Operations in (Hermitian) K-Theory and related topics

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Warwick Mathematics Institute

June 2019

THE UNIVERSITY OF
WARWICK

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Contents

Acknowledgments

I wish to express my deep gratitude and admiration towards my advisor, Marco Schlichting, who shared with me his experience, his passion, his insights, his ideas and his mastery of the field during many enlightening discussions and activities that not only made possible to write this thesis, but allowed me to become a mathematician. While giving me freedom to study and pursue my favourite research interests, he guided me by explaining not only how to do mathematics but more importantly how to *think* mathematics. He was incredibly supportive, patient and understanding as a human being: I think I was very lucky to have him as my advisor. Thanks, *Maestro*.

Together with Marco, I would like to thank my academic brothers Daniel, Dylan, Heng and James for creating a stimulating working group that enriched me both as a mathematician and as a person. I thank Christian Dahlhausen, Bernhard Köck, Heng Xie and Jens Hornbostel for inviting me to speak about my research at their institutions, for their support and useful discussions concerning my work. I also thank Denis Charles Cisinski, Alberto Navarro Garmendia, Adeel Khan, Husney Parvez Sarwar, Charles Weibel and Marcus Zibrowius for their interest in my work and for their support. I am grateful to Jens Hornbostel and John Greenlees for having accepted the daunting task of being the examiners of this thesis: their careful reading and their comments helped me to improve the overall quality of this work. The anonymous referee of the article extracted from Chapter 2 deserves special thanks for his comments and for having spotted several typos. I would also express

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. The content of Chapter 2 is adapted from an article which has been accepted for publication.

Abstract

Since the very beginning of K-theory, operations like the lambda or the Adams operations played a crucial role into the statement and the solution of many important problems. They followed the evolution of K-theory at any stage, generalizing and refining themselves as long as the theory was growing. The main objective of this thesis is to study a generalisation of the results contained in the works of Joel Riou from the category of smooth schemes to the category of schemes having an ample family of line bundles. In particular we show that it is possible to give a special lambda ring structure to K-theory seen as an element of the Zariski homotopy category of simplicial presheaves over the site of divisorial schemes over some regular base and that this structure is uniquely determined by the one we have on the level of the ordinary K-theory of vector bundles. This is done using homotopical methods and proving along the way that divisorial schemes can be embedded into smooth ones: result which is of independent interest. We then compare our construction with other older constructions and we deduce as an application of our main theorems some interesting results, including an Adams-Riemann-Roch theorem. Finally, we show that the methods of this thesis and the ones of Riou can be applied in some cases also to Hermitian K-theory.

Introduction

Among the fundamental theorems in geometry, the Riemann-Roch theorem certainly deserves a special mention, not only because of its intrinsic importance but also because of its prominent role in guiding the mathematics research during the last two centuries. In his celebrated 1857 masterpiece, Riemann studied his new complex functions, that we could call nowadays Riemann surfaces, establishing their general form once a finite number of simple poles is given and relating the number of this poles to a new geometric invariant he had discovered before: the genus. His seminal result was slightly generalized a few years later (1865) by one of his students, Roch, who interpreted analytically his mentor's results to reach what is nowadays known as the Riemann-Roch formula for a given Riemann surface Σ with canonical divisor K and a finite number of simple poles

$$l(D) - l(K - D) = deg(D) - p + 1$$

where p is the genus of Σ, D is the divisor associated with the prescribed poles and $l(D)$ is a quantity of interest (the dimension of the complex vector space of the functions on Σ whose associated divisor is D). This formula served as motivation for a generation of algebraic geometers who tried to improve and generalize it. Along the way, the mathematicians who were working on Riemann-Roch problems (many of them were part of the so called Italian School) discovered the notion of canonical classes, after seminal work of Noether, Segre, Severi and Todd among others. Meanwhile algebraic topology was growing quickly and homological and cohomological methods were discovered and appreciated. This eventually led to the study of characteristic classes of vector bundles in the '30 after the work of Stiefel, Whitney and Chern (only to cite a few) which was linked to the theory of canonical classes after the World War II thanks to the introduction of sheaf theory by

Leray and of the fundamental work by Kodaira and Serre, who envisioned a possible more general statement of the Riemann-Roch theorem involving both the algebraic and the topological side of the theory. This dream of Serre became quickly reality thanks to the work of Hirzebruch which was able to prove what is now called as the Hirzebruch-Riemann-Roch formula. But those were the years when geometry was going to be reshaped completely thanks to the genius of Grothendieck. While giving a very abstract and general framework for the study of algebraic geometry with the introduction of the notion of scheme, he found the definitive way to look at the Riemann-Roch problems as a defect of functoriality of a natural transformation between cohomology theories, one of these being a new extraordinary cohomology theory which became later known as K-theory. Grothendieck's insight came not only as a meaningful reinterpretation and generalization of the problem, but also led to a simpler proof of the appropriate statement (which is now called the Grothendieck-Riemann-Roch theorem) towards the introduction of several new concepts. In order to reach such a result, besides the discovery of K-theory, which is a remarkable achievement itself, Grothendieck introduced the notion of λ-ring to study it, originating from the behaviour of the exterior power operations and that turned out to be an extremely powerful algebraic notion that can be used in several contexts. Indeed one of the most striking applications of this notions was the simple solution by Adams of the Hopf invariant one problem (1966) where topological K-theory methods were used and the so called Adams operations were introduced. Lambda rings became then very important also in representation theory, class field theory and in the study of Witt vectors type constructions, convex polytopes and binomial rings. After a while, K-theory was vastly generalized by Quillen, who introduced higher K-theory groups in both topology and algebraic geometry as homotopy groups of a certain H-group K, whose group of path components was identified to the K-theory groups defined before, call it K_0. This new space K came as a loop space of a certain topological space Q that can be defined functorially out of some categories of interests, like the one of vector bundles over a scheme. Since Riemann-Roch type theorems and λ-structures played such an important role in classical K-theory, it became meaningful to study such structure on the higher K-theory groups. The most natural way to study maps between homotopy groups of topological spaces is to obtain them as maps induced by continuous maps between topological spaces. However, if we pursue this road in the naive way, we are doomed to failure, as remarked by Grayson: if we start from operations between vector bundles that should give rise to our λ-operations, and we plug in the Quillen's machinery to get a map between the K-theory spaces, we only end up with maps between topological spaces

originating *group* homomorphisms when we pass to the homotopy groups. Alas, λ-operations are far from being group homomorphisms so we cannot hope to obtain them in this way, at least not easily. Thus, we have to use subtler methods to build our operations for higher K-theory groups. One possible approach to circumvent the issue and to introduce such operations is to use representation theory. In this way many authors such as Hiller, Kratzer, Schechtman, Gillet, Soulé, Lecomte and Levine, to cite only a few of them, were able to define on various level of generality lambda and Adams operations for higher algebraic K-theory. They then used these operations to study Riemann-Roch problems (see the work of Gillet and Soulé for regular schemes) and to link certain decompositions on the K-theory groups with other fundamental algebraic invariant such as the (higher) Chow groups (see the seminal work of Levine), which were acquiring extreme relevance because of their use in intersection theory and in what became to be known as motivic cohomology. This latter theory found a final theoretical place in the (un)stable motivic homotopy theory envisioned in works of Suslin, Voevodsky, Morel, Levine and many others after the necessary homotopical maturity was reached thanks to the advances in simplicial homotopy theory. K-theory found then its final place as a "space" in the motivic homotopy categories. Joel Riou, in his PhD thesis (2006) was eventually able, using homotopical methods and the explicit computations of the algebraic K-theory of grassmannians, to build the lambda, Adams and virtually every operation we knew for classical K-theory in an homotopical way. In particular he was able to prove

Theorem 0.0.1 (Riou A.3.14). *If S is a regular separated scheme, denoting as* Sm/S *the category of smooth separated schemes over S one has the following isomorphisms*

$$\mathrm{Hom}_{\mathcal{H}(S)}(K^n, K) \cong \mathrm{Hom}_{\mathcal{H}(S)}((\mathbb{Z}\times\mathrm{Gr})^n, \mathbb{Z}\times\mathrm{Gr}) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S,\mathrm{Sets})}(K_0(-)^n, K_0(-))$$

where $\mathcal{H}(S)$ is the unstable motivic homotopy category of [MV99], K is the K-theory simplicial presheaf, Gr is the infinite Grassmannian and $K_0(-)$ is the presheaf of sets associating to every smooth scheme X its algebraic K-theory of vector bundles $K_0(X)$. The previous isomorphisms are also true if we consider the respective pointed categories.

We can use this theorem to lift the operations that we have on the K_0 level (for example the lambda operations) to operations on K theory in the unstable motivic homotopy category. After that we can use them to obtain general versions of Riemann-Roch type theorems, for example. The methods introduced by Riou

can be applied also to other cohomology theories. For example, one might wonder whether it is possible to do the same for Hermitian K-theory, another extraordinary cohomology theory which tries to study forms associated to vector bundles on geometric objects instead than simply vector bundles themselves. One objective of this work is then to begin to apply the methods of Riou to this cohomology theories, using recent advances into that theory. Moreover, the experienced reader will have noticed that the result of Riou only applies to *smooth* schemes, while lambda operations and Riemann-Roch type theorems in ordinary and higher algebraic K-theory go beyond that class of geometric objects. The main goal of this thesis is then to investigate how we can obtain a Riou type theorem if we move away from the context of smooth schemes and if such extensions can be applied also to the hermitian case. It turns out that if we restrict to the class of *divisorial* schemes the answer is indeed positive and many interesting facts arise along the way, including a general way to embed divisorial schemes into smooth ones and a very general version of the Adams-Riemann-Roch theorem that holds true in this context. Divisorial schemes, also known as schemes with an ample family of line bundles, satisfy the resolution property that makes the study of their algebraic K-theory particularly nice and pleasant. Indeed, for such schemes, after the groundbreaking results of Thomason (contained in the seminal [TT90]) all the various definitions of algebraic K-theory agree and the theory itself is very powerful thanks to descent results. It is also a quite general class of schemes: indeed every smooth separated scheme over a regular base scheme is divisorial and the same is true, for example, for every (possibly singular) quasi-projective variety over a field. This class of schemes has then a special place into the study of algebraic K-theory and therefore it is important to properly understand operations on K-theory in this context. The thesis is divided in two parts. the first part studies the problem of defining and studying the unstable operations on K-theory of divisorial schemes, while the second contains the applications of the general theory we develop together with the discussion of the hermitian side of the theory. We will now describe in more detail the content and the main results contained in the chapters of this thesis. Given a noetherian regular and divisorial base scheme S (for more, see assumptions 0.1), we will denote as Sch_S (Sm/S) the category of divisorial (smooth) schemes of finite type over S and by $\mathbf{sPre}(\mathcal{C})$ the category of simplicial presheaves over a Grothendieck site $\mathcal{C}$. We will denote as $\mathcal{H}(S)$ the unstable motivic homotopy category of Morel and Voevodsky which is the homotopy category of the model category obtained considering simplicial presheaves over Sm/S and localising the Nisnevich injective local model structure inverting $\mathbb{A}^1$-weak equivalences. With $\mathcal{I}$ ($\mathcal{I}^l_{Zar}$, $\mathcal{I}^l_{Nis}$) and $\mathcal{P}$ ($\mathcal{P}^l_{Zar}$, $\mathcal{P}^l_{Nis}$)

we will denote the global (local with respect to the Zariski or Nisnevich topology) injective or global (local with respect to the Zariski or Nisnevich topology) projective model structures respectively and we will use the notation $[,]$ to indicate Hom spaces in their respective model categories (the same will apply to $\mathcal{H}(S)$). For example, $\mathbf{sPre}(\mathrm{Sch}_S, \mathcal{I}^l_{Zar})$ will denote the model category of simplicial presheaves over Sch_S with the choice of the local injective model structure with respect to the Zariski topology and $\mathrm{Ho}(\mathbf{sPre}(\mathrm{Sch}_S, \mathcal{I}^l_{Zar}))$ will be its homotopy category. We denote by $\mathcal{S}$ the category of simplicial sets.

- Chapter 1 deals with the extension of the operations defined for K-theory in $\mathcal{H}(S)$ to the bigger homotopy category $\mathrm{Ho}(\mathbf{sPre}(\mathrm{Sch}_S, \mathcal{I}^l_{Zar}))$. We start in Section 1.1 with some recollections on the Bousfield-Kan $\mathbb{Z}_\infty$-completion which we need in the sequel deviating a little bit from the usual presentations. In Section 1.2 we use the Bousfield-Kan completion to show the following

  Proposition 0.0.2 (1.2.7, 1.2.8). *Given a Grothendieck site $\mathcal{C}$, let be $X \in$ $\mathbf{sPre}(\mathcal{C})$ a simplicial presheaf which is $\mathcal{P}^l$-fibrant and $\mathbb{Z}$-complete (meaning that it is sectionwise $\mathbb{Z}$-complete in the sense of [BK72]). Hence if $f : Y \to Y'$ is a map between $\mathcal{P}$-cofibrant presheaves inducing $H_*(-, \mathbb{Z})$-isomorphisms sectionwise, one has*

  $$[Y', X]_{\mathcal{P}^l} \cong [Y, X]_{\mathcal{P}^l} \cong [Y, X]_{\mathcal{I}^l} \cong [Y', X]_{\mathcal{I}^l}$$

  The same conclusion holds even if $f : Y \to Y'$ is a map between $\mathcal{I}^l$-cofibrant presheaves (so any map) inducing $H_*(-, \mathbb{Z})$-isomorphisms sectionwise.

  In Section 1.3 we make some remarks on what happens if we try to localise the global model structures on simplicial diagrams with respect to integral homology. This relates to some work of Goerss and Jardine ([GJ98]) and in some sense can be considered folklore, as explained in the text. In Sections 1.4 and 1.5 we recall some facts we need about bisimplicial sets and the classifying spaces: everything here is certainly well known except, perhaps, the exposition. With all these prerequisites, we can prove in 1.7 the following

  Proposition 0.0.3 (1.7.1,1.7.3). *For $n \geq 0$ we have*

  $$[\mathrm{BGL}^+, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [\mathrm{BGL}, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S}$$
  $$[\mathrm{BSp}^+, \mathrm{GW}^{[n]}]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [\mathrm{BSp}, \mathrm{GW}^{[n]}]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S}$$

where K (GW$^{[n]}$) is the (n-shifted Hermitian) simplicial K-theory presheaf (and $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ when we consider Hermitian K-theory).

The previous is used to show in Section 1.8 the following

Theorem 0.0.4 (1.8.10,1.8.11). *For any natural number n,*

$$[K^n, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [(\mathbb{Z} \times \mathrm{BGL})^n, K]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S} \cong [K^n, K]_{\mathcal{H}(S)}$$

Moreover if $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ it holds

$$[KSp^n, KSp]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [(\mathbb{Z} \times \mathrm{BSp})^n, KSp]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S} \cong [KSp^n, KSp]_{\mathcal{H}(S)}$$

where KSp denotes the symplectic hermitian K-theory simplicial presheaf (i.e. GW$^{[2]}$).

which is an interesting result itself and allows us to define a λ-ring structure on K-theory in the above homotopy categories building on the results of Riou. This is done by a study of the mapping spaces of the simplicial model categories involved together with the trivial but fundamental observation that the general and the symplectic linear groups are *smooth*. The same methods can be applied in more general situations, as we discuss in the text. In Section 1.9 we consider the case where our smooth schemes Sm/S are assumed to be separated (in the absolute sense) linking then our results with a more familiar and used class of schemes. Finally, in Section 1.10 we investigate which of our theorems hold true if we remove the hypothesis of being divisorial from our schemes.

- Chapter 2 is devoted to the proof of the following theorem

Theorem 0.0.5 (2.0.1). *Let X be a quasi-compact and quasi-separated scheme of finite type over a noetherian ring R having an ample family of line bundles. Then there exists a closed embedding $f : X \hookrightarrow W$ with W a smooth scheme over R admitting an ample family of line bundles. Moreover W arises as an open subscheme of the multihomogeneous spectrum of a suitable $\mathbb{Z}^n$-graded polynomial algebra.*

This is an interesting result on its own. When the divisorial scheme involved is a reduced scheme of finite type over an algebraically closed field, this theorem was proved by J.Hausen ([Hau02] Theorem 3.2) and multihomogeneous projective spaces were introduced by Brenner and Schroer ([BS03]), who were

able to show that divisorial schemes embed into them. We improve their results by showing that it is possible to refine their argument to actually get an embedding into a divisorial *smooth* scheme contained in such a multihomogeneous projective space. This refines classical arguments that can be found in the context of weighted projective spaces (see the work of Mori, Dolgachev and Reid among the others). We start in section 2.1 with some recollections on schemes having an ample family of line bundles (aka divisorial schemes) detailing a folklore technical result in Section 2.1.2 that we will need in the sequel. Section 2.2 contains a technical lemma that we need to show the smoothness of the final scheme we will consider in the statement of the emebedding theorem and in 2.3 we recollect some facts about multihomegeneous projective spaces that we need. Note we have given new constructions (equivalent to the ones given in [BS03]) and proofs whenever possible, to avoid the use of GIT quotients. Section 2.4 contains the proof of Theorem 2.0.1 that follows the argument of the proofs of the embedding theorem in [BS03] which follows the original argument of [Hau02]. Using it, we can prove a very interesting fact, which is the main result of Section 2.5

Theorem 0.0.6 (2.5.5). *Let X be a scheme of finite type over a noetherian affine scheme $S = \mathrm{Spec}(R)$ having an ample family of line bundles. Then given a finite number vector bundles $\mathcal{E}_1, ..., \mathcal{E}_n \in \mathrm{Vect}(X)$ there is a smooth divisorial scheme $Y_\mathcal{E}$ over S and vector bundles $\mathcal{E}_{1,Y_\mathcal{E}}, ..., \mathcal{E}_{n,Y_\mathcal{E}}$ over it together with a morphism $\psi_\mathcal{E} : X \to Y_\mathcal{E}$ such that $\psi_\mathcal{E}^*(\mathcal{E}_{i,Y_\mathcal{E}}) \cong \mathcal{E}_i$ for every $i = 1, ..., n$.*

- In Chapter 3 we prove the Main Theorem of this thesis

Theorem 0.0.7. *If S is a regular quasi-projective scheme over a noetherian affine scheme R, for any natural number n, if we consider the Thomason's K-theory presheaf K we have*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{Zar}(\mathrm{Sch}_S))}(K^n, K) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0^n, K_0)$$

where $\mathrm{Ho}(\mathbf{sPre}_{Zar}(\mathrm{Sch}_S))$ denotes the homotopy category of $\mathbf{sPre}_{Zar}(\mathrm{Sch}_S)$. Moreover we have that

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{Zar}(\mathrm{Sch}_S))}(K^n, K) \cong \mathrm{Hom}_{\mathcal{H}(S)}(K^n, K)$$

and that

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0^n, K_0) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0^n, K_0)$$

for any natural n. Passing to the sets of pointed morphisms, analogue isomorphisms hold.

This theorem asserts that the endomorphisms of K-theory in the model category of Zariski local simplicial presheaves over divisorial schemes depends only on their behaviour at the level of K_0 seen as a presheaf of sets and in addition these endomorphisms are in bijection with the ones obtained from the theorem of Riou. This is surprising not only because our schemes are allowed to be singular but also because we do not invert $\mathbb{A}^1$-weak equivalences so that we cannot see K theory as $\mathbb{Z} \times \mathrm{BGL}$ or as an the ind-scheme $\mathbb{Z} \times \mathrm{Gr}$ as we can do in the motivic homotopy category. More is true: these endomorphisms are uniquely determined by they behaviour on *affine* schemes as summarised by the following

Theorem 0.0.8 (Theorem 3.2.16)**.** *If S is a regular noetherian affine base scheme, all the arrows in the following commutative cube are isomorphisms*

$$
\begin{array}{ccc}
& \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{Zar}(\mathrm{Aff}/S))}(K^n,K) \longrightarrow \mathrm{Hom}_{\mathcal{H}^{aff}(S)}(K^n,K) \\
\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPre}_{Zar}(\mathrm{Sch}_S))}(K^n,K) \longrightarrow \mathrm{Hom}_{\mathcal{H}(S)}(K^n,K) \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}/S)}(K_0^n,K_0) \longrightarrow \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(K_0^n,K_0) \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0^n,K_0) \longrightarrow \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0^n,K_0)
\end{array}
$$

The pointed version of this theorem also holds.

The proof basically consists of two steps: first one considers the top face of the cube and shows that with the results contained in Chapter 1 together with some facts available in literature, all the arrows in that square are isomorphisms (for some of these maps one can assume S to be any divisorial regular base scheme). Then one notices that assuming S to be affine, using Theorem 2.5.5 and its variants, we can prove that also the lower horizontal maps are injective so that chasing the diagram gives the result (Section 3.1.2). We show in Section 3.1.3 that one can improve Theorem 3.1.6 and assume that S is regular divisorial quasi-projective of finite type over a noetherian ring R by improving the result 2.5.5 and arguing in the same way. In Section 3.2 we take care of the extension to affine schemes to complete the proof of Theorem 3.2.16. As in Chapter 1, in Section 3.3 we consider the case where our smooth schemes Sm/S are assumed to be separated (in the absolute sense) linking our

results with a more familiar and used class of schemes. Finally, in Section 3.4 we investigate which of our theorems hold true if we remove the hypothesis of being divisorial from our schemes.

