles over $X(\mathbb{C})$.

Lemma 5.3.3. *The following diagram is commutative*

$$\begin{tikzcd} GW^0(X) \arrow[r, "f"] & GW^0_{\mathrm{top}}(X(\mathbb{C})) \arrow[r, "g"] & KO^0(X(\mathbb{C})) \\ Q(X) \arrow[u] \arrow[r, "\tilde{f}"] & Q_{\mathrm{top}}(X(\mathbb{C})) \arrow[u, "u"] \arrow[r, "\tilde{g}"] & \mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C})) \arrow[u, "v"] \end{tikzcd}$$

where $\tilde{f}([\mathcal{L}, \phi])$ (resp. $\tilde{g}([L, \epsilon])$) is defined as $[\mathcal{L}(\mathbb{C}), \phi(\mathbb{C})]$ (resp. $[\mathfrak{R}(L, \epsilon)]$).

Proof. The square on the left-hand side is obviously commutative. It remains to show that the right-hand side square is commutative. Check that the map $\tilde{g}$ is well-defined. Note that, for each couple of complex bilinear spaces (L', ϵ') and (L, ϵ) on $X(\mathbb{C})$, if $\mathfrak{R}(L', \epsilon')$ is isomorphic to $\mathfrak{R}(L, \epsilon)$, then (L', ϵ') is isometric to (L, ϵ). Besides, the map $\tilde{g}$ has image in $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$. To see this, suppose $\tilde{g}([L, \epsilon]) = [\mathfrak{R}(L, \epsilon)]$ is not in $\mathrm{Pic}_{\mathbb{R}}(X(\mathbb{C}))$ for some $[L, \epsilon] \in Q_{\mathrm{top}}(X(\mathbb{C}))$. It follows that $X(\mathbb{C})$ has a point with an open neighborhood U such that $\mathfrak{R}(L, \epsilon)|_U$ is isomorphic to $U \times \mathbb{R}^n$ with $n \neq 1$. Then, $L|_U$ is isomorphic to $U \times \mathbb{C}^n$ ($n \neq 1$), since $\mathfrak{R}(L, \epsilon) \otimes_{\mathbb{R}} \mathbb{C} \cong L$. This contradicts the assumption that the bundle L has rank one. Then, it is clear that $g \circ u = v \circ \tilde{g}$. □

5.3.3 Comparison maps and cellular varieties

Let $\mathcal{H}(\mathbb{C})$ (resp. $\mathcal{SH}(\mathbb{C})$) be the unstable $\mathbb{A}^1$-homotopy category (resp. the stable $\mathbb{A}^1$-homotopy category) over $\mathbb{C}$. Let $\mathcal{H}_\bullet(\mathbb{C})$ be the pointed version of $\mathcal{H}(\mathbb{C})$. There are objects in $\mathcal{H}_\bullet(\mathbb{C})$:

$$\begin{array}{lcl} S^1_s & - & \text{the constant sheaf represented by } \Delta^1/\partial\Delta^1 \text{ pointed canonically;} \\ S^1_t & - & \text{the sheaf represented by } \mathbb{A}^1 - \{0\} \text{ pointed by 1;} \\ T & - & \text{the sheaf represented by the projective line } \mathbb{P}^1 \text{ pointed by } \infty. \end{array}$$

Set $S^{p,q} = (S^1_s)^{\wedge(p-q)} \wedge (S^1_t)^{\wedge q}$ with $p \geq q \geq 0$. Then, $S^{2,1}$ and T are $\mathbb{A}^1$-weakly equivalent. See [59, Section 3.2] for details and [93, Section 1.4] for discussion. One may take these objects to $\mathcal{SH}(\mathbb{C})$. The category $\mathcal{SH}(\mathbb{C})$ is triangulated with translation functor $S^{1,0} \wedge -$. Set $\widetilde{KO}^{p,q}(\mathcal{X}) \coloneqq [\Sigma^\infty \mathcal{X}, S^{p,q} \wedge KO]$ and $KO^{p,q}(X) \coloneqq [\Sigma^\infty X_+, S^{p,q} \wedge KO]$ where $\mathcal{X} \in \mathcal{H}_\bullet(\mathbb{C})$ and $X \in \mathcal{H}(\mathbb{C})$. The object $KO \in \mathcal{SH}(\mathbb{C})$ is the geometric model of Hermitian K-theory in the $\mathbb{A}^1$-homotopy theory defined by Schlichting and Tripathi (see [74] or [93, Section 1.5]). Moreover, there are isomorphisms $GW^q(X) \cong KO^{2q,q}(X)$ and $W^q(X) \cong KO^{2q-1,q-1}(X)$. One defines comparison maps (cf. [93, Section 2])

$$\begin{array}{rcl} \tilde{k}^{p,q}_h(\mathcal{X}): \ \widetilde{KO}^{p,q}(\mathcal{X}) & \to & \widetilde{KO}^p(\mathcal{X}(\mathbb{C})) \\ k^{p,q}_h(X): \ KO^{p,q}(X) & \to & KO^p(X(\mathbb{C})). \end{array}$$

In particular, when X is a complex smooth variety, we have

$$\begin{array}{rcl} gw^q = \ k^{2q,q}_h: \ GW^q(X) & \to & KO^{2q}(X(\mathbb{C})) \\ w^{q+1} = \ k^{2q+1,q}_h: \ W^{q+1}(X) & \to & KO^{2q+1}(X(\mathbb{C})). \end{array}$$

Theorem 5.3.1. *Let X be a complex smooth cellular variety. Assume further that Z is cellular and closed in X, and let $U \coloneqq X - Z$. Then, the map $k^{2q,q}_h(U)$ is an isomorphism and the map $k^{2q+1,q}_h(U)$ is injective.*

Proof. When $Z = \varnothing$, this theorem is a special case of [93, Theorem 2.6]. We slightly modify the proof of [93, Theorem 2.6] to show this theorem by induction on cells. Let $Z = Z_N \supset Z_{N-1} \supset \cdots \supset Z_0 = \varnothing$ be the filtration such that

$$Z_{k+1} - Z_k \cong \mathbb{A}^{n_k} =: C_k.$$

Set $U_k \coloneqq X - Z_k$ for each $0 \leq k \leq N$. Note that there is another filtration $X = U_0 \supset U_1 \supset \cdots \supset U_N = U$ with $U_k - U_{k+1} = Z_{k+1} - Z_k \cong C_k$ closed in U_k. Then, the normal bundle $\mathcal{N}_{U_k/C_k}$ of U_k in C_k is trivial. Hence, $\mathrm{Thom}(\mathcal{N}_{U_k/C_k})$ and $S^{2d,d}$ are $\mathbb{A}^1$-weakly equivalent, where d is the codimension of C_k in U_k, cf. [59, Proposition 2.17]. We can therefore deduce the commutative ladder diagram in [93, Figure 1 (p. 486)]. Assume by induction, the theorem is true for U_k, and we want to prove it for U_{k+1}. It is known that $\tilde{k}^{2q,q}_h(S^{2d,d})$ and $\tilde{k}^{2q+1,q}_h(S^{2d,d})$ are isomorphisms and that $\tilde{k}^{2q+2,q}_h(S^{2d,d})$ is injective, cf. [93, Proof of Theorem 2.6]. The results follow by the 5-lemma. □

5.3.4 Grothendieck-Witt group of a deleted quadric

In this subsection, we simply write $X = D_+(q_s), Q = Q_{s-2}$ and $DQ = DQ_{s-1}$. Note that Q is smooth and closed in $\mathbb{P}^{s-1}$ of codimension 1. The normal bundle $\mathcal{N}$ of Q in $\mathbb{P}^{s-1}$ is isomorphic to $\mathcal{O}_Q(2)$.

Theorem 5.3.2. *The comparison map $gw^q : GW^q(DQ_{\mathbb{C}}) \to KO^{2q}(DQ(\mathbb{C}))$ is an isomorphism for each $q \in \mathbb{Z}$.*

Proof. This theorem is a consequence of Theorem 5.3.1. □

Lemma 5.3.4. *The group $GW^0(DQ_{\mathbb{C}})$ is isomorphic to $GW^0(DQ_K)$.*

Proof. Applying Corollary 5.3.1 and the dévissage theorem, we observe that the vertical maps of W and GW-groups in the following commutative diagram are all isomorphisms