- Chapter 4 contains a side result which is not used in the remaining parts of the thesis and appears only during a variant of the proof of Theorem 3.2.16. Therefore this can be safely skipped at a first reading. The main result of this chapter is the following

Theorem 0.0.9 (4.2.2). *Let S be a regular noetherian scheme. There are weak equivalences (induced by an explicit map, not via zig-zags) $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}} \xrightarrow{\sim} \mathbb{Z} \times \mathrm{Gr}$ and $\mathbb{P}^{\infty}_{\mathrm{aff}} \xrightarrow{\sim} \mathbb{P}^{\infty}$ in $\mathcal{H}(S)$, where $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ and $\mathbb{P}^{\infty}_{\mathrm{aff}}$ are filtered colimits of affine (in the absolute sense) schemes.*

The proof of this theorem is obtained by first showing that in favourable situations we can make the Jouanolou device functorial enough to give rise to natural transformations between simplicial diagrams. This is done in full generality in Section 4.1. Section 4.2 then concludes the proof of the theorem by mean of simple homotopical algebra. Section 4.3 shows that the ind-schemes we have also satisfy some important technical properties.

- Chapter 5 contains the main applications of Theorem 3.2.16. In Section 5.1.4, after some recollections on lambda rings in Section 5.1.1 where we also emphasize the fact that ψ-rings, a close friend of λ-rings, also make sense in the noncommutative world (indeed we where not able to find a single suitable reference for all the material we use), we define λ, Adams and γ-operations for higher K-theory of divisorial schemes using 3.2.16. We also discuss the notion of lambda ring in a very abstract setting in Sections 5.1.2 and 5.1.3 where we spell out in some details how to prove that a given lambda ring in a suitable homotopy category of simplicial presheaves gives rise to lambda ring structures on the homotopy groups of simplicial presheaves. This is certainly subsumed in the work of Riou, although the statements and the details do not appear there. After that we can use Theorem 3.2.16 to prove the following

Theorem 0.0.10 (5.1.28,5.1.32). *Lambda, Adams and γ-operations $K_0(-) \to K_0(-)$ naturally induce maps on $K_n(\mathcal{X})$ for every $\mathcal{X} \in \mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)$. In particular this is true for the usual higher K-theory groups $K_n(X)$ for every*

$X \in \mathrm{Sch}_S$. *Moreover we have a natural multiplication law* $- \times - : K \times K \xrightarrow{\cup} K$ *which induces a graded ring structure on the graded* $K_0(X)$*-module*

$$K_*(X) := \bigoplus_{n \in \mathbb{N}} K_n(X)$$

for any scheme $X \in \mathrm{Sch}_S$, *different from the one we would get if we set the product of two homogeneous positive elements to be zero. Call the latter ring* $(K_*(X), \cdot)$ *and the former* $(K_*(X), \cup)$. *Then the datum* $(K_*(X), \cdot, \lambda^k)$ *is a lambda ring with associated* ψ*-ring* $(K_*(X), \cdot, \psi^k)$. *Moreover,* $(K_*(X), \cup, \psi^k)$ *is a noncommutative* ψ*-ring and the maps* $\psi^k : (K_*(X), \cup) \to (K_*(X), \cup)$ *are morphisms of noncommutative* ψ*-rings (see Definition 5.1.7). These structures are functorial.*

The argument used to prove this theorem is the analogue of the argument used by Riou in the smooth case. In Section 5.2 we compare the structures just defined to many others available in literature, and we show that they agree. Even in the smooth case, some of these comparisons, while certainly known to many experts, were mere folklore. In Section 5.3 we extend some additive results that were obtained by Riou for smooth schemes, in particular we are able to show the following theorem adapting the arguments of Riou and using our 3.2.16

Theorem 0.0.11 (5.3.3). *Denotig as* Ω^i_f *the ith right derived functor of the loop space functor, as* K_i *the ith higher algebraic* K*-theory presheaf and as* Pic *the presheaf associating to any scheme its Picard group, all the arrows in the following diagram are isomorphimsms*

$$
\begin{array}{ccc}
[\mathrm{B}\mathbb{G}_m, \Omega^i_f K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} & \longrightarrow & [\mathrm{B}\mathbb{G}_m, \Omega^i_f K]_{\mathcal{H}(S)} \\
\downarrow & & \pi_0 \downarrow \cong \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(\mathrm{Pic}, K_i) & \xrightarrow[\mathrm{res}]{} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{Pic}, K_i)
\end{array}
$$

Moreover, also all the arrows in the following commutative diagram are isomorphisms

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S),\mathrm{Ab}}(K_0, K_i) & \xrightarrow{\delta^*_{\mathrm{Sch}}} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(\mathrm{Pic}, K_i) \\
\beta \downarrow & & \downarrow \cong \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S),\mathrm{Ab}}(K_0, K_i) \xrightarrow[\cong]{\delta^*_{\mathrm{Sm}}} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{Pic}, K_i) & \cong & \varprojlim_n K_i(\mathbb{P}^n) \cong K_i(S)[[U]]
\end{array}
$$

where the maps δ_{Sm}^* and δ_{Sch}^* are induced from the presheaves maps $\delta_{\mathrm{Sm,Sch}}$: Pic $\to K_0$ given for any scheme X by the assignment $[L] \mapsto [L]$, for any L line bundle over X.

Section 5.4 recasts in a different language some formal Riemann-Roch algebra contained in [FL85] in such a way that it can be easily applied to the structures coming from our Theorem 0.0.10. This starts in Section 5.4.1 with with the modification of some definitions contained in [FL85], while in sections 5.4.2 and 5.4.3 we remark that some constructions contained in *op. cit.* can be performed in the category of schemes we consider. With these prerequisites, we can prove in Section 5.5 that we can use known results and the functorialities of the structures just built for higher K-theory using the machiery of [FL85] just generalized to prove a very general version of the Adams-Riemann-Roch theorem for higher K-theory

Theorem 0.0.12. *5.5.11 Fix a regular noetherian affine base scheme S. Let be $f : X \to Y$ a projective l.c.i. morphism in Sch_S. Then $(K_*, \cdot, \psi^j, f)$ $((K_*, \cup, \psi^j, f))$ is a RR datum and RR holds with respect to the datum $(\mathbb{Z}[1/j] \otimes K_*, \psi^j, f)$ for every j with multiplier $\tau_f \in K_0(X)$ given by Theorem 5.4.4. This means that the following diagram commutes for any j*

$$
\begin{array}{ccc}
\mathbb{Z}[1/j] \otimes K_*(X) & \xrightarrow{\;\tau_f \cdot \psi^j\;} & \mathbb{Z}[1/j] \otimes K_*(X) \\
{\scriptstyle f_*}\Big\downarrow & & \Big\downarrow{\scriptstyle f_*} \\
\mathbb{Z}[1/j] \otimes K_*(Y) & \xrightarrow[\;\psi^j\;]{} & \mathbb{Z}[1/j] \otimes K_*(Y)
\end{array}
$$

Having pursued the same path, this fact could have been shown before, say in the late nineties (and it was with a priori different operations, see [Kö98]) and might be considered by an expert aware of the Gillet-Soulé-Levine constructions of the lambda operations, for example, folklore. See also the work of Alberto Navarro Garmendia [Nav18] and [Kö98] for general Riemann-Roch formulas.

- Chapter 6 is devoted to the problem of which of the previous theorems can be brought to the context of Hermitian K-theory. In this case we do not have any Riou-like results so we have to provide them ourselves. We then start in Section 6.1 with some recollections concerning bilinear grassimannians, which are well studied in [ST15] and [PW10a]. The only novelty is to give a unified treatment of the facts proven in *op. cit.* which are usually spelt out only

in the symmetric or the symplectic case. In Section 6.2 we then consider the symmetric hermitian analogue of the theorem by Riou. Denoting as GW the element of $\mathcal{H}(S)$ representing symmetric hermitian K-theory, our main positive result is the following

Theorem 0.0.13 (6.2.8). *Fix S is a regular base scheme such that 2 is invertible in $\Gamma(S, \mathcal{O}_S)$. The map*

$$\pi_0 : [\mathrm{GW}, \mathrm{GW}]_{\mathcal{H}(S)} \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{GW}_0, \mathrm{GW}_0)$$

is surjective.

Unfortunately, lacking the computations of the Grothendieck-Witt groups of the orthogonal grassmannians, we are not able to go further to find a proof of the following, that we leave as an open conjecture

Conjecture 0.0.14. $[\mathrm{GW}^n, \mathrm{GW}]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{GW}_0^n, \mathrm{GW}_0)$. $\mathrm{GW} \in \mathcal{H}(S)$ *has a structure of λ-ring and this structure lift to a structure on every* $\mathrm{GW}_n(X)$ *for every* $X \in \mathrm{Sm}/S$.

We only notice that once we had the hermitian Riou theorem, we could define the λ-ring structure in the statement of the conjecture using the lambda ring structure defined at the GW_0 level by Zibrowius in [Zib18]. For symplectic K-theory, however, we have the computations we need to run Riou's machinery so we can obtain the analogue of Theorem 3.2.16. The main step, besides the achievement of a Riou like theorem for smooth schemes, which is obtained in Section 6.3, is the appropriate analogue of 2.5.5 for forms, the homotopical algebra required for everything else being already handled in Chapter 2. We succeed in producing such a theorem which asserts that we can pullback forms in Section 6.4.1 where using bilinear grassmannians we prove

Theorem 0.0.15 (6.4.4). *Consider S quasi-projective scheme over a noetherian affine scheme R where 2 is invertible and let X be a divisorial scheme of finite type over S. Then given a finite number of ϵ-inner product spaces over X, $V_1 = (\mathcal{E}_1, \varphi_1), ..., V_n = (\mathcal{E}_n, \varphi_n)$, there is a smooth scheme Y_V over S and ϵ-inner product spaces $V_{1,Y_V}, ..., V_{n,Y_V}$ over it together with a morphism $\psi_V : X \to Y_V$ such that $\psi_V^*(V_{i,Y_V}) \cong V_i$ for every $i = 1, ..., n$. If X and S are affine schemes, then we can take Y_V to be affine.*

This readily gives

Theorem 0.0.16 (6.4.5). *The natural restriction maps*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(\mathrm{GW}_0(-), \mathrm{GW}_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{GW}_0(-), \mathrm{GW}_0(-))$$

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}/S)}(\mathrm{GW}_0(-), \mathrm{GW}_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(\mathrm{GW}_0(-), \mathrm{GW}_0(-))$$

and the restriction maps

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(KSp_0(-), KSp_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(KSp_0(-), KSp_0(-))$$

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}/S)}(KSp_0(-), KSp_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(KSp_0(-), KSp_0(-))$$

are injective, where S is a quasi-projective (affine if we consider the categories of affine schemes) noetherian scheme of finite type over a noetherian affine scheme R where 2 is invertible. We have denoted here by KSp_0 the usual symplectic hermitian K-theory presheaf.

We can then conclude the story in Section 6.4.2 by proving the symplectic hermitian analogue of Theorem 3.2.16 (denote by KSp the object representing symplectic hermitian K-theory in the homotopy categories that appear)

Theorem 0.0.17 (6.4.7). *Fix S an affine regular noetherian base scheme with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Then all the arrows in the following commutative cube are isomorphisms*

with the obvious meaning of the terms involved. The pointed version of this theorem also holds.

In Section 6.5 we consider the case were our smooth schemes Sm/S are assumed to be separated (in the absolute sense) linking our results with a more familiar and used class of schemes. Finally, in Section 6.6 we investigate which of our theorems hold true if we remove the hypothesis of being divisorial from our schemes.

- There are two appendices. Appendix A recollects some facts from the works of Riou [Rio06] and [Rio10] and some preliminaries we need as the Jouanolou's

device or some logic facts concerning algebraic structures. We also sketch some arguments of the works in *op.cit.* spelling out some details that were abridged or that were written only in french. We hope this could allow the less expert reader to appreciate more the above mentioned works. Explicitly note that while essentially all the arguments are due to Riou, we sometimes deviate from the original proofs and definitions giving a discussion more tailored to this work. In Appendix B we recollect some technical facts concerning K-theory and descent that we use in the main text that can be considered certainly known, except perhaps some technicalities involving divisorial schemes and the exposition. We also give a very brief dictionary of Hermitian K-theory.

0.1 Assumptions and notations

All the schemes will be always assumed to be Noetherian of finite dimension unless otherwise stated. Whenever we will say that a base scheme S is regular, we will mean that it is noetherian, regular in the sense of [Sta18, Tag 02IS] and that it is divisorial (see Definition 2.1.2) unless otherwise stated. The only reason the divisorial assumption is needed is that we find convenient to have S as final object of many of the categories we will consider. We say that a scheme X is smooth over a base S if its structure map is smooth ([GD67] IV 6.8.6, 17.3.1, [GW10] 6.14, [Sta18, Tag 01V5]). Explicitly note that we do not require any separation hypothesis. We also make the blanket assumption that unless otherwise specified, all the schemes in the categories Sch_S and Sm/S of (smooth) schemes of finite type over a chosen base scheme S have an ample family of line bundles. We detail in specific sections what can be said if we go out of this context.

Notation 0.1.1. Throughout this thesis, given any category $\mathcal{C}$, we will denote as $[-, -]_{\mathcal{C}}$ the Hom sets $\mathrm{Hom}_{\mathcal{C}}(-, -)$. If we are considering $\mathcal{C}$ as a model category, by $[-, -]_{\mathcal{C}}$ we will always denote $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(-, -)$, i.e. the hom sets in the *homotopy category* of $\mathcal{C}$. If we will speak about pointed homotopy categories of a given model category $\mathcal{C}$ we will mean the homotopy category of the model category obtained by considering the pointed category $\mathcal{C}_\bullet$ and giving to it the pointed model structure induced from $\mathcal{C}$ (see [Hov99] Proposition 1.1.8). With $\mathcal{P}$ or $\mathcal{I}$ we will denote the global injective or projective model structure on a given category of simplicial diagrams. Whenever these simplicial diagrams are categories of simplicial presheaves over a Grothendieck site $(\mathcal{C}, \tau)$ we will denote as $\mathcal{P}_\tau^l \mathcal{C}$ or $\mathcal{I}_\tau^l \mathcal{C}$ (or by $(\mathrm{s}\mathbf{Pre}(\mathcal{C}), \mathcal{I}_\tau^l)$ and $(\mathrm{s}\mathbf{Pre}(\mathcal{C}), \mathcal{P}_\tau^l)$) the model categories of simplicial presheaves over $\mathcal{C}$ with the local projective or injective model structure relative to the Grothendieck topology

τ. When the site is clear we will abbreviate such notations with $\mathcal{P}^l_\tau$, $\mathcal{I}^l_\tau$ or even $\mathcal{P}^l$ and $\mathcal{I}^l$. Thus, for example, $\mathcal{I}^l_{Zar}\mathrm{Sch}_S$ will denote the model category of simplicial presheaves over Sch_S where we consider the injective local model structure with respect to the Zariski topology. If the choice of the injective structure is assumed, we will use the notation $\mathbf{sPre}_\tau(\mathcal{C})$. In addition, we will use the notation $\mathcal{H}^{\mathcal{C}}_\tau$ to denote $\mathrm{Ho}(\mathcal{I}^l_\tau \mathcal{C})$ while we will reserve the notations $\mathcal{H}(S)$ and $\mathcal{H}^{aff}(S)$ for the unstable motivic homotopy category over Sm/S and its full subcategory of smooth affine schemes SmAff/S respectively. Finally, we will denote as Aff/S the full subcategory of Sch_S of affine schemes.

Part I

Unstable operations on K-theory

Chapter 1

Endomorphisms in Higher K-Theory: homotopy theory

1.1 Some recollections. Topology, completion and simplicial sets

In this Section we recollect some useful facts on the Bousfield-Kan completion and some topological notions. Everything here is well known, except perhaps the exposition or some observations. For us the category $\mathrm{Top}(\mathrm{Top}_*)$ of (pointed) topological spaces will be a convenient category for homotopy theory, such as the category of compactly generated Hausdorff spaces, see for examples [Vog71]. For background on model categories see [Hir03],[Hov99],[BK72], [GJ09], [Qui67], [DS95].

Definition 1.1.1. An H-space is a pointed space (Y, p) having the type of a CW complex together with a map $\mu : Y \times Y \to Y$ of pointed spaces such that p is a pointed homotopy identity (if we see the point p as as a constant map $Y \to Y$ this means that $id, \mu \circ (id, p), \mu \circ (p, id)$ are pointed homotopic) and μ is homotopy associative. If such structure has a pointed homotopy inverse then (Y, p) is called H-group. More generally, given a model category $\mathcal{C}$ we will say that an object in it is an H-space (group) if it is a monoid (group) object in the pointed homotopy category $\mathrm{Ho}(\mathcal{C}_\bullet)$.

In this section we will be interested in H-spaces (groups) only in the category of topological spaces and of simplicial sets.

Example 1.1.2. Given a pointed CW complex (Y, y_0), then the loop space $(\Omega Y, \omega_0)$ is an H-group (see [Swi02]2.15, for example).

1

From the definition it follows that for an H-space (Y, y_0), its fundamental group $\pi_1(Y, y_0)$ is abelian and $\pi_0(Y)$ is a group if in addition Y is an H-group. Now, it is a general fact that in any space with a homotopy associative multiplication which makes the set of path components into a group, left or right multiplications by points induce homotopy equivalences between the path components. As a corollary (which can be easily proved also directly, see [Hat02] page 291, or [Dug66] page 387) we have

Lemma 1.1.3. *All the path connected components of every H-group $(G, *)$ are homotopy equivalent. In particular the same is true for all the path connected components of $(\Omega Y, \omega_0)$ for a given pointed CW complex Y.*

Definition 1.1.4 ([Spa95] page 384)**.** A path connected space Y is called n-simple $(n \geqslant 1)$ if for some $y_0 \in Y$, $\pi_1(Y, y_0)$ acts trivially on $\pi_n(Y, y_0)$. Y is called simple if it is n-simple for all $n \geqslant 1$.

Remark that the action could be non trivial even for $n = 1$.

Theorem 1.1.5. *A path connected H-space is simple.*

For the proof see [Spa95], Theorem 9 page 384.

Definition 1.1.6. If R is a solid ring (i.e. a commutative unital ring such that the multiplication $R \otimes_{\mathbb{Z}} R \to R$ is an iso, see [BK72] page 20) we say that a group G is R-nilpotent if it has a finite central series

$$1 = G_0 \vartriangleleft G_1 \vartriangleleft \cdots \vartriangleleft G_n = G$$

such that for every $1 \leq i \leq n$, G_i/G_{i-1} admits an R-module structure (unique, see [BK72] pag.82).

Note that a $\mathbb{Z}$-nilpotent group is a nilpotent group in the standard sense.

Remark 1.1.7. In this section we stick to solid rings in order to be faithful with the standard references where the theory is developed, howewer we explicitly note that this assumption can be safely removed for homotopy theoretic purposes in virtue of the so called core lemma, see [BK72] I 4.5 for a discussion, and [BK72] I Section 9 for a proof of that lemma.

Denote as $\mathcal{S}$ the category of simplicial sets, and as $\mathcal{S}_*$ its pointed version (see for example [GJ09], [May92], [FP90], [BK72]). Let $\mathcal{S}_*^c$ denote the category of connected simplicial pointed sets (i.e. objects with trivial $\pi_0 X = \pi_0|X|$, where $|-|$ denotes the geometric realization).

Definition 1.1.8 ([BK72] III 5.2). A simplicial set $X \in \mathcal{S}_*^c$ is called R-nilpotent (R solid ring) if

1) X is nilpotent, i.e. the action of $\pi_1 X$ on $\pi_n X$ is nilpotent for every $n \geq 1$. Recall that a group H acts on G nilpotently if there is a finite sequence of H-equivariant normal subgroups

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that for every $1 \leq i \leq n$, G_i/G_{i-1} is abelian and the induced H-action on it is trivial.

2) $\pi_i X$ is R-nilpotent for every $i \geq 1$.

Remarks 1.1.9. 1) $X \in \mathcal{S}_*^c$ is nilpotent if $|X|$ is simple (see [MP12] page 49), by the definition of homotopy groups of a simplicial set.

2) If $R = \mathbb{Z}$, $X \in \mathcal{S}_*^c$ is $\mathbb{Z}$-nilpotent if it is nilpotent.

We notice that a simplicial set $X \in \mathcal{S}_*$ is an H-space if and only if $|X| \in \mathrm{Top}_*$ is an H-space. Hence $X \in \mathcal{S}_*$ is an H-space if it is a monoid in $\mathrm{Ho}(\mathcal{S}_*)$. The same can be said concerning H-groups. Recall that Bousfield and Kan defined for every solid ring R, a functor $R_\infty : \mathcal{S} \to \mathcal{S}$, the so called Bousfield-Kan completion (see [BK72] I 4.2,[BK71] and [GJ09], for more on localization of topological spaces see [MP12], and [Ger73] for a comparison with +-construction). The main feature of this functor is that if a simplicial map $f : X \to Y$ induces an isomorphism on $H_*(-, R)$, then $R_\infty f$ is a weak equivalence (see [BK72] I 5.5). We recall briefly the definition of the Bousfield-Kan completion since it is not common to find it outside [BK72].

Definition 1.1.10 ([BK72] I 4.2). For a simplicial set X, we denote as RX the reduced free R-module generated by X. Remind that its n-simplices are finite formal linear combinations $\sum_i r_i x_i$ with $x_i \in X_n$, $r_i \in R$ and $\sum_i r_i = 1$. Then we have maps $\varphi : X \to RX$, $\varphi(x) = x$ and $\psi : RRX \to RX$ given by $\psi(\sum_i a_i(\sum_j b_{ij} x_{ij})) = \sum_{i,j}(a_i b_{ij}) x_{ij}$. Using these maps we can build a cosimplicial space $R^\bullet X$ by letting $(R^\bullet X)_n = R^{n+1} X$ and defining codegeneracy maps $s^i = R^i \psi R^{n-1-i}$ and coface maps $d^i = R^i \varphi R^{n+1-i}$. We then define $R_\infty X := \mathrm{Tot} R^\bullet X$ where we denoted as Tot the total space of a cosimplicial space, i.e. the simplicial set built out a cosimplicial set $X^\bullet$ by taking the cosimplicial mapping space $\mathfrak{Map}(\Delta^\bullet, X^\bullet)$ as defined in [BK72] I.3.3 for example. $R_\infty X$ is always a Kan complex (see [BK72] I 4.2).