$$\begin{tikzcd}
GW^0_{Q_K}(\mathbb{P}^{s-1}_K) \arrow[r] \arrow[d] & GW^0(\mathbb{P}^{s-1}_K) \arrow[r] \arrow[d] & GW^0(DQ_K) \arrow[r] \arrow[d, "\Omega"'] & W^1_{Q_K}(\mathbb{P}^{s-1}_K) \arrow[r] \arrow[d] & W^1(\mathbb{P}^{s-1}_K) \arrow[d] \\
GW^0_{Q_{\mathbb{C}}}(\mathbb{P}^{s-1}_{\mathbb{C}}) \arrow[r] & GW^0(\mathbb{P}^{s-1}_{\mathbb{C}}) \arrow[r] & GW^0(DQ_{\mathbb{C}}) \arrow[r] & W^1_{Q_{\mathbb{C}}}(\mathbb{P}^{s-1}_{\mathbb{C}}) \arrow[r] & W^1(\mathbb{P}^{s-1}_{\mathbb{C}})
\end{tikzcd}$$

where all vertical maps are induced from the left-hand side of the diagram (5.2) (use the 5-lemma to see the middle map Ω is an isomorphism). □

Recall the isomorphism of varieties $i_K : DQ_K \to X_K$ in Remark 5.1.1 (i). Note that $i_{\mathbb{C}} : DQ_{\mathbb{C}} \to X_{\mathbb{C}}$ gives a homeomorphism $i_{(\mathbb{C})} : DQ(\mathbb{C}) \to X(\mathbb{C})$ by taking $\mathbb{C}$-rational points. Besides, let $\upsilon : \mathbb{R}P^{s-1} \to X(\mathbb{C})$ be the natural embedding. The space $\mathbb{R}P^{s-1}$ is a deformation retract of the space $X(\mathbb{C})$ in the category of real spaces, cf. [57, Lemma 6.3]. These maps that induce isomorphisms in KO-theory or GW-theory are described in the diagram (5.4).

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$$\begin{tikzcd}
 & GW^0(DQ_{\mathbb{C}}) \arrow[r, "gw^0"] & KO^0(DQ(\mathbb{C})) \arrow[d, "i^*_{(\mathbb{C})}"] & \\
GW^0(X_K) \arrow[r, "i^*_K"] & GW^0(DQ_K) \arrow[u, "\Omega"] & KO^0(X(\mathbb{C})) \arrow[r, "\upsilon^*"] & KO^0(\mathbb{R}P^{s-1})
\end{tikzcd} \tag{5.4}$$

Proof of Theorem 5.2.1. Let ξ_{top} denote the tautological line bundle over $\mathbb{R}P^{s-1}$. Recall that there is an isomorphism of rings

$$KO^0(\mathbb{R}P^{s-1}) \cong \mathbb{Z}[\nu_{\text{top}}]/(\nu_{\text{top}}^2 + 2\nu_{\text{top}}, 2^{\delta(s)}\nu_{\text{top}})$$

where ν_{top} represents the class $[\xi_{\mathrm{top}}]-1$, cf. [40, Chapter IV]. Note that $\mathrm{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1})$ is isomorphic to $\mathbb{Z}/2$. Let $\vartheta : GW^0(X_K) \to KO^0(\mathbb{R}P^{s-1})$ be the composition of maps in the diagram (5.4). We have known ϑ is an isomorphism. Therefore, to prove Theorem 5.2.1, we only need to show $\vartheta(\nu) = \nu_{\mathrm{top}}$. To achieve this, we give the following lemma.

Lemma 5.3.5. *The group $Q(X_K)$ (cf. Section 5.3.2) is isomorphic to $\mathbb{Z}/2$.*

Proof. There is an exact sequence (cf. [46, Chapter IV.1])

$$1 \longrightarrow \mathcal{O}(X_K)^*/\mathcal{O}(X_K)^{2*} \longrightarrow Q(X_K) \xrightarrow{F} {}_2\mathrm{Pic}(X_K) \longrightarrow 1$$

where ${}_2\mathrm{Pic}(X_K)$ means the subgroup of elements of order ≤ 2 in $\mathrm{Pic}(X_K)$ and where F is the forgetful map. Note that ${}_2\mathrm{Pic}(X_K) \cong \mathbb{Z}/2$, cf. [77]. In addition, observe that $\mathcal{O}(X_K)^* \cong R^* = K^*$ and that the group K^*/K^{2*} is trivial. It follows that the forgetful map F is an isomorphism. In fact, it sends the non-trivial element $[\xi, \sigma]$ (in Lemma 5.2.1) to the non-trivial element $[\xi]$. $\square$

Proof of Theorem 5.2.1 (Continued). In light of Lemma 5.3.3, there is a map

$$\tilde{\vartheta} : Q(X_K) \to \mathrm{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1})$$

(obtained in an obvious way) such that the following diagram is commutative

$$\begin{tikzcd} GW^0(X_K) \arrow[r, "\vartheta"] & KO^0(\mathbb{R}P^{s-1}) \\ \mathbb{Z}/2 \cong Q(X_K) \arrow[u, "i"] \arrow[r, "\tilde{\vartheta}"] & \mathrm{Pic}_{\mathbb{R}}(\mathbb{R}P^{s-1}) \cong \mathbb{Z}/2. \arrow[u, "j"] \end{tikzcd}$$

The map i is injective (note that $[\xi]$ and 1 are distinct elements in $K_0(X_K)$ by its computation in [22, Proposition 2.4]). The map j is injective by the computation of $KO^0(\mathbb{R}P^{s-1})$. Then, we see that $\tilde{\vartheta}$ is bijective and must send $[\xi, \sigma]$ to $[\xi_{\mathrm{top}}]$. Therefore, $\vartheta([\xi, \sigma]) = [\xi_{\mathrm{top}}]$, so that $\vartheta(\nu) = \nu_{\mathrm{top}}$. $\square$

5.4 Operations on the Grothendieck-Witt group

The γ^i-operations on GW^0 of an affine scheme are analogous to those on the topological KO-theory which have been explained in [5, Section 1 and 2]. For readers' convenience, details have been added.

Let $\mathrm{Bil}(X)$ be the set of isometry classes of bilinear spaces over a scheme X. The orthogonal sum and the tensor product of bilinear spaces over the scheme X make

Bil(X) a semi-ring with a zero and a multiplicative identity. Then, by taking the associated Grothendieck ring $K(\mathrm{Bil}(X))$, we have a homomorphism of the underlying semi-rings

$$\iota : \mathrm{Bil}(X) \to K(\mathrm{Bil}(X))$$

satisfying the universal property (see [46, Chapter I.4] for details).

Remark 5.4.1. For an affine scheme X, the ring $GW^0(X)$ is identified with $K(\mathrm{Bil}(X))$, cf. [46, Chapter I.4 Proposition 1].

Definition 5.4.1 (Chapter IV.3 (p. 235) [46])**.** Let $(\mathcal{F}, \phi)$ be a bilinear space over a scheme X. Let i be a strictly positive integer. The *i-th exterior power of* $(\mathcal{F}, \phi)$, denoted by $\Lambda^i(\mathcal{F}, \phi)$, is the symmetric bilinear space $(\Lambda^i\mathcal{F}, \Lambda^i\phi)$ over X, where $\Lambda^i\mathcal{F}$ is the i-th exterior power of the locally free sheaf $\mathcal{F}$ and where

$$\Lambda^i\phi : \Lambda^i\mathcal{F} \times_X \Lambda^i\mathcal{F} \to \mathcal{O}_X$$

is a morphism of sheaves consisting of a symmetric bilinear form

$$\Lambda^i\phi(U) : \Lambda^i\mathcal{F}(U) \times \Lambda^i\mathcal{F}(U) \to \mathcal{O}_X(U)$$

defined by

$$\Lambda^i\phi(U)(x_1 \wedge \cdots \wedge x_i, y_1 \wedge \cdots \wedge y_i) = \det(\phi(U)(x_i, y_j))$$

for each open subscheme U of X. The exterior power $\Lambda^0(\mathcal{F}, \phi)$ for every bilinear space $(\mathcal{F}, \phi)$ (over X) is defined as $1 = (\mathcal{O}, \mathrm{id})$.

Lemma 5.4.1. *Let $(\mathcal{F}, \phi), (\mathcal{G}, \psi)$ be bilinear spaces over X. Then, we have that*

(a) $\Lambda^1(\mathcal{F}, \phi) = (\mathcal{F}, \phi)$*;*

(b) $\Lambda^k((\mathcal{F}, \phi) \oplus (\mathcal{G}, \psi)) \cong \bigoplus_{r+s=k} \Lambda^r(\mathcal{F}, \phi) \otimes \Lambda^s(\mathcal{G}, \psi)$*;*

(c) *If $(\mathcal{F}, \phi)$ is of constant rank Θ, $\Lambda^i(\mathcal{F}, \phi) = 0$ whenever $i > \Theta$.*