Remark 1.1.11. From this definition it is easy to get al the needed functorialities.

Definition 1.1.12 ([BK72] I 5.1)**.** A simplicial set $X \in \mathcal{S}$ is called

1) R-good if the natural map $X \to R_\infty X$ induces an isomorphism on $H_*(-, R)$. R-bad if it is not R-good.

2) R-complete if the map $X \to R_\infty X$ is a weak equivalence.

Theorem 1.1.13. *([BK72] III 5.4 or [BK71] 4.2) Every R-nilpotent space $X \in \mathcal{S}_*^c$ is R-complete.*

If a topological space is locally path connected, hence its connected components are exactly the path connected components and in this case the space is homeomorphic to the disjoint union of its (path) connected components (for the behaviour of a space having the homotopy type of a CW complex under this respect, the reader may look at [FP90] Proposition 1.4.14 and 5.1.1). We define the connected components of a simplicial set as follows (in some literature the definition is slightly different, see [GJ09]). Let $v_\alpha \in X_0$, define X_α as the smallest subcomplex of X such that its zero skeleton consists of vertices w with the property $w \sim v_\alpha$ in $\pi_0|X|$. One can see that $\pi_0(X) = \text{colim}(\, X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0 \,)$. This definition gives us for every simplicial set X a decomposition

$$X \cong \bigsqcup_{v_\alpha \in \pi_0 X} X_\alpha =: \bigsqcup_\alpha X_\alpha$$

For the details of this decomposition and a more extensive discussion of the path components of a simplicial set the reader is referred to [Lur19, Subsection 00G5] where this decomposition appears as Proposition 1.1.6.13 (Tag 00GJ)

Theorem 1.1.14. *([BK72]I.7.1-7.5)*

1) *(R_∞ commutes with $\bigsqcup$) Let $X \in \mathcal{S}$, then the inclusion*

$$\bigsqcup_{v_\alpha \in \pi_0 X} R_\infty X_\alpha \hookrightarrow R_\infty X$$

is an homotopy equivalence.

2) *Let $X, Y \in \mathcal{S}$, then the projections of $X \times Y \to X$ and $X \times Y \to Y$ induce a homotopy equivalence of simplicial sets $R_\infty(X \times Y) \to R_\infty X \times R_\infty Y$ which has a natural left inverse φ that is associative, commutative and compatible with the triple structure of R_∞ (see [BK72]).*

3) *A multiplication* $m : X \times X \to X$ *in* $\mathcal{S}$ *induces a multiplication*

$$m' : R_\infty X \times R_\infty X \xrightarrow{\varphi} R_\infty(X \times X) \xrightarrow{R_\infty m} R_\infty X$$

Moreover if m *is associative (commutative or has a left right unit) then so does* m'.

4) Let $X \in \mathcal{S}_*$ *be an* H-*space (group, etc.), then* $R_\infty X \in \mathcal{S}_*$ *is also an* H-*space (group, etc.).*

We come now to our definition of completion for a topological space.

Definition 1.1.15. Let X be a CW-complex. For a solid ring R define $R_\infty X :=$ $|R_\infty \mathrm{Sing}(X)|$ where Sing is the singular functor right adjoint to the geometric realisation $|-|$ (recall from [GJ09] the adjunction $|-| \dashv \mathrm{Sing}$). We say that X is R-complete if the map

$$\Psi_X : X \xrightarrow{\ \sim\ } |\mathrm{Sing}(X)| \xrightarrow{\ |\psi_{\mathrm{Sing}(X)}|\ } |R_\infty \mathrm{Sing}(X)| = R_\infty X$$

is a weak equivalence, where $\psi_{\mathrm{Sing}(X)}$ is the canonical arrow $\mathrm{Sing}(X) \to R_\infty \mathrm{Sing}(X)$ and the first arrow is a homotopy inverse of the canonical weak equivalence of CW-complexes $|\mathrm{Sing}(X)| \to X$. If X is any topological space having the type of a CW complex X', we say that it is R-complete if X' is such.

Remark 1.1.16. Notice that a topological space X having the homotopy type of a CW complex is R-complete if and only if $\mathrm{Sing}(X)$ is R-complete.

Proposition 1.1.17. *Let be* X *any CW complex such that its path components are nilpotent or* $\mathbb{Z}$-*complete. Then* X *is* $\mathbb{Z}$-*complete as element in* Top.*As a consequence, this applies to* H-*groups and so to loop spaces* ΩY *for any given pointed CW complex* Y*. The same applies for any* X *simplicial set with* $\mathbb{Z}$-*complete connected components.*

Proof. The proof is an exercise in topology using the functoriality of the $\mathbb{Z}_\infty$ functor and 1.1.14. Just remark that the path connected component of the identity of an H-group is an H-group (this holds more generally for H-spaces, see [Dug66] page 383). $\square$

Remark 1.1.18. If X is a pointed CW complex then all the connected components $(\Omega X)_\alpha$ of its loopspace are homotopy equivalent by 1.1.3 and they are simple, hence they are $\mathbb{Z}$-complete since $\mathrm{Sing}(\Omega X)_\alpha$ have to be nilpotent. The same can be said for every H-group.

We immediately obtain the following corollaries

Corollary 1.1.19. *Given a a CW complex Y such that its path components are nilpotent or $\mathbb{Z}$-complete, we have that the completion map $\psi_Y : Y \to \mathbb{Z}_\infty Y$ is a weak equivalence.*

Corollary 1.1.20. *If X is an H-group (could be a topological space or a simplicial set) then it is $\mathbb{Z}$-complete.*

Arguing as above, one can also have the following (which is proved in [BK72] II.2.7)

Lemma 1.1.21. *Every simplicial R-module is R-complete.*

1.2 Completion for simplicial diagrams

The results of this section should be regarded as a variation of the methods contained in the article of Levine [Lev97], and they might be folklore as somewhat subsumed in [GS99] and [Sou85], although they never appeared in this form nor they were ever explicitly written down to the knowledge of the author.

Suppose I is a small category, and consider the category of simplicial presheaves on it, a.k.a. the category of functors $I^{op} \to \mathcal{S}$, denoted $\mathcal{S}^{I^{op}}$ or s$\mathbf{Pre}(I)$. We can put several model structures on this category (general references are [BK72], [Jar87], [Jar04], [Jar15], [Dug01b]).

- The Bousfield-Kan projective global model structure $\mathcal{P}$ ([BK72]) where weak equivalences are sectionwise weak equivalences, fibrations are sectionwise fibrations and cofibrations are induced by LLP (Left Lifting Property). This model structure is simplicial (see for example [GJ09]), with the standard internal mapping space $\mathfrak{Map}_{s\mathbf{Pre}(I)}(-, -)$ (whose simplices are $\mathfrak{Map}_{s\mathbf{Pre}(I)}(X, Y)_n = \mathrm{Hom}_{s\mathbf{Pre}(I)}(X \times \Delta^n, Y)$, see for example [Lev97]).

- The injective Heller global model structure $\mathcal{I}$: as before but in this case the cofibrations are defined sectionwise and fibrations by lifting property.

Both structures are definable in the same way on an arbitrary category of small simplicial diagrams $\mathcal{S}^I$ and they are Quillen equivalent. One can Bousfield localize model structures with respect to certain classes of arrows, see [Hir03] Chapter 3, and in particular in the case of simplicial sets it is very important to localize with respect to homology with coefficients in a group G as explained in the classical

and influential [Bou75] or in modern terms in [GJ09]. We are interested into $h_*^{\mathbb{Z}}$-localizations, i.e. into localization with respect to integral homology. For $\mathcal{S}$, the $h_*^{\mathbb{Z}}$-local model structure will be the one where an object is fibrant if and only if it is a fibrant simplicial set Y which is $h_*^{\mathbb{Z}}$-local ([GJ09] X Corollary 3.3), i.e. for every map of simplicial sets $f : X \to Z$ such that $H_*(f, \mathbb{Z})$ is an isomorphism, then the induced map $\mathfrak{Map}_{\mathcal{S}}(Z, Y) \to \mathfrak{Map}_{\mathcal{S}}(X, Y)$ is a weak equivalence. Remark that our definition of $h_*^{\mathbb{Z}}$-local coincides with the one in *loc. cit.* because of the properties of the simplicial model structures and of the Bousfield localisations (also see the beginning of the proof of Corollary 3.3 in *op. cit.* or 2.5 in [Bou97]). We start with the following

Lemma 1.2.1. *Let X be a simplicial set. Then $\mathbb{Z}_\infty X$ is $h_*^{\mathbb{Z}}$-local.*

This is proved in [GJ09], X Remark 3.7. The following lemma is then interesting

Lemma 1.2.2. *Any fibrant $\mathbb{Z}$-complete simplicial set X is also $h_*^{\mathbb{Z}}$-local.*

Proof. We need to prove that for every homologism $f : A \to B$ between simplicial sets the map $f^* : \mathfrak{Map}(B, X) \to \mathfrak{Map}(A, X)$ is a weak equivalence. Now recall that the map $\varphi : X \to \mathbb{Z}_\infty X$ is a weak equivalence between fibrant objects by assumption. Then the following commutes

$$\begin{array}{ccc} \mathfrak{Map}_{\mathcal{S}}(B, X) & \xrightarrow{\varphi_*} & \mathfrak{Map}_{\mathcal{S}}(B, \mathbb{Z}_\infty X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathfrak{Map}_{\mathcal{S}}(A, X) & \xrightarrow{\varphi_*} & \mathfrak{Map}_{\mathcal{S}}(A, \mathbb{Z}_\infty X) \end{array}$$

and the horizontal arrows are weak equivalences by [Hir03] Corollary 9.3.3. The right vertical map is a weak equivalence because $\mathbb{Z}_\infty X$ is $h_*^{\mathbb{Z}}$-local by 1.2.1 and so by the 2/3 property we conclude. $\qquad\square$

The following is a variation of a theorem by Levine.

Theorem 1.2.3. *Suppose Z is a $\mathcal{P}$-fibrant object of $\mathrm{s}\mathbf{Pre}(I)$ (or $\mathcal{S}^I$) such that for every $i \in I$, $Z(i)$ is a $h_*^{\mathbb{Z}}$-local simplicial set. Then given a map of $\mathcal{P}$-cofibrant objects $f : X \to Y$ such that for every $i \in I$ the map $f(i) : X(i) \to Y(i)$ induces an $H_*(-, \mathbb{Z})$-isomorphism, we have that the map*

$$f^* : \mathfrak{Map}(Y, Z) \to \mathfrak{Map}(X, Z)$$

is a weak equivalence.

Proof. For every objects i, j of I, since $Z(j)$ is $h_*^{\mathbb{Z}}$-local, we have that

$$f^*(i) : \mathfrak{Map}_{\mathcal{S}}(Y(i), Z(j)) \to \mathfrak{Map}_{\mathcal{S}}(X(i), Z(j))$$

is a weak equivalence. Hence the result follows from Corollary B.4 of [Lev97]. □

We now continue to focus on the $\mathcal{P}$-model structure on $\mathcal{S}^I$, unless otherwise stated.

Definition 1.2.4. An element X of $\mathcal{S}^I$ is called $\mathbb{Z}$-complete if for every $i \in I$, $X(i)$ is a $\mathbb{Z}$-complete simplicial set.

Lemma 1.2.5. *Given a $\mathbb{Z}$-complete simplicial diagram X, there exists a map $\varphi_X : X \to X_{h_*^{\mathbb{Z}}f}$ which is a sectionwise weak equivalence and such that $X_{h_*^{\mathbb{Z}}f}(i)$ is $h_*^{\mathbb{Z}}$-local for any $i \in I$.*

Proof. We define $X_{h_*^{\mathbb{Z}}f}$ by applying sectionwise the $\mathbb{Z}_\infty$-completion functor. This means that we obtain a family of canonical completion maps $\{\varphi_{X(i)} : X(i) \to \mathbb{Z}_\infty X(i)\}_{i \in \mathrm{Ob}(I)}$ which are weak equivalences since X is $\mathbb{Z}$-complete and that form the required natural transformation φ_X since $\mathbb{Z}_\infty$ is functorial, where we set for every $i \in \mathrm{Ob}(I)$, $X_{h_*^{\mathbb{Z}}f}(i) := \mathbb{Z}_\infty X(i)$. $X_{h_*^{\mathbb{Z}}f}$ is then sectionwise $h_*^{\mathbb{Z}}$-local by 1.2.1. □

Proposition 1.2.6. *Assume $X \in \mathrm{Ob}(\mathcal{S}^I)$ is $\mathcal{P}$-fibrant and $\mathbb{Z}$-complete. If $f : Y \to Y'$ is a map between cofibrant diagrams inducing $H_*(-, \mathbb{Z})$-isomorphisms sectionwise, one has that*
$$[Y', X]_{\mathcal{P}} \cong [Y, X]_{\mathcal{P}}$$

Proof. This follows from the characterization of $[-, -]_{\mathcal{P}}$ as $\pi_0 \mathfrak{Map}(-, -)$ and Theorem 1.2.3. □

We turn now to the local case. Consider the case where $I = \mathcal{C}$ is a Grothendieck site (i.e. a small category $\mathcal{C}$ together with the choice of a specified Grothendieck topology τ, see [SGA72] or [Jar15] for one discussion in this context). One can put model structures on the category $s\mathbf{Pre}(\mathcal{C})$ such that weak equivalences becomes local weak equivalences (local weak equivalences being defined in [Jar15] page 64, for example). The most known is the Jardine's injective local model structure (described in [Jar86] or [Jar15]), denote it as $\mathcal{I}^l$, or as $\mathcal{I}_\tau^l \mathcal{C}$ if the site is not clear, where all presheaves are cofibrant. The second one is the Blander's local projective model structure (described for example in [Dug01b] or [Bla01]), denote it as $\mathcal{P}^l$, or as $\mathcal{P}_\tau^l \mathcal{C}$ if the site is not clear. These two structures are homotopy equivalent (see [Dug01b]

for an explanation or [DHI04] for a full proof) and $\mathcal{P}^l$-fibrant objects are also sectionwise fibrant, since $\mathcal{P}^l$ is obtained by $\mathcal{P}$ by left Bousfield localizing at the class of all hypercovers. One then get

Corollary 1.2.7. *Let X be a simplicial presheaf which is $\mathcal{P}^l$-fibrant and $\mathbb{Z}$-complete. Hence if $f : Y \to Y'$ is a map between $\mathcal{P}$-cofibrant presheaves inducing $H_*(-,\mathbb{Z})$-isomorphisms sectionwise, one has*

$$[Y', X]_{\mathcal{P}^l} \cong [Y, X]_{\mathcal{P}^l} \cong [Y, X]_{\mathcal{I}^l} \cong [Y', X]_{\mathcal{I}^l}$$

Proof. By the properties of Bousfield localization (see for example [Dug01b] Def. 5.4), $\mathcal{P}$-cofibrant objects are also $\mathcal{P}^l$-cofibrant and $\mathcal{P}^l$-fibrant objects are in particular $\mathcal{P}$-fibrant. So $[Y, X]_{\mathcal{P}^l} \cong [Y, X]_{\mathcal{P}}$ (same for Y') and the last two isomorphisms follow from the fact that the local injective and the local projective model structures are Quillen equivalent together with 1.2.6. $\qquad\square$

Corollary 1.2.8. *Under the hypothesis of Corollary 1.2.7 the same conclusion holds even if $f : Y \to Y'$ is a map between $\mathcal{I}^l$-cofibrant presheaves (so any map) inducing $H_*(-,\mathbb{Z})$-isomorphisms sectionwise.*

Proof. In fact, we can take $\mathcal{P}$-cofibrant replacements $\varphi : \tilde{Y} \to Y$ and $\varphi' : \tilde{Y'} \to Y'$ for Y and Y' respectively (these are functorial).These arrows are sectionwise weak equivalences so one has that $\tilde{f} : \tilde{Y} \to \tilde{Y'}$ induce a sectionwise $H_*(-,\mathbb{Z})$-isomorphism. In fact if we consider the commutative diagram

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{Y'} \\
\varphi \downarrow \sim & & \sim \downarrow \varphi' \\
Y & \xrightarrow{f} & Y'
\end{array}
$$

we have that f, φ and φ' are sectionwise $H_*(-,\mathbb{Z})$-isomorphisms and $\tilde{f}$ has also this property because $H_*(-,\mathbb{Z})$-isomorphisms satisfiy the 2/3 property on simplicial sets. So now one can apply 1.2.7 and get $[\tilde{Y}, X]_{\mathcal{I}^l} \cong [\tilde{Y'}, X]_{\mathcal{I}^l}$, but φ and φ' are in particular local weak equivalences and so $[Y, X]_{\mathcal{I}^l} \cong [Y', X]_{\mathcal{I}^l}$. $\qquad\square$

1.3 Localization with respect to homology

The results of the previous section can tell us something about what it could happen if we could localize the model category $\mathcal{S}^I$ with the projective or injective model structure with respect to integral homology. For the injective model structure,

localisation with respect to homology theories is studied in [GJ98]. However since the projective analogue of their work has not been spelt out together with its relation with the the Bousfield-Kan completion, we find there is no harm to write down some remark. First, we need to remind some facts about the Bousfield localization of a simplicial model category. We follow [Lur09] Appendix A.3.7 that we find very well written.

Definition 1.3.1. Let $\mathcal{C}$ a simplicial (left proper combinatorial) model category with mapping space $\mathfrak{Map}_{\mathcal{C}}(-,-)$ and cofibrant and fibrant replacements $Q, R : \mathcal{C} \to \mathcal{C}$. We denote $\mathbf{R}\mathfrak{Map}_{\mathcal{C}}(-,-) := \mathfrak{Map}_{\mathcal{C}}(Q(-), R(-))$ and we note that if Z is fibrant, then $\mathbf{R}\mathfrak{Map}_{\mathcal{C}}(X, Z) = \mathfrak{Map}_{\mathcal{C}}(QX, Z)$ for any $X \in \mathcal{C}$. Assume S is a class of arrows in $\mathcal{C}$. We will say that

1) $Z \in \mathcal{C}$ is S-local if it is fibrant and for every $f : X \to Y$ in S the map $f^* : \mathbf{R}\mathfrak{Map}_{\mathcal{C}}(Y, Z) \to \mathbf{R}\mathfrak{Map}_{\mathcal{C}}(X, Z)$ is a weak equivalence.

2) A map $f : X \to Y$ is an S-equivalence if for every S-local-object Z the induced map $f^* : \mathbf{R}\mathfrak{Map}_{\mathcal{C}}(Y, Z) \to \mathbf{R}\mathfrak{Map}_{\mathcal{C}}(X, Z)$ is a weak equivalence

Notice we can replace weak equivalence with homotopy equivalence in the definitions before since all the mapping spaces we are considering are fibrant because of the properties of a simplicial model category. Indeed we could have defined in an equivalent way $\mathbf{R}\mathfrak{Map}_{\mathcal{C}}(-,-) := \mathrm{Ex}^{\infty}\mathfrak{Map}_{\mathcal{C}}(Q(-), R(-))$ as done by Lurie in [Lur09]. Hence, notice that our definition is equivalent with the one one finds in [Lur09] A.3.7 or to the one in [Hir03] Definition 3.1.4 provided in the last one one is careful to notice that we might take into account the properties of what Hirschorn calls *homotopy function complex* (notice that usually one forgets this subtelty because in many model categories all the objects are cofibrant: in general it is not the case). We can have the following theorem, which follows from [Lur09] Proposition A.3.7.3 and [Hir03] Theorem 4.1.1

Theorem 1.3.2. *Let $\mathcal{C}$ be a simplicial (left proper combinatorial) model category as in the previous definition. Then the left Bousfield localization of $\mathcal{C}$ at a certain small set of arrows S exists as a simplicial model category $S^{-1}\mathcal{C}$. If the left Bousfield localisation of a simplicial model category $S^{-1}\mathcal{C}$ at a class of arrows S exists then it is a simplicial model category with the same mapping space, the same cofibrations and with weak equivalence the S-equivalences. The fibrant objects in $S^{-1}\mathcal{C}$ are precisely the S-local objects.*

Now given the category $\mathrm{s}\mathbf{Pre}(\mathcal{C})$ for some small category $\mathcal{C}$ endowed with the global projective or injective model structure, we note that it is a simplicial

left proper (for the projective model structure this is Lemma 1.7 in [Bla01], see also the discussion in page 11 of [Isa05]) combinatorial model category by definition so that we might apply the previous theorem with S_h the class of maps which are sectionwise $H_*(-, \mathbb{Z})$ isomorphism, i.e. they are the sectionwise homologisms. A priori, since S_h is a class, $S_h^{-1}(\text{sPre}(\mathcal{C}))$ might not exist. For the global injective model structure, such localisation exists because of Theorem 1.1 in [GJ98] and that result (or rather its proof) should imply that such a localisation exists as well in the projective case, although this seems to be folklore and the author has not written down all the details of this. We then have as a consequence of Theorem 1.2.3 the following result

Theorem 1.3.3. *Consider the category* $\text{sPre}(\mathcal{C})$ *endowed with the global injective (projective) model structure and let be* S_h *the collection of the sectionwise homologisms. Then if the Bousfield localization* $S_h^{-1}(\text{sPre}(\mathcal{C}))$ *exists (this is true in the injective case) the injective (projective) fibrant diagrams which are levelwise* $h_*^{\mathbb{Z}}$*-local simplicial sets are fibrant objects in it. This implies that "* $\mathbb{Z}$*-complete fibrant diagrams satisfy* $h_*^{\mathbb{Z}}$*-descent". In this case, for projective* $\mathbb{Z}$*-complete diagrams, the assignment* $X \mapsto X_{h_*^{\mathbb{Z}}}$ *is an explicit fibrant replacement in* $S_h^{-1}(\text{sPre}(\mathcal{C}))$.