Proof. (a) and (c) are clear. For (b), it is enough to show that the canonical isomorphism of locally free sheaves

$$\varrho : \bigoplus_{r+s=k} \Lambda^r\mathcal{F} \otimes \Lambda^s\mathcal{G} \to \Lambda^k(\mathcal{F} \oplus \mathcal{G})$$

respects the symmetric bilinear forms. This may be checked locally. Let U be an affine open subset of the scheme X. One may choose elements

$$x^{(t)} = x_{1,t} \wedge \cdots \wedge x_{r,t} \in \Lambda^r\mathcal{F}(U) \text{ and } y^{(t)} = y_{1,t} \wedge \cdots \wedge y_{s,t} \in \Lambda^s\mathcal{G}(U)$$

for $t \in \{1,2\}$. Let $a_{i,j} := \phi(U)(x_{i,1}, x_{j,2})$ and $b_{k,l} := \psi(U)(y_{k,1}, y_{l,2})$. We have matrices $A = [a_{i,j}]_{r\times r}$ and $B = [b_{k,l}]_{s\times s}$. On the one hand, we get that

$$\Lambda^r \phi(U) \otimes \Lambda^s \psi(U)(x^{(1)} \otimes y^{(1)}, x^{(2)} \otimes y^{(2)}) = \det(A) \times \det(B). \tag{5.5}$$

On the other hand, set

$$u^{(t)} := \varrho(U)(x^{(t)} \otimes y^{(t)}) \in \Lambda^{r+s}(\mathcal{F}(U) \oplus \mathcal{G}(U))$$

for $t \in \{1,2\}$. Consider the elements

$$(x_{j,t}, 0), (0, y_{k,t}) \in \mathcal{F}(U) \oplus \mathcal{G}(U)$$

for $1 \le j \le r, 1 \le k \le s$ and $t \in \{1,2\}$. It is clear that

$$u^{(t)} = (x_{1,t}, 0) \wedge \cdots \wedge (x_{r,t}, 0) \wedge (0, y_{1,t}) \wedge \cdots \wedge (0, y_{s,t})$$

for $t \in \{1,2\}$. Then, we deduce

$$\Lambda^{r+s}(\phi(U) \oplus \psi(U))(u^{(1)}, u^{(2)}) = \det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{5.6}$$

Note (5.5) = (5.6). The result follows. □

Let $A(X)$ denote the group $1 + tK(\mathrm{Bil}(X))[[t]]$ of formal power series with constant term 1 (under multiplication). Consider a map

$$\Lambda_t : \mathrm{Bil}(X) \to A(X), [\mathcal{F}, \phi] \mapsto 1 + \sum_{i\ge 1} \Lambda^i([\mathcal{F}, \phi]) t^i.$$

If $I : (\mathcal{F}, \phi) \to (\mathcal{G}, \psi)$ is an isometry of bilinear spaces, so is the natural map

$$\Lambda^i I : \Lambda^i(\mathcal{F}, \phi) \to \Lambda^i(\mathcal{G}, \psi).$$

Then, the map Λ_t is well-defined. Furthermore, Lemma 5.4.1 (b) implies that Λ_t is a homomorphism of the underlying monoids. By the universal property of K-theory, we can lift Λ_t to a homomorphism of groups

$$\lambda_t : K(\mathrm{Bil}(X)) \to A(X)$$

such that $\lambda_t \circ \iota = \Lambda_t$. Taking coefficients of λ_t, we get operators (not homomorphisms in general)

$$\lambda^i : K(\mathrm{Bil}(X)) \to K(\mathrm{Bil}(X)).$$

Set $\gamma_t = \lambda_{t/(1-t)}$ and write $\gamma_t = 1 + \sum_{i\ge 1} \gamma^i t^i$. Again, we obtain operators

$$\gamma^i : K(\mathrm{Bil}(X)) \to K(\mathrm{Bil}(X)).$$

Explicitly, we deduce

$$\sum_{i\geq 0}\gamma^i t^i = \sum_{i\geq 0}\lambda^i t^i(1-t)^{-i} = 1+\sum_{i\geq 1}(\sum_{i\geq s\geq 1}\lambda^s\binom{i-1}{s-1})t^i.$$

Hence, the γ^i are certain $\mathbb{Z}$-linear combinations of the λ^s. By definition, the map γ_t is a homomorphism of groups. Hence, for all $x, y \in K(\mathrm{Bil}(X))$, we have

Corollary 5.4.1. (a) $\gamma_t(x+y) = \gamma_t(x)\gamma_t(y)$;

(b) $\gamma_t([\eta]-1) = 1+t([\eta]-1)$ *where* η *is a bilinear space of rank* 1 *over X;*

(c) *If* $(\mathcal{F},\phi) \in \mathrm{Bil}(X)$ *is of constant rank* Θ, $\gamma^i((\mathcal{F},\phi)-\Theta) = 0$ *if* $i > \Theta$.