Proof. This is a simple application of the previous general theorem and of Theorem 1.2.3 for the projective case, while for the injective case if we consider X $\mathcal{I}$-fibrant diagram which is levelwise $h_*^{\mathbb{Z}}$-local and $f : Y \to Y'$ any homologism, to show that $f^* : \mathfrak{Map}(Y', X) \to \mathfrak{Map}(Y, X)$ is a weak equivalence we apply $\mathfrak{Map}(-, X)$ to the diagram contained in the proof of 1.2.8 and then we chase using 1.2.3 and [Hir03] Corollary 9.3.3. □

1.4 Bisimplicial sets and Homotopy theory

We need some facts concerning bisimplicial sets, whose category will be denoted as $\mathcal{S}^2$ (good references are [GJ09], [BK72],[BF78]) and concerning homotopy limits and colimits (same references and [Hir]). These are functors of the form $\Delta^{op} \times \Delta^{op} \to$ Sets. One can define the diagonal functor $\text{diag} : \mathcal{S}^2 \to \mathcal{S}$ by precomposing with the diagonal functor $\Delta^{op} \to \Delta^{op} \times \Delta^{op}$, $[n] \mapsto [n] \times [n]$. One can also see bisimplicial sets as simplicial objects in the category of simplicial sets, i.e. as functors $\Delta^{op} \to \mathcal{S}$. We can define a functor $T : \mathcal{S} \to \mathcal{S}^2$ which sends $X_\bullet$ to the functor $X_{\bullet\bullet} : \Delta^{op} \to \mathcal{S}$ which is obtained by seeing the n-simplices of $X_\bullet$ as constant simplicial sets and

linking them via the faces and the cofaces maps of $X_\bullet$. This means that T is defined by the assignment

$$(X_\bullet : \Delta^{op} \to \mathrm{Sets}) \mapsto (T(X_\bullet) : \Delta^{op} \to \mathrm{Sets} \to \mathcal{S})$$

where the map $\mathrm{Sets} \to \mathcal{S}$ sends a set to its associated constant simplicial set (in fancier words, $T(X_\bullet)$ is $X_\bullet \boxtimes \Delta^0$ as in [GG03]). In other words $T(X_\bullet)$ is the following bisimplicial set

$$
\begin{array}{ccccc}
X_0 & \xrightarrow{\ id\ } & X_0 & \xrightarrow{\ id\ } & X_0 \ldots \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X_1 & \xrightarrow{\ id\ } & X_1 & \xrightarrow{\ id\ } & X_1 \ldots \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
X_2 & \xrightarrow{\ id\ } & X_2 & \xrightarrow{\ id\ } & X_2 \ldots
\end{array}
$$

Also in this case, by inspection, $\mathrm{diag}(T(X_\bullet)) \cong X_\bullet$. We can now turn any simplicial set into an homotopy colimit by the following theorem, whose proof follows by the description of the homotopy colimits in terms of coend ([Hir]) and by the fact that the diagonal of a bisimplicial set is isomorphic to its total space (see literature before). This is the following theorem

Theorem 1.4.1. *Assume $X : \Delta^{op} \to \mathcal{S}$ is a bisimplicial set. Then $\mathrm{diag}(X)$ is weakly equivalent to $\underset{[n] \in \Delta^{op}}{\mathrm{hocolim}} X_n$*

Corollary 1.4.2. *Given a simplicial set $X_\bullet \in \mathcal{S}$, then $X_\bullet \simeq \underset{[n] \in \Delta^{op}}{\mathrm{hocolim}} X_n$, where the X_n are seen as constant simplicial sets and $\simeq$ denotes a weak equivalence.*

Proof. Then by the previous construction and 1.4.1 we get

$$X_\bullet \cong \mathrm{diag}(T(X_\bullet)) \simeq \underset{[n] \in \Delta^{op}}{\mathrm{hocolim}} T(X_\bullet)_\bullet \cong \underset{[n] \in \Delta^{op}}{\mathrm{hocolim}} X_n$$

$\square$

We can now pass to the category of small (bi)simplicial diagrams. Let I be a small category then we can define the categories of simplicial and bisimplicial objects on it. The arguments of the proof of the above results gives (see nlab)

Theorem 1.4.3. *Let $X_\bullet$ be an element in $\mathcal{S}^I$ (category of covariant functors $I \to \mathcal{S}$ seen as a model category with the global injective model structure). Then we have $X_\bullet \simeq \underset{[n] \in \Delta^{op}}{\mathrm{hocolim}} X_n$ where the X_n are seen as constant simplicial diagrams. The same is true if we Bousfield localise $\mathcal{S}^I$ at some class of morphisms.*

As a particular case, the previous theorem holds for simplicial presheaves, (by letting considering I^{op} instead of I). Assume now we are in this last case, i.e we consider elements X of $\mathbf{sPre}(I)$. If $\mathfrak{Map}(-,-)$ is the standard function complex in $\mathbf{sPre}(I)$, we recall that (see [BK72], XII) it takes homotopy colimits in the first variable to homotopy limits, i.e. $\mathfrak{Map}(\operatorname*{hocolim}_{i \in J} X_i, Y)$ is weakly equivalent to $\operatorname*{holim}_{i \in J^{op}} \mathfrak{Map}(X_i, Y)$ and these two simplicial spaces are actually isomorphic if one uses the definition of homotopy limits and colimits given in [Hir03] which we assume to adopt in our work. See [Hir03] 18.1.10 for the proof of this fact and 18.1.11 for a comparison with the definition of [BK72] (warning, there is a minor error in [BK72], see the reference just given for a discussion). Now, if R is a representable presheaf, then one can calculate that $\mathfrak{Map}(R, Y) \cong Y(R)$. Suppose we give to $\mathbf{sPre}(I)$ a simplicial model structure $\mathfrak{M}$ coming from some Bousfield localisation of the global injective model structure with $\mathfrak{Map}(-,-)$ as function complex. Under this assumptions we are then able to prove the following

Proposition 1.4.4. *Let $R_\bullet$ be a simplicial presheaf representing a simplicial object in I, i.e. such that R_n is representable for any $[n] \in \Delta^{op}$. Then if Y is $\mathfrak{M}$-fibrant we have*

$$[R_\bullet, Y]_{\mathfrak{M}} \cong \pi_0 \operatorname*{holim}_{[n] \in \Delta} Y(R_n)$$

Proof. By the previous reasonings using [Hir03] 9.3.3 we have

$$[R_\bullet, Y]_{\mathfrak{M}} \cong \pi_0 \mathfrak{Map}(R_\bullet, Y) \cong \pi_0 \mathfrak{Map}(\operatorname*{hocolim}_{[n] \in \Delta^{op}} R_n, Y)$$
$$\cong \pi_0 \operatorname*{holim}_{[n] \in \Delta} \mathfrak{Map}(R_n, Y) \cong \pi_0 \operatorname*{holim}_{[n] \in \Delta} Y(R_n)$$

$\square$

In particular this holds for $\mathcal{I}^l \mathcal{C}$, the local injective model category of simplicial presheaves over a site $\mathcal{C}$.

1.5 Remarks on classifying spaces in algebraic geometry

We partially follow the exposition of [Lev98] which is very clear. For this section only we let Sch_S to be the category of schemes over a given base S (we do not require any extra assumption on these schemes). Given a set X and a finite set T, we define $X^T := \operatorname{Hom}_{\operatorname{Sets}}(T, X) \cong \prod_{|T|} X$. We also define the simplicial set $\mathrm{E}X : \Delta^{op} \to \operatorname{Sets}$, $[n] \mapsto X^{|n|}$ with the obvious choices for the faces and the degeneracies (partial diagonals and projections). This construction, if X is a group, coincides with the

usual WG construction as detailed in [GJ09] V 4, pag. 269 (this construction in fact assumes G to be a group). In fact if one consider G as a constant simplicial group, he gets that $\mathrm{W}G_n = \prod_{|[n]|} G = \mathrm{E}G_n$ and the same faces and degeneracies. The choice of a point gives a contraction of $\mathrm{E}X$ so that it is contractible as simplicial set (this is detailed in [Lev98] page 357). One call BG the simplicial set $\mathrm{E}G/G$ where the action of G is given by right (or left) multiplication, e.g. $G \times \mathrm{E}G_n \to \mathrm{E}G_n$, $(h,(g_n,...,g_0)) \mapsto (hg_n,...,g_0)$ (one can also see this as an action of the *simplicial* constant group G on $\mathrm{E}G$, but we do not need this generality). The space BG can be also described as $\overline{\mathrm{W}}G$ as in [GJ09] pag.269 or [May92] pag. 87, and it is a simplicial Eilenberg-Mac Lane object of type $K(G,1)$. We remember that $\overline{\mathrm{W}}G_n \cong G^n(:= \underbrace{G \times_S \cdots \times_S G}_{n-1 \text{ times}})$. For a more general situation (G a simplicial group) its structure is detailed in [May92] pag.87. It is then a simple comparison of the definitions that allows us to see that the simplicial set $\overline{\mathrm{W}}G$ is isomorphic to NG, the nerve of G seen as a category of one element. Assume one has a sequence $G_0 \hookrightarrow G_1 \hookrightarrow ...$ of group homomorphism inclusions, and denote $G := \mathop{\mathrm{colim}}\limits_{n\in\mathbb{N}} G_n = \bigcup_{n\in\mathbb{N}} G_n$. Consider now $\overline{\mathrm{W}}G$. One has

$$(\overline{\mathrm{W}}G)_i = \prod_{|i|} G = \prod_{|i|} \mathop{\mathrm{colim}}_{n\in\mathbb{N}} G_n \cong \mathop{\mathrm{colim}}_{n\in\mathbb{N}} \prod_{|i|} G_n \cong \mathop{\mathrm{colim}}_{n\in\mathbb{N}} (\overline{\mathrm{W}}G_n)_i$$

and using this it is possible to check that $\overline{\mathrm{W}}G \cong \mathop{\mathrm{colim}}\limits_{n\in\mathbb{N}} \overline{\mathrm{W}}G_n$. The same applies to the nerve, i.e. N$G \cong \mathop{\mathrm{colim}}\limits_{n\in\mathbb{N}} \mathrm{N}G_n$. Note also that the nerve functor preserves directed colimits since the simplices Δ^n are compact objects in Cat. If one has a sequence of groups as above, one can invoke standard arguments on filtered homotopy colimits of simplicial sets to conclude that $\mathop{\mathrm{hocolim}}\limits_{n\in\mathbb{N}} \mathrm{N}G_n \simeq \mathop{\mathrm{colim}}\limits_{n\in\mathbb{N}} \mathrm{N}G_n$ (weak equivalence). In fact filtered colimits of simplicial sets are homotopy equivalent to their homotopy colimits via the standard map (this is proved in [Hir], Proposition 14.11, see also [BK72]). So in this situation one gets

Proposition 1.5.1. *Let $G_0 \hookrightarrow G_1 \hookrightarrow ...$ a sequence of group homomorphisms inclusions, then*

$$\mathrm{N}G \cong \overline{\mathrm{W}}G \cong \mathop{\mathrm{colim}}_{n\in\mathbb{N}} \mathrm{N}G_n \simeq \mathop{\mathrm{hocolim}}_{n\in\mathbb{N}} \mathrm{N}G_n$$

If we now consider an S-scheme X and a finite set T, the representable functor

$$\mathrm{Hom}_{\mathrm{Sch}_S}(-,X)^T : \mathrm{Sch}_S \to \mathrm{Sets}$$

is represented by the scheme $X \underbrace{\times_S \cdots \times_S}_{|T| - 1 \text{ times}} X$, we denote this representing object

by $X^{T/S}$. One thus get a simplicial scheme $X^{(-)/S} : \Delta^{op} \to \mathrm{Sch}_S$ called $\mathrm{E}X$ and similarly to the sets case, if one is given a group scheme $X = G$, it is possible to form $\mathrm{E}G$ and $\mathrm{B}G$ as simplicial schemes as above. By switching from simplicial schemes to simplicial presheaves on $\mathcal{C} = \mathrm{Sch}_S$ one sees that the construction of $\mathrm{E}X$ here gives rise to a simplicial presheaf which is represented by a simplicial scheme, same for $\mathrm{E}G$ and $\mathrm{B}G$. To obtain a description of an analogue of 1.5.1, one notes that what is a subgroup injection $G \hookrightarrow H$ for groups corresponds, given two group schemes F, H, to a map of presheaves of groups $F \to H \in \mathbf{Pre}(\mathrm{Sch}_S, \mathrm{Grp})$ (the latter being the category of presheaves valued in groups) such that for any S-scheme A, $F(A) \hookrightarrow H(A)$ is a subgroup or in other terms $F \hookrightarrow H$ is a closed S-immersion homomorphism. Hence given a sequence of closed subgroups embeddings of S-group schemes $G_0 \hookrightarrow G_1 \hookrightarrow ...$ one gets for every $A \in \mathrm{Sch}_S$ a sequence of subgroups $G_0(A) \hookrightarrow G_1(A) \hookrightarrow ....$ The functor $\mathrm{B}G : \mathrm{Sch}_S \to \Delta^{op}\mathrm{Sets}$, $A \mapsto \mathrm{B}G(A)$ (obtained as $\mathrm{Sch}_S \xrightarrow{G} \mathrm{Grp} \xrightarrow{\mathrm{B}} \Delta^{op}\mathrm{Sets}$) is represented by the simplicial scheme $\mathrm{B}G$ for any group scheme G (which has $\mathrm{B}G_n \cong G \underbrace{\times_S \cdots \times_S}_{n - 1 \text{ times}} G$), see [Lev98] for this part).

Remember that filtered colimits of simplicial sets are homotopy equivalent to their homotopy colimits via the standard map. This is true for filtered colimits in any combinatorial model category, because of [Dug01a] Proposition 7.3. Then one has, forming the homotopy colimit, or the usual colimit of a sequence of group schemes as above (i.e. by considering the $\mathrm{B}G_i$ as simplicial presheaves and taking the colimits sectionwise obtaining an ind-object, see [SGA72] for this last notion) the following result

Proposition 1.5.2. *If $G_0 \hookrightarrow G_1 \hookrightarrow ...$ is a sequence of S-group schemes as above (or simplicial presheaves represented by them) then one has*

$$\mathrm{B}G \cong \mathrm{B}\underset{n \in \mathbb{N}}{\mathrm{colim}}G_n \cong \underset{n \in \mathbb{N}}{\mathrm{colim}}\mathrm{B}G_n \simeq \underset{n \in \mathbb{N}}{\mathrm{hocolim}}\mathrm{B}G_n$$

where the last equivalence is a weak equivalence.

Remark 1.5.3. A way to prove that $\underset{n \in \mathbb{N}}{\mathrm{colim}}\mathrm{B}G_n \simeq \underset{n \in \mathbb{N}}{\mathrm{hocolim}}\mathrm{B}G_n$ in the previous Proposition is to notice that the diagram of which we are taking the colimit is a tower of cofibrations where each object is cofibrant and such diagrams are cofibrant so we do not need to take any cofibrant replacement to compute their homotopy colimit.

Consider now the sequence

$$\mathbb{G}_{m,S} \hookrightarrow \mathrm{GL}_{2,S} \hookrightarrow ...$$

which for every $A \in \mathrm{Sch}_S$ translates into (see also [GS99] page 37)

$$\Gamma(A, \mathcal{O}_A)^* \hookrightarrow \mathrm{GL}_2(\Gamma(A, \mathcal{O}_A)) \hookrightarrow \mathrm{GL}_3(\Gamma(A, \mathcal{O}_A)) \hookrightarrow ...$$

Then by 1.5.1, $\mathrm{BGL}(A) \cong \underset{n \in \mathbb{N}}{\mathrm{colim}} \mathrm{BGL}_n(A)$ for any commutative ring A, so because the geometric realization functor commutes with colimits (being a left adjoint) one gets that $\mathrm{BGL}(A)$ is homeomorphic to $\underset{n \in \mathbb{N}}{\mathrm{colim}} \mathrm{BGL}_n(A)$ if by $\mathrm{B}(-)$ we denote the topological classifying space (say $|-| \circ \mathrm{N}$). By 1.5.2 one also gets the same results for the ind-object BGL and the simplicial schemes BGL_n, where for example GL_n represents the functor $A \mapsto \mathrm{GL}_n(\Gamma(A, \mathcal{O}_A))$. In particular

$$\mathrm{BGL} \cong \underset{n \in \mathbb{N}}{\mathrm{colim}} \mathrm{BGL}_n \simeq \underset{n \in \mathbb{N}}{\mathrm{hocolim}} \mathrm{BGL}_n \tag{1.1}$$

We explicitly point out that one can do the same for symplectic linear groups. In fact if we denote as $\mathrm{Sp}_{n,S}$ the scheme representing the functor which associates to a scheme X the set of rank $2n$ symplectic matrices with coefficients in $\Gamma(X, \mathcal{O}_X)$, we have a chain of embeddings $\mathrm{Sp}_{1,S} \hookrightarrow \mathrm{Sp}_{2,S} \hookrightarrow ...$ as well and taking colimits we have

$$\mathrm{BSp} \cong \underset{n \in \mathbb{N}}{\mathrm{colim}} \mathrm{BSp}_n \simeq \underset{n \in \mathbb{N}}{\mathrm{hocolim}} \mathrm{BSp}_n \tag{1.2}$$

as in the case of the general linear group.

1.5.1 Relation with the classical topological notion of classifying space

In this section we freely follow [BS08]. The materian of this section is not needed in the thesis and we include it only for completeness, therefore the reader can skip this section if he wants. In topology the classifying space of a (discrete or topological) group G is given by a weakly contractible space EG together with a free (right) action of G on it such that, denoting as $BG := EG/G$ we obtain a map $\pi : EG \to BG$ with the property that, for every topological space X, every principal G bundle over it is obtained as pullback of the bundle π. In other words there is a bijection between homotopy classes of maps $X \to BG$ and homotopy equivalent classes of principal G bundles over X. It can be proved that all the bundles satisfying this

property are homotopy equivalent so that the classifying space of a group is unique up to homotopy equivalence. The first notable construction of a classifying space was given by Milnor and its construction became a standard tool, referred as "the Milnor's construction", which we will not recall here (see [Sel97] page 101 for the details). However we give here a quick sketch of the proof that, for *discrete* topological groups, the BG we have defined is indeed a classifying space, leaving some details to the reader. In the previous section we gave the following definition

Definition 1.5.4. Starting with a group G, we define its *simplicial Bar construction* as follows. One defines the simplicial set EG using the assignment $[n] \mapsto \mathrm{Hom}_{\mathrm{Sets}}([n], G) \cong G^{n+1}$ which is a simplicial group. G acts on EG by right multiplication on the first factor so that we can take the quotient EG/G for this action to get a simplicial set $B'G$ and a map $\psi : EG \to B'G$ between them.

Example 1.5.5. Given a group G, define the category $\mathfrak{E}_G$ as the groupoid having as objects the elements of G and as Hom-sets $\mathrm{Hom}_{\mathfrak{E}_G}(g, h) = g^{-1}h$. One can also consider G as a category $\mathfrak{B}_G$ with on object e and $\mathrm{Hom}_G(e, e) = G$. We have a functor $\varphi : \mathfrak{E}_G \to \mathfrak{B}_G$ given by $g \mapsto *$ on objects and $g^{-1}h \mapsto g^{-1}h$ on arrows. We can prove that $N(\mathfrak{E}_G) \cong EG$ and that $N(\mathfrak{B}_G) \cong B'G$. Moreover, $N(\varphi)$ is isomorphic to ψ of the previous definition.

We now remind the following simple proposition

Proposition 1.5.6. *Suppose we have two functors $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation τ between them. Then τ induces an homotopy between the maps $B(f)$ and $B(G)$. In particular we have that an equivalence of categories $\mathcal{C} \simeq \mathcal{D}$ induces an homotopy equivalence $B(\mathcal{C}) \simeq B(\mathcal{D})$.*

Proposition 1.5.7. *The geometric realization of $\psi : EG \to B'G$ (or $N(\varphi) : N(\mathfrak{E}_G) \to N(\mathfrak{B}_G)$) characterizes BG (intended as $|N(\mathfrak{B}_G)|$) as a classifying space of G.*

Proof. We will only explain how to see that $N(\mathfrak{E}_G)$ is weakly contractible, leaving all the other details to the reader. We consider the trivial category $*$ having only one object $\bullet$ and one arrow $id_\bullet$. We define the functor $\eta : * \to \mathfrak{E}_G$ via $\bullet \mapsto 1$ and $id_\bullet \mapsto id_1$. One easily check that this functor is an equivalence of categories. Using the fact that the geometric realization of the nerve of $*$ is the point, we conclude using Proposition 1.5.6. $\square$

1.6 Application to BGL and BSp

The results of this section concerning K-theory might be seen as folklore from some experts. Indeed although the author was not able to find a precise reference, it seems that they are "in the air" in the articles [Sou85], [GS99].