Proof. (a) is proved. For (b), we deduce

$$\gamma_t([\eta]-1) = \frac{\gamma_t([\eta])}{\gamma_t(1)} = \frac{\lambda_{t/(1-t)}([\eta])}{\lambda_{t/(1-t)}(1)} = \frac{1+[\eta]t/(1-t)}{(1-t)^{-1}} = 1+t([\eta]-1).$$

For (c), see the proof of [5, Lemma 2.1]. □

5.5 Further investigation

In Theorem 5.2.1, we have only computed $GW^0(D_+(q_s))$ over $\bar{F}$, which is enough for deducing Theorem 5.0.4.

Question 1: Can we generalize Theorem 5.0.4 by computing $GW^0(D_+(q_s)_F)$ over any field F of char. $\neq 2$ (eg. $F = \mathbb{R}$)?

It is valuable to consider this question, because Shapiro's conjecture (cf. Remark 5.0.3) concerns if the base fields of characteristic $\neq 2$ matter. Next, we show that although $GW^0(D_+(q_s)_F)$ may provide more information than $GW^0(D_+(q_s)_{\bar{F}})$ do, the extra information is not necessary to generalize Theorem 5.0.4. This assertion provides a little evidence to support the Shapiro's conjecture.

Recall the number $\delta(s)$ in the beginning of this chapter and the Hurwitz-Radon number $\rho(n)$ in Chapter 2.

Lemma 5.5.1. *A formula of type* $[\rho(n), n, n]$ *exists. Hence, a formula of type* $[s, 2^{\delta(s)}, 2^{\delta(s)}]$ *exists.*

Proof. See [29, Theorem 1] for an explicit construction of $[\rho(n), n, n]$, and [78, Exercise 6, Chapter 0] for the relation between these numbers. □

Applying the formula $[s, 2^{\delta(s)}, 2^{\delta(s)}]$ and Lemma 5.2.1, we deduce

Corollary 5.5.1. *There is an embedding of symmetric bilinear forms*

$$(P,\sigma)^{2^{\delta(s)}} \to (R,\mathrm{id})^{2^{\delta(s)}}.$$

Thus, $(P,\sigma)^{2^{\delta(s)}}$ *is trivial and* $2^{\delta(s)}v = 0$ *in* $GW^0(D_+(q_s)_F)$.

It is also clear $(\xi,\sigma)\otimes(\xi,\sigma) = (\mathscr{O},\mathrm{id})$, so that $v^2 + 2v = 0$ in $GW^0(D_+(q_s)_F)$. Thus, we find a well-defined map

$$i : \mathbb{Z}[v]/(v^2 = -2v, 2^{\delta(s)}v) \to GW^0(D_+(q_s)_F).$$

Lemma 5.5.2. *The map* $i : \mathbb{Z}[v]/(v^2 = -2v, 2^{\delta(s)}v) \to GW^0(D_+(q_s)_F)$ *is split injective.*

Proof. There is a map $GW^0(D_+(q_s)_F) \to GW^0(D_+(q_s)_{\bar{F}})$ obtained by the base-change $F \to \bar{F}$. Note that there is a commutative diagram.

$$\begin{tikzcd}
\mathbb{Z}[v]/(v^2 = -2v, 2^{\delta(s)}v) \arrow[r, "i"] \arrow[dr, "\approx"'] & GW^0(D_+(q_s)_F) \arrow[d] \\
 & GW^0(D_+(q_s)_{\bar{F}})
\end{tikzcd}$$

The isomorphism $\mathbb{Z}[v]/(v^2 = -2v, 2^{\delta(s)}v) \cong GW^0(D_+(q_s)_{\bar{F}})$ constructed in Theorem 5.2.1 gives the commutativity. The result follows. $\square$

Observe that the coefficients $\binom{r+i-1}{i}2^{i-1}\nu$ in the proof of Theorem 5.0.4 land in the image of the injective map i. It follows that $\binom{r+i-1}{i}2^{i-1}\nu = 0$ (for $i > n-r$) in the ring $\mathbb{Z}[v]/(v^2 = -2v, 2^{\delta(s)}v)$, which implies $2^{\delta(s)} | \binom{r+i-1}{i}2^{i-1}$. Thus, we see sums-of-squares formulas ignore the base field information under the standard trick of γ-operations.

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