Let now consider a site $\mathcal{C} = (\mathrm{Sch}_S, \tau)$, where Sch_S is some small category of schemes over some base S (the set theoretical issues will be fixed once we bound the size of the underlying topological spaces of the schemes involved by a large cardinal $\mathfrak{k}$, see [Jar15] example 3.5, for example) and τ is a Grothendieck topology (with enough points for simplicity). We consider the simplicial presheaves BGL and BSp over this site, i.e. the simplicial presheaves defined by

$$\mathrm{Sch}_S \to s\mathrm{Sets}, \qquad U \mapsto \mathrm{BG}(\Gamma(U, \mathcal{O}_U)) \quad G = \mathrm{GL}, \mathrm{Sp}$$

as in the previous section. Notice that these presheaves are sectionwise fibrant since the nerve of any groupoid is fibrant (see [GJ09]I 3.5). Denote as BGL^+, BSp^+ the presheaves

$$\mathrm{Sch}_S \to \mathrm{Sets}, \qquad U \mapsto \mathrm{BG}^+(\Gamma(U, \mathcal{O}_U))$$

where $+$ denotes the application of the $\mathbb{Z}_\infty$ completion functor. Indeed these presheaves evaluated at any scheme give us a simplicial set which has the same homotopy groups of the one one could get using the Quillen's $+$ construction relative to the commutator subgroup (references are [Wei13], [Aut], [Ros94], [Hat02]; for a discussion of the link between $+$ and $\mathbb{Z}_\infty$ at the level of spaces see [Wei13] or better [Ger73]). However since the reference given consider just the case of BGL, it makes no harm to remind why this is true. For any ring R one can in fact consider the following commutative diagram (with $+$ we mean the Quillen's construction here)

$$
\begin{array}{ccc}
\mathrm{BG}(R) & \xrightarrow{\ f\ } & \mathrm{BG}^+(R) \\
\downarrow & & \downarrow \\
\mathbb{Z}_\infty(\mathrm{BG}(R)) & \xrightarrow{\mathbb{Z}_\infty(f)} & \mathbb{Z}_\infty(\mathrm{BG}^+(R))
\end{array}
$$

with G being equal to GL or Sp and f being the canonical map given with the Quillen's $+$ construction. Then f induces an isomorphism on integral homology so $\mathbb{Z}_\infty(f)$ is a weak equivalence. But now $\mathrm{BG}^+(R)$ is a connected H-space (this can be also seen as a consequence of the general machinery in [Sch17] Appendix A) so it is simple then 1.1.17 applies and the right vertical map is a weak equivalence, so that we get the claim. Using the previous diagram one has that this simplicial presheaves

comes together with a map $i : \mathrm{BG} \to \mathrm{BG}^+$ which induces an $H_*(-, \mathbb{Z})$-isomorphisms sectionwise. This is also true for $(\mathrm{BG})^n$ and $(\mathrm{BG}^+)^n \simeq (\mathrm{BG}^n)^+$ because of Theorem 1.1.14 2). So 1.2.8 applies and we get

Proposition 1.6.1. *Let $\mathcal{C} = (\mathrm{Sch}_S, \tau)$ be a site as above and let be X a $\mathcal{P}^l$-fibrant simplicial presheaf which is $\mathbb{Z}$-complete. Then $[(\mathrm{BG}^+)^n, X]_{\mathcal{I}_\tau^l \mathcal{C}} \cong [\mathrm{BG}^n, X]_{\mathcal{I}_\tau^l \mathcal{C}}$ for any $n \in \mathbb{N}$ where G can be both GL or Sp.*

Proof. Indeed, since X is $\mathcal{P}^l$-fibrant we can apply 1.2.8. $\qquad\square$

Remark 1.6.2. The previous proposition is the generalization at level of presheaves of the classical universal property of the $+$ construction. Moreover, the same results hold for more general algebraic groups, but we will not need that.

1.7 Application to (hermitian) K-theory

Fix now the site $\mathcal{C} = (\mathrm{Sch}_S, \tau)$ where τ is the Zariski topology, S is a noetherian scheme of finite dimension and Sch_S is the category of schemes of finite type over S admitting an ample family of line bundles (see Section 2.1.1 for this notion). In this case the $\mathcal{P}^l$-fibrant simplicial presheaves are the sectionwise fibrant presheaves which satisfy the Brown-Gersten property (because of [BG73], [Dug01b], [DHI04]; see discussion in Appendix E.1). We define the K-theory presheaf on $\mathcal{C}$ following Quillen as the presheaf $U \mapsto \Omega\mathrm{Ex}^\infty(Q\mathrm{Vect}(U))$, where for any scheme U we have assigned to it $\mathrm{Vect}(U)$ the category of vector bundles over U, assignment that can be made functorial using big vector bundles as explained in the appendix, and Q denotes the Quillen's Q-construction (see [Qui73], [Wei13] or [Sch11] among the others, notice that the resulting simplicial set is naturally pointed). For a pointed simplicial set X, ΩX is the simplicial loop space defined in [GJ09] page 31 (this can be defined also as $\mathfrak{Map}_{\mathcal{S}_*}((\mathbb{S}^1, *), X)$ using the pointed mapping space). Moreover, we have chosen the Ex^∞ as functorial fibrant replacement instead of $\mathrm{Sing} \circ |-|$ because it does not change the set of zero simplices. Since all the schemes we are considering are divisorial, this is a good definition of K-theory because of the seminal work of Thomason and Trobaugh as explained in the appendix. Note this presheaf can be pointed in a natural way (sectionwise there is a natural choice, see [Sch11] pag.173) and it is going to be an H-group in every homotopy category we will consider. As a side fact, remark that if we consider the projective global model structure on simplicial presheaves, the K-theory simplicial presheaf we just defined is actually the image of the simplicial presheaf $U \mapsto \mathrm{Ex}^\infty(Q\mathrm{Vect}(U))$ under the loop functor as defined in [Hov99] Definition 6.1.1, as it can be checked using the

simplicial model structure that we have for simplicial presheaves (see [Jar15] pages 99 and 106 for some details) and [GJ09] II Lemma 2.3. Now, by its definition K is $\mathcal{P}$-fibrant (remind that the loop space of a fibrant simplicial set is fibrant because of its definition and [GJ09] Lemma I 7.5) and moreover by 1.1.20, K is $\mathbb{Z}$-complete in the sense of Definition 1.2.4. In addition, the discussion contained in Appendix E.2 tells us that K has the Brown Gersten property in $\mathcal{C}$, and so K is also $\mathcal{P}^l$-fibrant. Hence 1.6.1 applies and we get the following

Proposition 1.7.1. $[(\mathrm{BGL}^+)^n, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [\mathrm{BGL}^n, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S}$ *for any* $n \in \mathbb{N}$.

Remark 1.7.2. One could be tempted to to use the classical universal property of the plus construction to prove the previous proposition directly using the fact that K-theory is an H-space. Indeed it seems that this fact is somewhat subsumed into the papers of Gillet, Soulé and Loday [GS99], [Lod76] and [Sou85] and as we said is basically contained in the paper of Levine [Lev97]. However, Levine does not make any use of local homotopy theory nor he uses the Bousfield-Kan completion and the former authors do not write down statements as general as ours nor they write down details or the former theorem so we think that we are doing some service to the community by writing down this "folklore".

We now pass to discuss the analogue of the previous proposition for Symplectic hermitian K-theory.

1.7.1 The case of Symplectic hermitian K-theory

In this section we will assume that we have as a base scheme S a scheme where $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ although recent progresses indicate that this might be unecessary. We will stick to the category Sch_S of divisorial schemes of finite type over a noetherian S. As in the case of K-theory, we can define a simplicial presheaf representing hermitian n-shifted hermitian K-theory. Denote it as $\mathrm{GW}^{[n]}$, $\mathrm{GW}^{[2]}$ being symplectic hermitian K-theory. Roughly speaking one start from a presheaf of dg categories with weak equivalences and dualities and then one applies the construction made explicit in [Sch17] 9.1, see Appendix B.5. What is relevant to our discussion is that we end up with a simplicial presheaf which is $\mathcal{P}$-fibrant and $\mathbb{Z}$-complete since it is an H-group for any n as remarked for example in [Sch10] 2.7 Remark 2. Moreover, this presheaf also satisfies Zariski and Nisnevich descent and it is $\mathbb{A}^1$-homotopy invariant on regular schemes([Sch17] Theorems 9.7, 9.8 and 9.9, see also Appendix B.5). So in particular it is $\mathcal{P}^l$-Zariski fibrant. We then have because of 1.6.1

Proposition 1.7.3. $[(\mathrm{BSp}^+)^m, \mathrm{GW}^{[n]}]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [\mathrm{BSp}^m, \mathrm{GW}^{[n]}]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S}$ *for any* $m \in \mathbb{N}$, $n \in \mathbb{Z}$. *In particular this holds for* $n = 2$.

1.8 Endomorphisms of K-theory: Part I

In this section, we denote as $\mathcal{H}(S)$ the unstable motivic homotopy category over a regular noetherian scheme S (i.e. the homotopy category of simplicial presheaves over Sm/S with the Nisnevich $\mathbb{A}^1$-localized injective local model structure) as defined in the seminal [MV99] for example. See also appendix E for more. We let Sm/S be the category of divisorial smooth schemes of finite type over S. We keep the previous notations concerning model structures, and we let Sch_S be the category of divisorial S-schemes of finite type. Accordingly we will denote, for example, as $\mathcal{I}^l_{Zar}\mathrm{Sch}_S$ the category of simplicial presheves over the Zariski site of S-schemes as above equipped with the local injective model structure. We write in that expression Nis instead of Zar to denote the use of the Nisnevich topology. Let K be the K-theory simplicial presheaf defined as in 1.6 (our schems are divisorial so it does not matter which model of K-theory we use). One has that there is a local Zariski-weak equivalence $K \simeq \mathbb{Z} \times \mathrm{BGL}^+$ as proved in [GS99] for example. This is nothing more than the fact that K_0 of a local ring is $\mathbb{Z}$ and the Quillen's theorem $+ = Q$. The same holds for the Nisnevich topology. Our aim is to show that

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{I}^l_{Zar}\mathrm{Sch}_S)}(K, K) \cong \mathrm{Hom}_{\mathcal{H}(S)}(K, K)$$

It suffices to prove that $[\mathrm{BGL}^+, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [\mathrm{BGL}^+, K]_{\mathcal{H}(S)}$ by a direct application of 1.2.8 or because $K \simeq \mathbb{Z} \times \mathrm{BGL}^+ \cong \mathrm{II}_{n\in\mathbb{Z}}\mathrm{BGL}^+$ in the homotopy categories considered (these three objects are in fact locally weakly equivalent) and disjoint unions (finite products) of cofibrant (fibrant) objects are still coproducts (or finite products) in these homotopy categories so that Hom takes coproducts to products (more is true, see [Hov99] Example 1.3.11).

Proposition 1.8.1. *Under the previous notation if F is any $\mathcal{I}$-fibrant simplicial presheaf in $\mathrm{s}\mathbf{Pre}(\mathrm{Sm}/S)$*

$$\mathfrak{Map}_{\mathrm{s}\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{BGL}, F) \simeq \lim_{n\in\mathbb{N}^{op}} \operatorname*{holim}_{i\in\Delta} F(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

$$\simeq \operatorname*{holim}_{n\in\mathbb{N}^{op}} \operatorname*{holim}_{i\in\Delta} F(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

in particular

$$[\mathrm{BGL}^+, K]_{\mathcal{H}(S)} \cong [\mathrm{BGL}, K]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S} \cong \pi_0 \operatorname*{holim}_{n\in\mathbb{N}^{op}} \operatorname*{holim}_{i\in\Delta} K(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

$$\cong \pi_0 \lim_{n\in\mathbb{N}^{op}} \operatorname*{holim}_{i\in\Delta} K(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

We have denoted as $\cong$ the isomorphisms and as $\simeq$ the weak equivalences.

Proof. To start with we prove the first result. We have

$$\mathfrak{Map}_{\mathbf{sPre}(\mathrm{Sm}/S)}(\mathrm{BGL}, F) \cong \mathfrak{Map}_{\mathbf{sPre}(\mathrm{Sm}/S)}(\underset{n \in \mathbb{N}}{\mathrm{colim}}\mathrm{BGL}_n, F)$$

$$\cong \underset{n \in \mathbb{N}^{op}}{\lim} \mathfrak{Map}_{\mathbf{sPre}(\mathrm{Sm}/S)}(\mathrm{BGL}_n, F)$$

$$\simeq \underset{n \in \mathbb{N}^{op}}{\lim} \underset{i \in \Delta}{\mathrm{holim}}\, F(\mathrm{GL}_n \times_S^i \mathrm{GL}_n)$$

We used in the first isomorphism (1.1), in the second the fact that $\mathfrak{Map}$ takes colimits to limits ([Hir03] 9.2.2) and the third weak equivalence comes from the description of classifying space carried on in Section 4, in particular the fact that BGL_n is a simplicial scheme (recall that GL_n is a *smooth* scheme over S because of [DG11] 4.5 or [GD71] I Proposition 9.6.4, and $(\mathrm{BGL}_n)_i \cong \mathrm{GL}_n \times_S^i \mathrm{GL}_n)$, 1.4.3, [Hir03] Theorem 18.1.10 and the properties of representable presheaves. The fact that

$$\mathfrak{Map}_{\mathbf{sPre}(\mathrm{Sm}/S)}(\mathrm{BGL}, F) \simeq \underset{n \in \mathbb{N}^{op}}{\mathrm{holim}}\, \underset{i \in \Delta}{\mathrm{holim}}\, F(\mathrm{GL}_n \times_S^i \mathrm{GL}_n)$$

follows from a similar calculation (use again (1) and the behaviour of $\mathfrak{Map}$ with respect to homotopy colimits). We turn now to the second assertion. It can be proved (see [MV99]) that $\mathrm{BGL}^+ \simeq^{\mathbb{A}^1} \mathrm{BGL}$. So $[\mathrm{BGL}^+, K]_{\mathcal{H}(S)} \cong [\mathrm{BGL}, K]_{\mathcal{H}(S)}$. Since K is $\mathbb{A}^1$-homotopy invariant and satisfies Nisnevich descent, i.e. there is a sectionwise weak equivalence $d: K \to K_f$ with K_f $\mathcal{I}_{Nis}^l$-fibrant we have $[\mathrm{BGL}, K]_{\mathcal{H}(S)} \cong [\mathrm{BGL}, K]_{\mathcal{I}_{Nis}^l \mathrm{Sm}/S}$. We can also write

$$[\mathrm{BGL}, K]_{\mathcal{I}_{Nis}^l \mathrm{Sm}/S} \cong [\underset{n \in \mathbb{N}}{\mathrm{hocolim}}\mathrm{BGL}_n, K]_{\mathcal{I}_{Nis}^l \mathrm{Sm}/S}$$

$$\cong [\underset{n \in \mathbb{N}}{\mathrm{hocolim}}\mathrm{BGL}_n, K_f]_{\mathcal{I}_{Nis}^l \mathrm{Sm}/S}$$

$$\cong \pi_0 \mathfrak{Map}_{\mathbf{sPre}(\mathrm{Sm}/S)}(\underset{n \in \mathbb{N}}{\mathrm{hocolim}}\mathrm{BGL}_n, K_f)$$

$$\cong \pi_0 \underset{n \in \mathbb{N}}{\mathrm{holim}}\, \underset{i \in \Delta}{\mathrm{holim}}\, K_f(\mathrm{GL}_n \times_S^i \mathrm{GL}_n)$$

$$\cong \pi_0 \underset{n \in \mathbb{N}}{\mathrm{holim}}\, \underset{i \in \Delta}{\mathrm{holim}}\, K(\mathrm{GL}_n \times_S^i \mathrm{GL}_n)$$

Where we have used the simplicial model structure on $\mathcal{I}_{Nis}^l \mathrm{Sm}/S$ together with the result just proved and the fact that d is a sectionwise fibrant replacement (and a map between $\mathcal{P}$-fibrant presheaves). A minor modification in the argument

together (use differently (1.1) in the first step) with the result just proved for $\mathfrak{Map}$ gives us

$$[\mathrm{BGL}^+, K]_{\mathcal{H}(S)} \cong [\mathrm{BGL}, K]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S} \cong \pi_0 \varprojlim_{n \in \mathbb{N}^{op}} \operatorname*{holim}_{i \in \Delta} K(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

$$\square$$

The following is then very similar to prove

Proposition 1.8.2. *If F is any $\mathcal{I}$-fibrant simplicial presheaf in $\mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)$*

$$\mathfrak{Map}_{\mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)}(\mathrm{BGL}, F) \simeq \varprojlim_{n \in \mathbb{N}^{op}} \operatorname*{holim}_{i \in \Delta} F(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

$$\simeq \operatorname*{holim}_{n \in \mathbb{N}^{op}} \operatorname*{holim}_{i \in \Delta} F(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

Moreover

$$[\mathrm{BGL}^+, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [\mathrm{BGL}, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong \pi_0 \operatorname*{holim}_{n \in \mathbb{N}} \operatorname*{holim}_{i \in \Delta} K(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

$$\cong \pi_0 \varprojlim_{n \in \mathbb{N}} \operatorname*{holim}_{i \in \Delta} K(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

Proof. The proof of the first assertion goes exactly as the proof of the previous proposition. Going to the proof of the second part we have from 1.7.1 that $[\mathrm{BGL}^+, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong$ $[\mathrm{BGL}, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S}$. Then, since K also satisfies descent for the Zariski topology there is a sectionwise $\mathcal{I}^l_{Zar}$-fibrant replacement $d : K \to K_f$ as above we can repeat almost verbatim the argument of 1.8.1 to get

$$[\mathrm{BGL}^+, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong \pi_0 \operatorname*{holim}_{n \in \mathbb{N}} \operatorname*{holim}_{i \in \Delta} K(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

$$\cong \pi_0 \varprojlim_{n \in \mathbb{N}} \operatorname*{holim}_{i \in \Delta} K(\mathrm{GL}_n \times^i_S \mathrm{GL}_n)$$

$$\square$$

Remark 1.8.3 (important remark)**.** We could actually prove the previous two propositions in a more general situation, with exactly the same argument. In 1.8.1 we can substitute BGL with any filtered colimit of simplicial smooth schemes over S and K with any sectionwise fibrant simplicial presheaf satisfying Nisnevich descent (i.e. $\mathcal{P}^l_{Nis}$-fibrant) and $\mathbb{A}^1$-homotopy invariant. In 1.8.2 we can substitute BGL with any filtered colimit of simplicial objects over Sch_S and K with any simplicial presheaf satisfying Zariski descent. One can even push this further. We do this in the following Proposition, giving a proof for the convenience of the reader.

Proposition 1.8.4. *Let $\mathcal{C}$ be any Grothendieck site. If F is any $\mathcal{I}$-fibrant simplicial presheaf in $\mathrm{sPre}(\mathcal{C})$, J a small filtered set, $(X_j)_{j \in J}$ a directed family of simplicial objects of $\mathcal{C}$ and $X \cong \operatorname*{colim}_{j \in J} X_j \simeq \operatorname*{hocolim}_{j \in J} X_j$, we have*

$$\mathfrak{Map}_{\mathrm{sPre}(\mathcal{C})}(X, F) \simeq \lim_{j \in J^{op}} \operatorname*{holim}_{i \in \Delta} F((X_j)_i)$$

$$\simeq \operatorname*{holim}_{j \in J^{op}} \operatorname*{holim}_{i \in \Delta} F((X_j)_i)$$

in particular, if G is any sectionwise fibrant simplicial presheaf satisfying descent (i.e. $\mathcal{P}^l$-fibrant)

$$[X, G]_{\mathcal{I}^l \mathcal{C}} \cong \pi_0 \lim_{j \in J^{op}} \operatorname*{holim}_{i \in \Delta} G((X_j)_i)$$

$$\cong \pi_0 \operatorname*{holim}_{j \in J^{op}} \operatorname*{holim}_{i \in \Delta} G((X_j)_i)$$

Let A be a class of maps s.t. we can perform the left Bousfield localization on $\mathcal{P}^l_\tau \mathcal{C}$ and $\mathcal{I}^l_\tau \mathcal{C}$ in order to obtain the model categories $\mathcal{I}^l_A \mathcal{C}$ and $\mathcal{P}^l_A \mathcal{C}$. Then if a sectionwise weakly equivalent fibrant replacement of G is also A-local, one has in addition that $[X, G]_{\mathcal{I}^l_A \mathcal{C}} \cong [X, G]_{\mathcal{I}^l \mathcal{C}}$

Proof. We have to mimic 1.8.1.

$$\mathfrak{Map}_{\mathrm{sPre}(\mathcal{C})}(X, F) \cong \mathfrak{Map}_{\mathrm{sPre}(\mathcal{C})}(\operatorname*{colim}_{j \in J} X_j, F)$$

$$\cong \lim_{j \in J^{op}} \mathfrak{Map}_{\mathrm{sPre}(\mathcal{C})}(X_j, F)$$

$$\simeq \lim_{j \in J^{op}} \operatorname*{holim}_{i \in \Delta} F((X_j)_i)$$

As before we used in the first isomorphism the definition of X, in the second the fact that $\mathfrak{Map}$ takes colimits to limits ([Hir03] 9.2.2) and the third weak equivalence comes from the properties of representable presheaves. The fact that

$$\mathfrak{Map}_{\mathrm{sPre}(\mathcal{C})}(X, F) \simeq \operatorname*{holim}_{j \in J^{op}} \operatorname*{holim}_{i \in \Delta} F((X_j)_i)$$

follows similarly. We turn now to the second assertion. Because of the assumptions, there is a sectionwise weak equivalence $d: G \to G_f$ with G_f $\mathcal{I}^l$-fibrant and A-local,

so by Corollary 6.3 of [DHI04] we have $[X,G]_{\mathcal{I}^l_A\mathcal{C}} \cong [X,G]_{\mathcal{I}^l\mathcal{C}}$. Moreover we can write

$$
\begin{aligned}
[X,G]_{\mathcal{I}^l\mathcal{C}} &\cong [\operatorname*{hocolim}_{j\in J} X_j, G]_{\mathcal{I}^l\mathcal{C}} \\
&\cong [\operatorname*{hocolim}_{j\in J} X_j, G_f]_{\mathcal{I}^l\mathcal{C}} \\
&\cong \pi_0 \, \mathfrak{Map}_{\mathbf{sPre}(\mathcal{C})}(\operatorname*{hocolim}_{j\in J} X_j, G_f) \\
&\cong \pi_0 \operatorname*{holim}_{j\in J} \operatorname*{holim}_{i\in\Delta} G_f((X_j)_i) \\
&\cong \pi_0 \operatorname*{holim}_{j\in J} \operatorname*{holim}_{i\in\Delta} G((X_j)_i)
\end{aligned}
$$

Where we have used the simplicial model structure on $\mathcal{I}^l\mathcal{C}$ together with the result just proved and the fact that d is a sectionwise fibrant replacement (between $\mathcal{P}$-fibrant presheaves). $\qquad\square$

Now we come back to K-theory and we obtain

Proposition 1.8.5. $[\mathrm{BGL}^+, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [\mathrm{BGL}^+, K]_{\mathcal{H}(S)}$

Proof. Immediate from 1.8.1 and 1.8.2. $\qquad\square$

Remark 1.8.6. The previous result also holds if we replace BGL^+ by a colimit of simplicial presheaves representing simplicial smooth schemes, because of the previous two propositions. Moreover we can replace K by a simplicial presheaf having the properties detailed in the previous propositions. The argument is exactly the same. We resume this in the following proposition

Proposition 1.8.7. *Assume J a small filtered set $(X_j)_{j\in J}$ a directed family of simplicial objects of Sm/S and $X \cong \operatorname*{colim}_{j\in J} X_j$. Let G be a sectionwise fibrant simplicial presheaf in $\mathbf{sPre}(\mathrm{Sch}_S)$ satisfying Zariski descent and whose restriction G_s to $\mathbf{sPre}(\mathrm{Sm}/S)$ satisfies Nisnevich descent (i.e. it is both $\mathcal{P}^l_{Zar}$ and $\mathcal{P}^l_{Nis}$-fibrant). Then abusing the notation*

$$[X,G]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [X,G]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S}$$

If moreover G_s has a $\mathcal{I}^l_{Nis}$-fibrant replacement which is also $\mathbb{A}^1$-local, we get

$$[X,G]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [X,G]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S} \cong [X,G]_{\mathcal{H}(S)}$$

By what we said at the beginning of this section, from what precedes it follows

Theorem 1.8.8. $[K, K]_{\mathcal{I}_{Zar}^l \mathrm{Sch}_S} \cong [\mathbb{Z} \times \mathrm{BGL}, K]_{\mathcal{I}_{Nis}^l \mathrm{Sm}/S} \cong [K, K]_{\mathcal{H}(S)}$

After this result was proved, the author discovered that the the previous statement was basically included in some unpublished 2013 notes by Cisinski, who sketches a different but simpler argument to reach essentially the same conclusion. However our method is different and in some extent, "more explicit".

Remark 1.8.9. The bijections $[K, K]_{\mathcal{I}_{Zar}^l \mathrm{Sch}_S} \cong [K, K]_{\mathcal{I}_{Nis}^l \mathrm{Sm}/S} \cong [K, K]_{\mathcal{H}(S)}$ can be seen to be induced by the functor

$$\mathrm{Ho}(\mathbf{sPre}(\mathrm{Sch}_S), \mathcal{I}_{Zar}^l) \to \mathrm{Ho}(\mathbf{sPre}(\mathrm{Sm}/S), \mathcal{I}_{Zar}^l)$$

induced by deriving the restriction functor $\mathbf{sPre}(\mathrm{Sch}_S) \to \mathbf{sPre}(\mathrm{Sm}/S)$ and then applying the localization functors

$$\mathrm{Ho}(\mathbf{sPre}(\mathrm{Sm}/S), \mathcal{I}_{Zar}^l) \to \mathrm{Ho}(\mathbf{sPre}(\mathrm{Sm}/S), \mathcal{I}_{Nis}^l)$$

and

$$\mathrm{Ho}(\mathbf{sPre}(\mathrm{Sm}/S), \mathcal{I}_{Nis}^l) \to \mathcal{H}(S)$$

as revealed from an adequate analysis of the arguments given. Indeed, the only non trivial step can be to convince yourself that the restriction functor res : $\mathbf{Pre}(\mathrm{Sch}_S) \to \mathbf{Pre}(\mathrm{Sm}/S)$ induces a bijection $\pi_0 \mathfrak{Map}_{\mathbf{Pre}(\mathrm{Sch}_S)}(X, F) \cong \pi_0 \mathfrak{Map}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{res}(X), \mathrm{res}(F))$ for any $X \in \mathrm{Sm}/S$ and $F \in \mathbf{sPre}(\mathrm{Sch}_S)$, but this follows from the Yoneda lemma. The same holds in the generality of the previous remark.

Now by noticing that $(\mathbb{Z} \times \mathrm{BGL}^+)^n \cong \mathbb{Z}^n \times (\mathrm{BGL}^+)^n \cong \coprod_{i \in \mathbb{Z}^n} (\mathrm{BGL}^+)^n$, and that the same holds for BGL, using 1.7.1 we can argue as above to get

Theorem 1.8.10. *For any natural number n, $[K^n, K]_{\mathcal{I}_{Zar}^l \mathrm{Sch}_S} \cong [(\mathbb{Z} \times \mathrm{BGL})^n, K]_{\mathcal{I}_{Zar}^l \mathrm{Sm}/S} \cong [K^n, K]_{\mathcal{H}(S)}$*

1.8.1 The case of Symplectic K-theory

In this subsection S will be a regular (remind that we assume all our regular schemes to be divisorial unless otherwise stated) Noetherian scheme such that $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. We will consider in this Section the simplicial presheaf $\mathrm{GW}^{[2]} =: KSp$ over Sch_S. We already recalled its descent properties and there is a local weak equivalence $\mathbb{Z} \times \mathrm{BSp}^+ \simeq KSp$ in $\mathbf{sPre}(\mathrm{Sch}_S, \mathcal{I}_{Zar}^l)$ (argue as in the case of K-theory using [Sch17] Theorem A.1 and Corollary A.2) so that if our schemes are divisorial, all the possible definitions of KSp are equivalent. Also, recall that in the paper [ST15],

it is shown that $KSp \simeq \mathbb{Z} \times \mathrm{BSp}$ in $\mathcal{H}(S)$ and we do not need to consider the étale classifying space in this context because of the equivalence between symplectic vector bundles and fppf Sp_{2n}-torsors (the proof of this fact is contained in [AHW18] page 1205, see also [PW10a] page 25). Moreover, we explicitly note that the symplectic groups are smooth schemes over our chosen base S. This fact usually given for granted by many authors, can be proven explicitly when S is a field (see [Wat79]) and for the general case one can reduce to fields because of [DG80] page 289. Then we can repeat the arguments of the previous sections because of the discussion in Section 1.8 on BSp and BSp$^+$ so that, noticing that Proposition 1.8.4 and that we have Proposition 1.6.1 we have proved the following theorem and that has to be considered as the analogue of 1.8.5 and 1.8.10

Theorem 1.8.11. *We have* $[\mathrm{BSp}^+, KSp]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [\mathrm{BSp}, KSp]_{\mathcal{H}(S)}$. *Moreover for any natural number n it holds* $[KSp^n, KSp]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [(\mathbb{Z} \times \mathrm{BSp})^n, KSp]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S} \cong [KSp^n, KSp]_{\mathcal{H}(S)}$

Also, we observe that the analogue of Remark 1.8.9 holds.

1.9 Separated schemes

We notice that if we had chosen a separated base scheme S and considered the full subcategory $\mathrm{Sm}/S^{sep} \subseteq \mathrm{Sm}/S \subseteq \mathrm{Sch}_S$ of separated (in the absolute sense) smooth schemes over S instead than the category of divisorial schemes nothing would have changed in the proof of the above theorems. We summarise this into the following theorem

Theorem 1.9.1. *Let S be a regular separated noetherian base scheme (with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ if we consider hermitian K-theory) and $\mathrm{Sm}/S^{sep} \subseteq \mathrm{Sm}/S \subseteq \mathrm{Sch}_S$ the category of separated (in the absolute sense) smooth schemes over S. Then for any natural number n we have*

$$[K^n, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [(\mathbb{Z} \times \mathrm{BGL})^n, K]_{\mathcal{I}^l_{Zar}\mathrm{Sm}/S^{sep}} \cong [K^n, K]_{\mathcal{H}(S)^{sep}}$$

and

$$[KSp^n, KSp]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [(\mathbb{Z} \times \mathrm{BSp})^n, KSp]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S^{sep}} \cong [KSp^n, KSp]_{\mathcal{H}(S)^{sep}}$$

1.10 Non-divisorial schemes

We remark in this section that many of the theorems we have proved hold in the case where the schemes considered are not divisorial. Indeed all we have used is that we can define K-theory as an H-group for the model categories we have considered, that it satisfies Nisnevich and Zariski descent, that it is homotopy invariant as simplicial presheaf on Sm/S and that it is locally weakly equivalent to $\mathbb{Z} \times \mathrm{BGL}^+$. Now, we keep S to be a regular noetherian base scheme (in this case by regular we admit also non divisorial schemes, i.e. we only ask that the local rings are regular) and we define K-theory using the definition of Thomason and Trobaugh, i.e. we define for any scheme X the simplicial set $K(X)$ to be $\Omega \iota S_\bullet Perf(X)$ (this is the the simplicial version of zeroth space of the spectrum defined in [TT90] Definition 3.1) where $S_\bullet$ denotes the Waldhausen $S_\bullet$-construction and $Perf(X)$ is the category of perfect complexes on X. We then drop the hypothesis of our schemes to be divisorial, while still of finite type over S. Accordingly we still have Zariski and Nisnevich descent because of the work of Thomason ([TT90] 10.8) and K is still $\mathbb{A}^1$-homotopy invariant over Sm/S ([TT90] Proposition 6.8). Moreover we also have a Zariski local weak equivalence between $\mathbb{Z} \times \mathrm{BGL}^+$ and K as simplicial presheaves on Sch_S (indeed we can check locally that the natural map is a weak equivalence). So we can use 1.8.4 and 1.8.7 and still get 1.8.10 dropping the assumption of divisoriality for the schemes we are considering. Similar considerations apply to KSp. If one wants to have $\mathbb{A}^1$-homotopy invariance for non regular schemes, then one could employ homotopy invariant K-theory, but this will not be discussed here. We summarise all these remarks in the following theorem

Theorem 1.10.1. *Let* Sm/S *and* Sch_S *the categories of smooth noetherian schemes and noetherian schemes of finite type over a regular base scheme* S *(so that* $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ *when hermitian* K*-theory is considered). Consider* K *and* KSp *the Thomason's* K*-theory and the Schlichting's* $\mathrm{GW}^{[2]}$. *Then for any* $n \in \mathbb{N}$ *we have*

$$[K^n, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [(\mathbb{Z} \times \mathrm{BGL})^n, K]_{\mathcal{I}^l_{Zar}\mathrm{Sm}/S} \cong [K^n, K]_{\mathcal{H}(S)}$$

and

$$[KSp^n, KSp]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S} \cong [(\mathbb{Z} \times \mathrm{BSp})^n, KSp]_{\mathcal{I}^l_{Nis}\mathrm{Sm}/S} \cong [KSp^n, KSp]_{\mathcal{H}(S)}$$

Moreover, since for generic schemes we have a well define Nisnevich topology, we can replace in the above isomorphisms $\mathcal{I}^l_{Zar}\mathrm{Sch}_S$ *with* $\mathcal{I}^l_{Nis}\mathrm{Sch}_S$ *everywhere.*

Proof. The proof goes as the one of Theorems 1.8.10 and 1.8.11 mutatis mutandis using Proposition 1.8.4. □

Chapter 2

Embedding divisorial schemes into smooth ones

In this chapter, unless otherwise indicated, R will always denote a commutative noetherian ring. We are devoted to prove the following theorem.

Theorem 2.0.1. *Let X be a quasi-separated scheme of finite type over $\mathrm{Spec}(R)$ having an ample family of line bundles $\{(L_i, s_i)\}_{i=1}^{n}$. Then there exists a closed embedding $f : X \hookrightarrow W$ with W noetherian smooth scheme over R admitting an ample family of line bundles.*

2.1 Schemes having an ample family of line bundles

2.1.1 Definition and main properties

We start by fixing some terminology. For schemes locally of finite type over some base, we adopt the definition found in [GD71] I 6.2.1 or in [GW10] Definition 10.5. We say that a scheme X is smooth over a base S if its structure map is smooth ([GD67] IV 6.8.6, 17.3.1, [GW10] 6.14, [Sta18, Tag 01V5]). Explicitly note that a smooth morphism is locally of finite presentation ([GD67] IV 1.4.2) and so locally of finite type. We do not assume that a smooth morphism is separated. Hence smooth schemes over a base S are a full subcategory of schemes locally of finite type over S. Notice also that a morphism $X \to Y$, if Y is at least locally noetherian, is locally of finite presentation if and only if it is locally of finite type ([GW10] Remark 10.36). We now recall a simple fact

Lemma 2.1.1. *Let X be a locally noetherian scheme. Then its connected components $\{U_\alpha\}_{\alpha \in I}$ are open and we have an isomorphism of schemes $(X, \mathcal{O}_X) \cong \amalg_{\alpha \in I}(U_\alpha, \mathcal{O}_{X|U_\alpha})$*

Proof. If a topological space is at least locally noetherian, then its connected components are open ([GD67] I 6.1.9). Then denoting as $\{U_\alpha\}_{\alpha \in I}$ the open connected components of X, the spaces $(U_\alpha, \mathcal{O}_{X|U_\alpha})$ are well defined open subschemes of X. If we glue this schemes forming their disjoint union (as in [GW10] 3.10 for example) it is a simple check to obtain the isomorphism stated in the lemma. $\square$

Definition 2.1.2. ([SGA71] II 2.2.3 or [TT90] Definition 2.1) A scheme X has an ample family of line bundles (or it is called *divisorial*) if it is quasi-separated and there is a finite family of line bundles $L_1, ..., L_n$ on it such that there are finitely many global sections $s_i \in \Gamma(X, L_i)$ with the property that their non vanishing loci X_{s_i} ([GD71]0, 4.1.9) form an open affine cover of X.

Remark 2.1.3. Schemes having an ample family of line bundles are quasi-compact. In fact, if a scheme has an ample family of line bundles, then by definition it has a finite cover by affine schemes, so that it is quasi-compact. Moreover, allowing repetitions of the same line bundle, we can assume that an ample family of line bundles comes into the form $\{(L_i, s_i)\}_{i=1}^n$ where $s_i \in \Gamma(X, L_i)$ for every i.

Examples of divisorial schemes are $\mathbb{P}_R^n$, Grasmannians, etc. The definition of divisorial scheme is equivalent to ask (for schemes quasi-compact and quasi-separated) that there exists a finite family of line bundles L_i such that the open non vanishing loci X_f form a basis for the topology of X, where f varies over the global sections $\Gamma(X, L_i^{\otimes l})$ for $i = 1, ..., n$ and $l \geqslant 1$, see [TT90] 2.1.1. Every family of line bundles satisfying this last property can be in fact twisted (if our scheme is quasi-compact) to a family of the form given in our definition and sometimes we will abuse the terminology calling ample family a family of line bundles satisfying one of these two conditions. More equivalent definitions could be given, and a remarkable feature of such schemes is that they have the resolution property (see [TT90] 2.1.3). For our purposes it is relevant to say that if a scheme X admits an ample family of line bundles $L_1, ..., L_n$ then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type there exists (see the proof of [SGA71] II 2.2.3.1 or [TT90] Lemma 2.1.3) natural numbers $p, _j m_1, ..., _j m_n$ where $j = 1, ..., p$ and a surjective map

$$\bigoplus_{j=1}^{p}(\otimes_{i=1}^{n} L_i^{\otimes -(_j m_i)}) \twoheadrightarrow \mathcal{F}$$

Remark 2.1.4. A very important property of divisorial schemes is that they have affine diagonal, i.e. their diagonal embedding ([Sta18, Tag 01KJ])is affine. For a simple proof see [BS03] Proposition 1.2.

Many important classes of schemes have an ample family of line bundles. Notice that noetherian schemes or schemes having affine diagonal are both quasi-separated.

Lemma 2.1.5. *([SGA71] II 2.2.7.1) Every regular (or locally factorial) noetherian scheme having affine diagonal has an ample family of line bundles*

Corollary 2.1.6. *Every quasi-compact scheme X that have affine diagonal and that is smooth over a noetherian regular base scheme S (and so in particular it is noetherian) has an ample family of line bundles.*

Remark 2.1.7. In [SGA71] the hypothesis of having affine diagonal is replaced by the stronger separated hypothesis. However the separated hypothesis is used in the proof of [SGA71]II 2.2.7 in order to apply [SGA71]II 2.2.6; in particular it is required that an open embedding of an affine scheme into our scheme X is an affine morphism. But this is true if X has affine diagonal, so the proof goes through (see also [BS03] Proposition 1.3). Notice also that if we remove the hypothesis of having affine diagonal, Lemma 2.1.5 fails as the example of the affine plane with double origin shows ([SGA71]II 2.2.7.2): this is a scheme smooth, quasi-compact, and quasi-separated that does not have an ample family of line bundles.

Remark 2.1.8. Notice that Lemma 2.1.5 is used in thesis only to justify the fact that we have a well behaved Nisnevich topology over the category of divisorial smooth base over a divisorial base S, see Remark B.3.3. Also, we explicitly point out hat the property of a morphism of shemes of having affine diagonal is stable under composition and base change because the same proof used for the property of being separated found for example in [Sta18, Tag 01KH] goes through, affine morphisms being stable under composition and base change because of [Sta18, Tag 01SC] and [Sta18, Tag 01SD], for example.

In this work, by quasi-compact immersion we mean a morphism which is both quasi-compact ([GD67] I 6.6.1) and an immersion ([GD67] I 4.2.1, see also [GW10] Definition 3.43). We can prove the following.

Proposition 2.1.9. *Let X be a scheme admitting a quasi-compact immersion $i : X \to Y$ in a divisorial scheme Y. Denoting as $L_1, ..., L_n$ an ample family of line bundles of Y, we have that $i^*L_1, ..., i^*L_n$ is an ample family of line bundles over X so that X is itself divisorial.*

Proof. For every $f \in \Gamma(Y, L_i^{\otimes l})$ if $i_f : Y_f \to Y$ is the open embedding of the non vanishing locus Y_f in Y, we can draw the following pullback diagram

$$
\begin{array}{ccc}
X_{i^*f} & \xrightarrow{\ i_{i^*f}\ } & X \\
\downarrow & & \downarrow{\scriptstyle i} \\
Y_f & \xrightarrow[\ i_f\ ]{} & Y
\end{array}
$$

where $i^*f \in \Gamma(X, i^*L_i^{\otimes l})$ and i_{i^*f} is an open embedding. But now we have that i is a quasi-compact immersion, so factoring it as composition of a closed embedding followed by an open embedding we notice that, by point set topology, the preimage of a basis for Y under i is a basis for the topology on X. This together with the explicit description of the preimage of a non vanishing locus Y_f for $f \in \Gamma(Y, L_j^{\otimes l})$ allows us to see that X_{i^*f}, $i^*f \in \Gamma(X, i^*L_j^{\otimes l})$ with $j = 1, ..., n$ and $l \geq 1$ are a basis for the topology of Y, so $i^*L_1, ..., i^*L_n$ is an ample family of line bundles for X. $\qquad\square$

We explicitly note that the definition of immersion given by Hartshorne in [Har77] 2.4 is different, but for quasi-compact maps they agree, see [Sta18, Tag 01QV] or [GW10] Remark 10.31. Since in what is left of this chapter we will work only with schemes which are locally of finite type over a noetherian base S, they are at least locally noetherian ([GD71] I 6.2.2) and also quasi-separated because of [Sta18, Tag 01OY] so we can speak simply of immersion without any difficulty using the definition of Hartshorne or of Grothendieck because for any immersion $f : X \to Y$, if Y is at least locally noetherian, then f is quasi-compact for [GD67] I 6.6.4 (i). If such schemes are divisorial, they are indeed quasi-compact and so we can drop the distinction between locally of finite type and of finite type. We also have the following

Proposition 2.1.10. *Let X be a divisorial scheme and let be $\mathcal{E}$ a vector bundle on it. Then $\mathbb{P}(\mathcal{E})$ has an ample family of line bundles.*

Proof. One consider the usual projection $\pi : \mathbb{P}(\mathcal{E}) \to X$ and notice that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is relatively π-ample (see [GD67] II 4.6.1 for this notion): in fact if one considers a finite trivializing open affine cover $\{U_i = \operatorname{Spec}R_i\}_{i=1}^n$ made by connected affine schemes, we get that $\pi^{-1}(U_i) \cong \mathbb{P}_{R_j}^{n_i}$, where the n_i are the ranks of $\mathcal{E}$ on U_i and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ restricts to $\mathcal{O}_{\mathbb{P}_{R_j}^{n_i}}(1)$ on $\pi^{-1}(U_i)$, which is ample. Now one can apply [TT90] 2.1.2 (f) to conclude. $\qquad\square$

2.1.2 A particular family

In this subsection we show that it is possible, given any ample family on a scheme X, to build another ample family which has properties useful to our aims.

Proposition 2.1.11. *Let X be a scheme having an ample family $\{(L_i, s_i)\}_{i=1}^n$. Then there exists an ample family $\{(\mathcal{L}_j, f_j)\}_{j=1}^m$ such that $\{X_{f_j}\}$ forms an affine open cover of X and there are isomorphisms $\alpha_{ij} : \mathcal{O}_{X|X_{f_i}} \xrightarrow{\cong} \mathcal{L}_{j|X_{f_i}}$ for all i, j where α_{ii} is induced by the restriction of f_i to X_{f_i} and in the remaining cases α_{ij} are induced by sections $s_{ij} \in \Gamma(X_{f_i}, \mathcal{L}_{j|X_{f_i}})$.*

Proof. By the definition of ample family, we know that the open subschemes X_f, $f \in \Gamma(X, L_i^{\otimes n})$ form a basis for the topology of X for i, $n \geq 0$ varying. We notice that if an open subscheme of the form X_f is contained in an open affine subscheme of X, then X_f has to be affine by [SGA71]II 2.2.3.1. Now, for any $y \in X$, find an open affine subset $U_y \ni y$ of X such that $L_{i|U_y} \cong \mathcal{O}_{X|U_y}$ for every i. Since the X_f of the above form are a basis for the topology of X, we can then find sections $f_y \in \Gamma(X, L_{i_y}^{\otimes n_y})$ such that $y \in X_{f_y} \subseteq U_y$ are open affine subschemes of X and

$$\alpha_{i,y,n} : \mathcal{O}_{X|X_{f_y}} \xrightarrow{\cong} L_{i|X_{f_y}}^{\otimes n}$$

for every y, i ad n induced by sections $s_{i,y,n} \in \Gamma(X_{f_y}, L_{i|X_{f_y}}^{\otimes n})$ such that if $i = i_y$ and $n = n_y$, $s_{i_y,y,n_y} = f_{y|X_{f_y}}$. Since X is quasi-compact, we can take a finite number of X_{f_y} as an open affine cover of X corresponding to points $y_1, ..., y_m$. Denote $\mathcal{L}_j := L_{i_{y_j}}^{\otimes n_{y_j}}$ and $f_j := f_{y_j}$ for $j = 1, ..., m$. The family $\{(\mathcal{L}_j, f_j)\}_{j=1}^m$, together with the choice $\alpha_{ij} := \alpha_{i_{y_j}, y_i, n_{y_j}}$ and $s_{ij} := s_{i_{y_j}, y_i, n_{y_j}}$ has the desired properties. $\qquad\square$

Remark 2.1.12. If X is locally of finite type over a noetherian ring R, then all the rings $A_j = \Gamma(X_{f_j}, \mathcal{O}_X)$ coming from the previous Proposition are finitely generated R-algebras.

2.2 Lemma S

We prove in this section a technical lemma which is used in the proof of Theorem 2.0.1. We call this lemma Lemma S, where S stands for smooth. The proof is essentially combinatorics inspired by the case of weighted projective spaces: it is given since the author was not able to find in the literature a more general algorithm to use in order to derive results like the following.

Lemma 2.2.1. *Let d be a fixed element of $\mathbb{Z}_{\geqslant 0}$, R be any commutative unital ring and $A = R[y_{ik}, x_{ij}] = \bigoplus_{p \in \mathbb{Z}^n} A_p$ a $\mathbb{Z}^n$-graded polynomial ring such that, denoting as $e_1, ..., e_n$ the canonical basis of $\mathbb{Z}^n$ as $\mathbb{Z}$-module, the following hold*

- *$i, j \in \{1, ..., n\}$, $y_{ik} \in E_i$, where E_i is a set of variables of cardinality $l_i \in \mathbb{N}$.*

- *$A_0 = R$.*

- *$\deg(x_{ij}) = de_i + e_j$.*

- *$\deg(y_{ik}) = \deg(x_{ii})$ for every i.*

Let $T_k := \prod_{j=1}^n x_{kj} = x_{k1} \cdots x_{kn}$ then $\deg(T_k) = (1, ..., nd+1, 1, ..., 1)$ where the non 1 term is in the kth position and $T_k \in A$ is relevant for every $k = 1, ..., n$. Moreover $A_{(T_k)}$ is a smooth R-algebra isomorphic to a polynomial R-algebra in $n(n-1) + \sum_{i=1}^n |E_i|$ variables.

Proof. Suppose k is fixed. The calculation of the degree of T_k is simple and in particular T_k is homogeneous. To see that it is relevant we have to consider the subgroup D in $\mathbb{Z}^n$ generated by the degrees of $x_{k1}, ..., x_{kn}$ and check that it has finite index. The $n \times n$ matrix

$$\begin{pmatrix} \deg(x_{k1}) \\ \vdots \\ \deg(x_{kn}) \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} & d & \\ 0 & \vdots & 0 \\ & d & \end{pmatrix} \tag{2.1}$$

is invertible over $\mathbb{Q}$ hence has image D of finite index in $\mathbb{Z}^n$. This suffices to show that T_k is relevant. Now consider $A_{(T_k)}$. We note that the usual isomorphism $A_{T_k} \cong A_{x_{k1} \cdots x_{kn}} \cong R[y_{il}, x_{ij}, x_{kj}^{-1}]$ is graded (giving all the rings the grading induced from A) and we remark that it induces an isomorphism on the degree zero level i.e. $A_{(T_k)} \cong (R[y_{il}, x_{ij}, x_{kj}^{-1}])_0$. From now on we will identify A_{T_k} and $A_{(T_k)}$ with $R[y_{il}, x_{ij}, x_{kj}^{-1}]$ and $(R[y_{il}, x_{ij}, x_{kj}^{-1}])_0$ respectively via the canonical isomorphisms. By direct computation we easily check that the following are elements of $A_{(T_k)}$

- $y_{il} \cdot x_{ki}^{-(d+1)} \cdot x_{kk}^d =: y_{il} \cdot a_{il}$ for every $y_{il} \in E_i$ because

$$(d+1)e_i - (d+1)(de_k + e_i) + d(d+1)e_k = 0$$

- $x_{ij} \cdot x_{kj}^{-1} \cdot x_{ki}^{-d} \cdot x_{kk}^d =: x_{ij} \cdot b_{ij}$ for all i, j because

$$de_i + e_j - (de_k + e_j) - d(de_k + e_i) + d(d+1)e_k = 0$$

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Define a new $\mathbb{Z}^n$-graded ring as follows: as a ring it is the $R[x_{kj}^{\pm 1}]$-algebra

$$R[x_{kj}^{\pm 1}][\overline{y_{jl}}, \overline{x_{ij}}]$$

where $(i, j = 1, ..., n) \wedge (i \neq k)$ and for any j, $\overline{y_{jl}} \in \overline{E_j}$ with $|\overline{E_j}| = l_j = |E_j|$. We give to this ring the grading induced by letting elements in R to have degree 0 and by giving to the variables the following degrees

- $\deg(x_{kj}^{\pm 1}) = \pm(de_k + e_j)$.

- $\deg(\overline{y_{jl}}) = \deg(\overline{x_{ij}}) = 0$ where $(i, j = 1, ..., n) \wedge (i \neq k)$.

We now notice that we can see the ring $A_{T_k} = R[y_{il}, x_{ij}, x_{kj}^{-1}]$ as an graded $R[x_{kj}^{\pm 1}]$-algebra $R[x_{kj}^{\pm 1}][y_{jl}, x_{ij}]$ where $i \neq k$. We can now define the following $R[x_{kj}^{\pm 1}]$-algebra homomorphism

$$R[x_{kj}^{\pm 1}][\overline{y_{jl}}, \overline{x_{ij}}] \xrightarrow{\ \varphi\ } A_{T_k} = R[x_{kj}^{\pm 1}][y_{jl}, x_{ij}]$$

$$\overline{y_{jl}} \longmapsto y_{jl} \cdot a_{jl} = y_{jl} \cdot \frac{x_{kk}^d}{x_{kj}^{d+1}}$$

$$\overline{x_{ij}} \longmapsto x_{ij} \cdot b_{ij} = x_{ij} \cdot \frac{x_{kk}^d}{x_{kj} x_{ki}^d} \text{ if } i \neq k$$

$$x_{kj}^{\pm 1} \longmapsto x_{kj}^{\pm 1}$$

The homomorphism φ is graded and an isomorphism, so it induces an isomorphisms of rings on the zero degree parts of the two graded rings. This means, since $A_{(T_k)}$ is the degree zero part of A_{T_k}, that we have an isomorphism of rings $A_{(T_k)} \cong R[\overline{y_{jl}}, \overline{x_{ij}}]$, $i \neq k$, as wanted, so that the lemma is fully proved. This isomorphism is obtained by precomposing φ^{-1} with the inclusion $\alpha : A_{(T_k)} \hookrightarrow A_{T_k} \cong R[y_{il}, x_{ij}, x_{kj}^{-1}]$ and then by restricting to the image of the obtained map. $\qquad\square$

2.3 Recollections on multihomogeneous Proj

We are going to recall some facts about multihomogeneous localization as introduced and studied in [BS03]. Where it is possible, we take an equivalent but different approach to *op.cit.*. This theory is linked with toric geometry but the author has not been able to find a suitable different reference for this material from that perspective in the generality required for this work. If S is a $\mathbb{Z}^n$-graded ring and $f \in S$ is an homogeneous element, we can define a $\mathbb{Z}^n$-grading on S_f by defining $\deg(\frac{g}{f^n}) = \deg(g) - n \cdot \deg(f)$ for every $g \in S$ homogeneous element. In this case we will denote as $S_{(f)}$ the degree zero part of S_f.

Definition 2.3.1. ([BS03] pag. 6) Let S be a $\mathbb{Z}^n$-graded ring. We call such a ring *periodic* if the degrees of homogeneous units $f \in S^\times$ (a priori this is just a set, but it easy to verify that this is actually a group) form a subgroup of $\mathbb{Z}^n$ of finite index. We say that an element $f \in S$ is *relevant* if it is homogeneous and S_f is periodic.

The following is a simple lemma

Lemma 2.3.2. *Let $\varphi : A \to B$ a homogeneous ring homomorphism between $\mathbb{Z}^n$-graded rings A and B. Then if $f \in A$ is relevant, $\varphi(f) \in B$ is relevant.*

Note that if $f = g_1 \cdots g_n \in S$ is a homogeneous factorization and the degrees of g_i generate a subgroup of finite index in $\mathbb{Z}^n$, then f is relevant.

Lemma 2.3.3. *Let S be a $\mathbb{Z}^n$-graded periodic ring, and f and homegeneous element in it. Then S_f is periodic.*

Proof. One simply notices that the homogeneous units in S are still homogeneous units in S_f, so their degrees generate a subgroup of $\mathbb{Z}^n$ of finite index. $\square$

Corollary 2.3.4. *Let S be a $\mathbb{Z}^n$-graded ring, then if $f \in S$ is relevant and $g \in S$ is homogeneous it follows that fg is a relevant element.*

Proof. It is an immediate consequence of the previous lemma. $\square$

Remark 2.3.5. Notice that if S is a $\mathbb{Z}^n$-graded ring and $f \in S$ is homogeneous and relevant, then the subgroup $D \subseteq \mathbb{Z}^n$ of the degrees of the homogeneous units in S_f is generated by the degrees of a finite family of homogeneous units in S_f, say $u_1, ..., u_r$. Hence $f = f \cdot u_1 u_1^{-1} \cdots u_r u_r^{-1} = g_1 \cdots g_l$ where all g_i are homogeneous and units in S_f and $< \deg(g_i) > = D$.

Now consider $S = \bigoplus_{d \in \mathbb{Z}^n} S_d$ a $\mathbb{Z}^n$-graded ring. Assume that the degrees of the homogeneous units $f \in S^\times$ form a subgroup $D \subseteq \mathbb{Z}^n$ of finite index (i.e. S is periodic) and write $S' = \bigoplus_{d \in D} S_d$ for the induced graded subring of S. Then $S' = S_0[u_1^{\pm 1}, ..., u_r^{\pm 1}]$ where $u_i \in S^\times$ for every $i = 1, ..., r$.

Proof. First we let $u_1, ..., u_r$ to be some homogeneous units whose degrees $\deg(u_i) = d_i$ generate D. Then we prove the equality by showing the inclusions $S' \subseteq S_0[u_1^{\pm 1}, ..., u_r^{\pm 1}]$ and $S' \supseteq S_0[u_1^{\pm 1}, ..., u_r^{\pm 1}]$. The second one is trivial. For the first one, suppose $f \in S_d$ with $d \in D$. Note that by our assumptions we have $d = k_1 d_1 + ... + k_r d_r$ for some $k_1, ..., k_r \mathbb{Z}$. Define $\alpha_f := u_1^{-k_1} \cdots u_r^{-k_r} \in S_d \cap S^\times$. Then $f \cdot \alpha_f =: \beta_f \in S_0$. But $f = \beta_f \cdot \alpha_f^{-1} \in S_0[u_1^{\pm 1}, ..., u_r^{\pm 1}]$. Iterating for every $f \in S'$ gives the inclusion $\subseteq$ as we wanted. $\square$

The above reasoning shows that we can find an actual torus into every periodic ring. This fact can be prompted to the following lemma

Lemma 2.3.6. *([BS03] Lemma 2.1) For periodic rings S, the projection $\operatorname{Spec}(S) \to \operatorname{Spec}(S_0)$ is a GIT quotient for the action of the torus $\operatorname{Spec}(S_0[\mathbb{Z}^n])$ on $\operatorname{Spec}(S)$ induced by the grading.*

From this it follows that if we set $D_+(f) := \operatorname{Spec}(S_{(f)})$ for f relevant, then the map $\operatorname{Spec}(S_f) \to D_+(f)$ is a GIT quotient for the induced action. In general, given a $\mathbb{Z}^n$-graded ring, we have an induced action of the torus $\operatorname{Spec}(S_0[\mathbb{Z}^n]) =: T$ on it, represented by a map $\sigma : T \times \operatorname{Spec}(S) \to \operatorname{Spec}(S)$. This action in general is not well behaved. We can however take the quotient of S by this action (i.e. the cokernel of σ in the category of locally ringed spaces), which exists as a locally ringed space $\operatorname{Quot}(S)$ (see [DG70] Proposition 1.6). This space however is not always a scheme, so the following definition is motivated

Definition 2.3.7. ([BS03] 2.2) Let S be a $\mathbb{Z}^n$-graded ring as above. Define the *multihomogeneous projective spectrum of S* to be

$$\operatorname{Proj}(S) := \bigcup_{f \in S \text{ relevant}} D_+(f) \subseteq \operatorname{Quot}(S)$$

By 2.3.6 and the above discussion, it is a scheme.

We can give a more explicit description of multihomogeneous projective spaces by giving a gluing datum of affine schemes and gluing it, without any use of GIT quotients. In order to do this, We need the following lemma

Lemma 2.3.8. *Let be S a $\mathbb{Z}^n$-graded ring, $f \in S$ a relevant element and $g \in S$ an homogeneous element. Then*

$$S_{(fg)} \cong (S_{(f)})_{\frac{g^d u_1^{k-k_1} \cdots u_r^{k-k_r}}{u_1^{k_1} u_1^{k-k_1} \cdots u_r^{k_r} u_r^{k-k_r}}} \cong (S_{(f)})_{\frac{g^d u_1^{k-k_1} \cdots u_r^{k-k_r}}{(u_1 \cdots u_r)^k}}$$

where the degrees of $u_1, ..., u_r \in S$ and their degrees generate the subgroup D of $\mathbb{Z}^n$ of finite index generated by the degrees of the homogeneous units in S_f, there exists $m \in \mathbb{N}$ such that $f^m = u_1 \cdots u_r$, $d \cdot \deg(g) = k_1 \cdot \deg(u_1) + ... + k_r \cdot \deg(u_r)$ and $k = max\{k_i\}$.

Proof. We first notice that such $d, m, k_1, ..., k_r, u_1, ..., u_r$ exist because f is relevant and so the subgroup D of $\mathbb{Z}^n$ generated by the homogeneous units of S_f has finite index. Defining $b := u_1^{k_1} \cdots u_r^{k_r}$, which is still a relevant element, we can check that

$S_{(f)} \cong S_{(b)}$ by repeating the same argument used in these cases for $\mathbb{Z}$-gradings. In fact for every $h \in S$ homogeneous the map $S_{h^l} \to S_h$, $\frac{a}{h^l} \mapsto \frac{a}{h^l}$ is an isomorphisms for all i and all $l \in \mathbb{N}$ so that we first have $S_{(f)} \cong S_{(f^m)} \cong (...(S)_{u_1})_{u_2})...)_{u_r})_0 \cong S_{(u_1 \cdots u_r)}$ and then by induction $S_{(u_1 \cdots u_r)} \cong S_{(u_1^{k_1} \cdots u_r^{k_r})} \cong S_{(b)}$. So we have $S_{(f)} \cong S_{(b)}$, $S_{(fg)} \cong S_{(bg)}$ and to conclude we need to show that $S_{(bg)} \cong (S_{(b)})_{\frac{g^d}{b}}$. To achieve this we can argue as in EGA II 2.2.2. Since bg divides bg^d which divides $(bg)^d$ we have a canonical isomorphism $(S_{bg})_0 \cong (S_{bg^d})_0 \cong ((S_b)_{\frac{g^d}{1}})_0$. Using the fact that b is invertible in S_b we also get that the following isomorphisms are graded

$$S_{bg^d} \cong (S_b)_{\frac{g^d}{1}} \cong (S_b)_{\frac{g^d}{b}}$$

Finally by construction $\frac{g^d}{b}$ has degree zero in $(S_b)_{\frac{g^d}{b}}$ and then $((S_b)_{\frac{g^d}{b}})_0 \cong (S_{(b)})_{\frac{g^d}{b}}$. The lemma is now fully proved. $\qquad\square$

Using the previous lemma, starting from a $\mathbb{Z}^n$-graded ring, we consider the gluing datum $(D_+(f), D_+(fg), \varphi_{gf} : D_+(fg) \xrightarrow{\cong} D_+(gf))_{f,g \text{ relevant}}$ where $D_+(fg) \subseteq D_+(f)$ are open subschemes because the previous lemma, the isomorphisms φ_{gf} are induced from the canonical isomorphisms $S_{fg} \cong S_{gf}$ and the cocycle conditions are verified. The scheme resulting from this gluing datum is in fact isomorphic to $\mathrm{Proj}(S)$.

Remark 2.3.9. Note that for f, g relevant elements, we have by the previous lemma that $D_+(fg) \subseteq D_+(f)$ and $D_+(fg) \subseteq D_+(g)$ are actually open embeddings such that we can glue the $D_+(f)$ in order to get $\mathrm{Proj}(S)$ in such a way that $D_+(f) \cap D_+(g) = D_+(fg) \subseteq \mathrm{Proj}(S)$. For a counterexample of this behaviour when f, g are homogeneous but not relevant consider the following situation. Let $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ have the $\mathbb{Z}^2$-grading defined as $S_0 = \mathbb{C}$, $\deg(x_1) = \deg(x_2) = (1, 0)$ and $\deg(x_3) = \deg(x_4) = (0, 1)$. Then $S_{(x_1)} \cong \mathbb{C}[\frac{x_2}{x_1}]$, $S_{(x_3)} \cong \mathbb{C}[\frac{x_4}{x_3}]$, $x_1 x_3$ is relevant and $S_{(x_1 x_3)} \cong \mathbb{C}[\frac{x_2}{x_1}, \frac{x_4}{x_3}]$ but the graded map $S_{(x_1)} \hookrightarrow S_{(x_1 x_3)}$ does not induce an open embedding $\mathbb{A}^2_{\mathbb{C}} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$. It is true in general that, for a $\mathbb{Z}^n$-graded ring S, the isomorphism $(S_f)_g \cong S_{fg}$ is graded such that $((S_f)_g)_0 \cong S_{(fg)}$ and the maps $S_f \hookrightarrow S_{fg}$ are graded for any homogeneous f, g such that they induce maps $S_{(f)} \hookrightarrow S_{(fg)}$. Moreover, results such as [GD67]II 2.2.3 fails in this case if we abandon $\mathbb{Z}$-gradings for $\mathbb{Z}^n$-gradings.

Giving a $\mathbb{Z}^n$-graded ring S, defining $S_+ \subseteq S$ to be the ideal in S generated by all the relevant elements, we call $V(S_+)$ in $\operatorname{Spec}(S)$ the *irrelevant subscheme*. This way we obtain that the induced affine projection map

$$\operatorname{Spec}(S) - V(S_+) \to \operatorname{Proj}(S)$$

is a GIT quotient for the induced action of the torus $\operatorname{Spec}(S_0[\mathbb{Z}^n]) =: T$ on S.

Proposition 2.3.10. *([BS03]3.1, 3.5) Let S be a $\mathbb{Z}^n$-graded ring. Then $\operatorname{Proj}(S)$ has affine diagonal and if S is finitely generated as an S_0-algebra, $\operatorname{Proj}(S)$ is divisorial.*

Consider now X a noetherian divisorial scheme having $\{(L_i, s_i)\}_{i=1}^n$ as an ample family of line bundles. Define the $\mathcal{O}_X$-algebra

$$\mathcal{B} := \bigoplus_{d \in \mathbb{Z}^n} \mathcal{B}_d = \bigoplus_{d \in \mathbb{Z}^n} L^d$$

where $d = (d_1, ..., d_n) \in \mathbb{Z}^n$, $L^d = L_1^{\otimes d_1} \otimes ... \otimes L_n^{\otimes d_n}$ and $L^0 = \mathcal{O}_X$. We notice that the ring multiplication on $\mathcal{B}$ is induced by the tensor product and that, X being noetherian, we do not need to sheafify the direct sum presheaf (see [Har77] Ex.1.11 or [Sta18, Tag 01AI]), so that $\mathcal{B}$ is a $\mathbb{Z}^n$-graded quasicoherent $\mathcal{O}_X$-algebra taking value on every open subset U of X the graded $\Gamma(U, \mathcal{O}_X)$-algebra $\bigoplus_{d \in \mathbb{Z}^n} \Gamma(U, L^d)$. For every $f \in \Gamma(X, L^d)$ (for some $d \in \mathbb{Z}^n$) the multiplication $\mathcal{O}_X$-module map $f : \mathcal{O}_X \to L_X^d$ is an $\mathcal{O}_X$-module isomorphism over X_f. The same is true replacing f with f^n and L^d with $L^{n \cdot d}$ for every $\mathbb{N} \ni n \geq 0$. Notice that by [GD71]0, 4.1.10, $X_{f^n} = X_f$ for every n and these multiplication maps all together give us an $\mathcal{O}_{X_f}$-algebra isomorphism $\bigoplus_{n \geq 0} L_{X_f}^{nd} \cong \mathcal{O}_{X_f}[T]$, inducing an isomorphism $\operatorname{Spec}(\bigoplus_{n \geq 0} \mathcal{B}_{nd|X_f}) \cong \mathbb{A}_{X_f}^1$ (here by Spec we mean the relative spectrum). Notice further that in this case our standard definition of vanishing locus coincides with the definition of [BS03] in Section 4. The following is then a reformulation for our case of [BS03] Proposition 4.2.

Proposition 2.3.11. *Let X and $\mathcal{B}$ be as above. Let $\varphi : S \to \Gamma(X, \mathcal{B})$ be a homogeneous map between $\mathbb{Z}^n$-graded rings. Let $f_1, ..., f_l$ be a finite number of relevant elements $f_j \in S$. Note $\varphi(f_j) \in \Gamma(X, L^{\deg(f_j)})$ for every $j = 1, ..., l$. Assume $\bigcup_{j=1}^l X_{\varphi(f_j)} = X$. Then there is a natural morphism $\psi : X \to \bigcup_{j=1}^l D_+(f_j) \subseteq \operatorname{Proj}(S)$.*

Proof. For every f_j we have a map $\alpha_j : S_{(f_j)} \to \Gamma(X, \mathcal{B})_{(\varphi(f_j))}$. Moreover, for every j we have a homomorphism

$$\beta_j : \Gamma(X, \mathcal{B})_{(\varphi(f_j))} \to \Gamma(X_{\varphi(f_j)}, \mathcal{O}_X), \qquad g/\varphi(f_j)^n \mapsto (\varphi(f_j)^n|X_{\varphi(f_j)})^{-1}(g_{X_{\varphi(f_j)}})$$

where $(\varphi(f_j)^n|X_{\varphi(f_j)})^{-1}$ is the inverse mapping of the $\mathcal{O}_{X|X_{f_j}}$-module isomorphism $\mathcal{O}_{X|X_{\varphi(f_j)}} \cong \mathcal{B}_{nd|X_{\varphi(f_j)}}$ (note that $\varphi(f_j)$ is invertible over $X_{\varphi(f_j)}$). The map $\beta_j \circ \alpha_j : S_{(f_j)} \to \Gamma(X_{\varphi(f_j)}, \mathcal{O}_X)$ give us, since the f_j are relevant, maps $X_{\varphi(f_j)} \to D_+(f_j)$.These maps, using the fact that all the f_j are relevant, agree on the overlaps so that we get the desired morphism $\psi : X \to \mathrm{Proj}(S)$. The map $\beta_j \circ \alpha_j$, can be written also in the following way, denoting as d_j the degree of f_j

$$S_{(f_j)} = (\bigoplus_{n \in \mathbb{Z}} S_{d_j n})_{(f_j)} \longrightarrow \bigoplus_{n \in \mathbb{Z}} \Gamma(X, L^{d_j n})_{(\varphi(f_j))} \longrightarrow (\bigoplus_{n \in \mathbb{Z}} \Gamma(X_{\varphi(f_j)}, L^{d_j n}))_0 = \Gamma(X_{\varphi(f_j)}, \mathcal{O}_X)$$

$$\Big\uparrow \cong$$

$$\Gamma(X, \mathcal{B})_{(\varphi(f_j))}$$

Now one can check that this maps agree when restricted over the overlaps, i.e., denoting $\gamma_{f_j} = (\beta_j \circ \alpha_j)$, we have $\gamma_{f_j|D_+(f_i)} : D_+(f_i f_j) \to \Gamma(X_{\varphi(f_i)\varphi(f_j)}, \mathcal{O}_X)$ is really $\gamma_{f_i f_j}$. $\qquad\square$

Remark 2.3.12. If S is a finitely generated $\mathbb{Z}$-graded polynomial algebra, then as noticed in [BS03] Example 3.8, $\mathrm{Proj}(S)$ is a so called weighted projective space. These space, although quasi-projective, are not necessarily smooth as the following example from [BR86] (remark before Corollary 2.7 pag.121) shows. Suppose $S = \mathbb{C}[x_0, x_1, x_2]$ with $\deg(x_0) = \deg(x_1) = 1$ and $\deg(x_2) = 2$. Then $S_{(x_2)} \cong S^{(2)}_{(x_2)} = \mathbb{C}[x_0^2, x_0 x_1, x_1^2, x_2]_{(x_2)} \cong \mathbb{C}[x^2, xy, y^2] \cong \mathbb{C}[A, B, C]/B^2 = CA$ which is not smooth over $\mathbb{C}$.

Notation 2.3.13. From now on we shall denote with the arrows $\hookrightarrow\!\!\circ\!\!\to \hookrightarrow\!\!/\!\!\to$ open and closed embeddings respectively.

2.4 Proof of the Embedding Theorem 2.0.1

Let X be a scheme as in 2.0.1 and $\{(L_i, s_i)\}_{i=1}^n$ a family of line bundles, that we can assume of the form given by Proposition 2.1.11. As in Section 3 we define the $\mathcal{O}_X$-algebra

$$\mathcal{B} := \bigoplus_{d \in \mathbb{Z}^n} L^d$$

where for $d = (d_1, ..., d_n) \in \mathbb{Z}^n$, $L^d = L_1^{\otimes d_1} \otimes ... \otimes L_n^{\otimes d_n}$ and $L^0 = \mathcal{O}_X$. Let $\widehat{X} := \mathrm{Spec}(\mathcal{B})$ be its (relative) spectrum and

$$q : \widehat{X} \to X$$

the affine projection morphism induced by the $\mathcal{O}_X$-algebra morphism $\mathcal{O}_X \hookrightarrow \mathcal{B}$. Note that the $\mathbb{Z}^n$-grading on $\mathcal{B}$ induces an action of the diagonalizable group $G := \mathrm{Spec} R[\mathbb{Z}^n]$ on $\widehat{X}$ (see [DG11]) but we won't need that.

Due to our assumptions on the ample family of line bundles (2.1.11), for any $i = 1, ..., n$, we have that $X_{s_i} \cong \mathrm{Spec}(A_i) \subseteq X$ is an open affine subscheme of X and we have elements $x_{i1}^{\pm 1}, ..., x_{ij}^{\pm 1}, ..., x_{in}^{\pm 1} \in L_j^{\pm 1}(X_{s_i})$ given by $x_{ij} = \alpha_{ij}(1)$ (notice $x_{ij}^{-1} = (\alpha_{ij}^\vee)(1))$, with $x_{ii} = s_{i|X_{s_i}}$ such that $\deg(x_{ij}^{\pm 1}) = \pm(0, ..., 1, 0, ..., 0)$ with 1 in the jth place, giving an $\mathcal{O}_{X|X_{s_i}}$-algebra isomorphism $\mathcal{O}_{X|X_{s_i}}[x_1^{\pm 1}, ..., x_n^{\pm 1}] \xrightarrow{\cong} \mathcal{B}_{X_{s_i}}$, $x_j^{\pm 1} \mapsto x_{ij}^{\pm 1}$ and $A_i[x_1^{\pm 1}, ..., x_n^{\pm 1}] \xrightarrow{\cong} \mathcal{B}(X_{s_i})$.

This way, for any i, we have $q^{-1}(X_{s_i}) \cong \mathrm{Spec}(A_i[x_1^{\pm 1}, ..., x_n^{\pm 1}])$. We also get that q is a G-torsor so that it is a principal homogeneous bundle and then realizing X as a GIT quotient (for more on GIT quotients, the reader is referred to [MFK94]) of $\widehat{X}$ under the action of G induced by the grading.

Remark 2.4.1. For every $\mathfrak{p} \in X_{s_i}$, we have that $x_{ij}^{\pm 1}(\mathfrak{p}) \neq 0$. In fact, under the isomorphism (depending on j) of $\mathsf{k}(\mathfrak{p})$-vector spaces $\mathcal{B}_\mathfrak{p} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} \mathsf{k}(\mathfrak{p}) \cong \mathsf{k}(\mathfrak{p})[x_1^{\pm 1}, ..., x_n^{\pm 1}]$, $x_{ij}^{\pm 1}(\mathfrak{p})$ corresponds to $x_j^{\pm 1}$ and so it is different from 0.

Using the fact that X is locally of finite type over R, we have that for every i, A_i is a finitely generated R-algebra, generated say by a finite set $E_i \subset A_i$ of cardinality $l_i \in \mathbb{N}$ of elements e_{ik}. We look at these elements as elements of $\mathcal{O}_X(X_{s_i}) = \mathcal{B}_0(X_{s_i})$ so that $\mathcal{B}(X_{s_i})$ will be a finitely generated R-algebra generated by the elements $e_{ik}, x_{ij}^{\pm 1}$.

We now use [GD71]I 6.8.1 (or [GW10] 7.22) or better their proof to find, for every i, a positive integer d and elements $e_{ik}' \in \Gamma(X, L_i^{\otimes(d+1)})$, $x_{ij}' \in \Gamma(X, L_j \otimes L_i^d)$ such that $x_{ii}' = s_i^{\otimes d+1}$, $e_{ik|X_{s_i}}' = e_{ik} \otimes x_{ii}^{\otimes d+1}$ and $x_{ij|X_{s_i}}' = x_{ij} \otimes x_{ii}^{\otimes d}$. We note that we can take d to work for every i.

Remark 2.4.2. In view of Remark 2.4.1, by tensoring any element e_{ik}', x_{ij}' with s_i (i.e. increasing d by one), we can assume that for every i, j it is true that $X_{x_{ij}'} = X_{s_i}$. Moreover we have that in $\Gamma(X, \mathcal{B})$ they are all homogeneous and $\deg(e_{ik}') = (0, ..., d+1, 0, ...0)$ with $d+1$ in place i, $\deg(x_{ii}') = \deg(e_{ik}')$, $\deg(x_{ij}') = (0, ..., 1, 0, ..., d, 0, ..., 0)$ with 1 in the jth place and d in the ith place if $i \neq j$.

If we denote as $w_1, ..., w_n$ the canonical basis of $\mathbb{Z}^n$ as $\mathbb{Z}$-module, we can write $\deg(x'_{ij}) = dw_i + w_j$ and $\deg(e'_{ik}) = \deg(x'_{ii})$ for every i, j.

Now let $T_i := \prod_{j=1}^{n} x'_{ij} \in \Gamma(X, \mathcal{B})$. Note that $\deg(T_i) = (1, ..., nd + 1, 1, ..., 1)$ and that by [GD71]0, 4.1.10 and 2.4.2 $X_{T_i} = X_{s_i}$ for every $i = 1, ..., n$. We now consider B to be the R-subalgebra of $\Gamma(X, \mathcal{B})$ generated by e'_{ik}, x'_{ij}. Consider the inclusion $\varphi' : B \to \Gamma(X, \mathcal{B})$, and let B be graded in such a way that φ' is a homogeneous map. This map induces ([GW10] 3.4) a map

$$\varphi = \operatorname{Spec}(\varphi') : \widehat{X} \to \operatorname{Spec}(B) =: Y$$

since $\Gamma(X, \mathcal{B}) = \Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}})$.

We shall need the following lemma, which has general interest and puts together [GD71] 0.5.4.6 and I 6.8.1-6.8.2

Lemma 2.4.3. *Suppose X is a noetherian scheme, $\mathcal{F}$ an $\mathcal{O}_X$-algebra, $\mathcal{L}$ a line bundle, $\Gamma_*(\mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{L}^{\otimes n})$ and $M_* := \Gamma(\mathcal{L}, \mathcal{F})$ the $\Gamma_*(\mathcal{L})$-algebra $\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$. Then if $s \in \Gamma(X, \mathcal{L})$ we have a ring isomorphism*

$$(M_*)_s \cong \Gamma(X_s, \bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

Proof. We give two proofs of this fact. The first one goes as follows. We notice that we have a canonical restriction ring homomorphism $\varphi : (M_*)_s \to \Gamma(X_s, \bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$. This map is injective because if $\frac{a}{s^m} \in (M_*)_s$ maps to zero, then $a_{|X_s} = 0$ and using [GD71] I 6.8.1.(i) we have that for some natural number n, $a \otimes s^{\otimes n} = 0$ in M_* and so $a = 0$ in $(M_*)_s$ which conclude the injectivity part. Moreover φ is surjective because of [GD71] I 6.8.1. (ii). We can now give the second proof. Because of the definition of X_s, $\Gamma(X_s, \bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \cong \Gamma(X_s, \mathcal{F}) \otimes_{\Gamma(X_s, \mathcal{O}_X)} \Gamma(X_s, \mathcal{O}_X)[t, t^{-1}]$. Now, we use [GD71] I 6.8.1 to choose, for every $f \in \Gamma(X_s, \mathcal{F})$ a lift $f' \in M_*$ of $f \otimes s^{n_f}$ such that the following map is a ring homomorphism

$$\Gamma(X_s, \bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \to (M_*)_s, \qquad f \otimes t^d \mapsto f \otimes s^{d+n_f}$$

This map is surjective (every element in the target restricts to an element in the source via the canonical restriction map) and injective because if $f \otimes s^{d+n_f} = 0$ then $f_{|X_s} = 0$ since s does not vanish on X_s. $\square$

Proposition 2.4.4. $\varphi : \widehat{X} \to Y$ *is an open embedding.*

Proof. Let $\{\varphi_{T_i} : q^{-1}(X_{T_i}) = q^{-1}(X_{s_i}) \subseteq \widehat{X}\}$ be an open affine cover of $\widehat{X}$ (recall q is an affine map). We consider the open subscheme

$$i : \widehat{Y} := \cup_{i=1}^n \mathrm{Spec}(B_{T_i}) \lhook\joinrel\longrightarrow Y$$

and we want to show that φ factor as a composition $i \circ \beta : \widehat{X} \overset{\cong}{\longrightarrow} \widehat{Y} \lhook\joinrel\longrightarrow Y$. We want to check this locally, i.e. we want to show that for any i, in the following pullback diagram

$$
\begin{array}{ccc}
A & \longrightarrow & q^{-1}(X_{s_i}) \\
\downarrow & & \downarrow{\scriptstyle \varphi \circ \varphi_{T_i}} \\
\mathrm{Spec}(B_{T_i}) & \longrightarrow & \mathrm{Spec}(B)
\end{array}
$$

we have that both the upper horizontal arrow and the left vertical arrow are isomorphisms and this will conclude the proof. We want to show that $\Gamma(X_{T_i}, \mathcal{B}) \cong \Gamma(X, \mathcal{B})_{T_i} \cong B_{T_i}$, this way, considering the previous pullbacks, because for any i we have $q^{-1}(X_{s_i}) = q^{-1}(X_{T_i}) \cong \mathrm{Spec}(\Gamma(X_{T_i}, \mathcal{B}))$, the argument will be concluded. The first isomorphism exists because of 2.4.3, so that we are left to prove $B_{T_i} \cong \Gamma(X, \mathcal{B})_{T_i}$.

Remark 2.4.5. Notice that we are really allowed to use Lemma 2.4.3 because if in the statement we have that $\mathcal{F}$ is a $\mathbb{Z}^{n-1}$-graded $\mathcal{O}_X$-algebra ($n \geq 1$) then the $\mathcal{O}_X$-algebra $\bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is $\mathbb{Z}^n$-graded and $M_* := \Gamma(\mathcal{L}, \mathcal{F})$ is a $\mathbb{Z}^n$-graded algebra as well.

In what it is left of the proof, i is going to be fixed. The first step is to notice that B is a finitely generated R-algebra so that it is of the form $R[e'_{pk}, x'_{pj}]/I$ ($p, j \in \{1, ..., n\}$) with I ideal generated by finitely many (R is noetherian so by Hilbert basis Theorem, B is finitely presented) polynomials $p_1, ..., p_\alpha$. So $B_{T_i} \cong R[e'_{pk}, x'_{pj}, (x'_{ij})^{-1}]/I'$ where I' is generated by $p_1, ...p_\alpha, x'_{i1} \cdot (x'_{i1})^{-1} = 1, ..., x'_{in} \cdot (x'_{in})^{-1} = 1$. Now, polynomial relations on the form $p_i = 0$ which holds in B are true also in $\Gamma(X, \mathcal{B})$ since the former is a subalgebra of the latter, so they induces relations also on $\Gamma(X_{T_i}, \mathcal{B}) \cong \Gamma(X, \mathcal{B})_{T_i}$. We already know (using the notation introduced before) that $\Gamma(X_{T_i}, \mathcal{B}) \cong R[e_{ik}, x_{ij}^{\pm 1}]/J$ where J is an ideal presented by a finite number of polynomials $g_1, ...g_\gamma, x_{i1} \cdot x_{i1}^{-1} = 1, ..., x_{in} \cdot x_{in}^{-1} = 1$. Now using $\Gamma(X_{T_i}, \mathcal{B}) \cong \Gamma(X, \mathcal{B})_{T_i}$ we see that the elements $e'_{pk}, x'_{pj} \in \Gamma(X, \mathcal{B})$ maps surjectively to the family e_{ik}, x_{ij} (up to multiplication by x_{ii}^{-d} or by $x_{ii}^{-(d+1)}$) so that we see that $\Gamma(X, \mathcal{B})_{T_i} \cong R[e'_{pk}, x'_{pj}, (x'_{pj})^{-1}]/J'$ where J' is generated by $f_1, ..., f_\beta, x'_{i1} \cdot (x'_{i1})^{-1} = 1, ..., x'_{in} \cdot (x'_{in})^{-1} = 1$ where the polynomials f_l comes from the polynomials g_q via

the isomorphism $\Gamma(X_{T_i}, \mathcal{B}) \cong \Gamma(X, \mathcal{B})_{T_i}$. Now the restrictions of the relations $p_1 = 0, ..., p_\alpha = 0$ are implied by the relations $f_1, ..., f_\beta$ after inverting T_i, by definition, hence $(I' \subseteq J')$. Now, all the polynomial relations f_l in $\Gamma(X_{T_i}, \mathcal{B}) \cong \Gamma(X, \mathcal{B})_{T_i}$ can be lifted to polynomials f'_l in $\Gamma(X, \mathcal{B})$ in the variables $e'_{ik} \cdot (x'_{ii})^m, x'_{ij} \cdot (x'_{ii})^m$ for a suitable m(use again [GD71] I 6.8.1 (ii)) and with coefficients in R so that after inverting T_i, they will be implied by the relations in the set $q_1, ..., q_\alpha$. This shows $J' \subseteq I'$ so that $J' = I'$ and the proof is concluded. $\square$

From the previous proposition we actually find that the image of the open embedding φ is $\bigcup_{i=1}^n D(T_i)$. Now let be $E := R[t_{ik}, z_{ij}]$, i, k, j be as above and consider the surjective homogeneous R-algebra homomorphism

$$\psi' : E = R[t_{ik}, z_{ij}] \longrightarrow\!\!\!\!\!\rightarrow B$$
$$t_{ik} \longmapsto e'_{ik}$$
$$z_{ij} \longmapsto x'_{ij}$$

where by homogeneous we mean that we are giving to E a $\mathbb{Z}^n$-grading induced by ψ'. This homomorphism induces a closed immersion

$$\psi : Y = \mathrm{Spec}(B) \to \mathrm{Spec}(E) =: Z$$

Every $f \in E$ will induce a map $\psi'_f : E_f \to B_{\psi'(f)}$ such that

$$
\begin{array}{ccc}
\mathrm{Spec}(B_{\psi'(f)}) = D(\psi'(f)) & \xrightarrow{\quad\psi_f\quad} & D(f) \\
\cap\downarrow & & \cap\downarrow \\
Y & \xrightarrow{\quad\psi\quad} & Z
\end{array}
$$

is a pullback. Here we have denoted with $\hookrightarrow\!\!\circ\!\!\rightarrow \hookrightarrow\!\!/\!\!\rightarrow$ open and closed embeddings respectively. Hence letting $T'_i = \prod_{j=1}^n z_{ij}$ we have that $\psi'(T'_i) = T_i$ and so we get closed embeddings s.t.

$$
\begin{array}{ccc}
D(T_i) & \xrightarrow{\quad/\quad} & D(T'_i) \\
\cap\downarrow & & \cap\downarrow \\
Y & \xrightarrow{\quad\psi\quad} & Z
\end{array}
$$

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are pullbacks. Now using the fact that

$$
\begin{array}{ccc}
q^{-1}(X_{T_i}) & \xrightarrow{\;\cong\;} & D(T_i) \\
\cap\downarrow & & \cap\downarrow \\
\widehat{X} & \xhookrightarrow{\;\;\varphi\;\;} & Y
\end{array}
$$

are pullbacks, one can check that the following is also a pullback

$$
\begin{array}{ccccc}
\widehat{X} & \xrightarrow{\cong} & \bigcup_{i=1}^{n} D(T_i) & \lhook\joinrel\longrightarrow & \bigcup_{i=1}^{n} D(T_i') =: Z_{\widehat{X}} \\
{\scriptstyle\cong}\downarrow & & \downarrow & & \downarrow \\
\widehat{X} & \xhookrightarrow{\varphi} & Y & \lhook\joinrel\longrightarrow{\psi} & Z
\end{array}
$$

This means that the preimage of $Z_{\widehat{X}}$ under $\rho = \psi \circ \varphi$ is exactly $\widehat{X}$. Note also that all these maps come from ring maps which are homogeneous, i.e. preserve the $\mathbb{Z}^n$-grading. In other words, they are G-equivariant morphisms for the action of G induced naturally by the grading. Moreover, ρ is actually a (quasi-compact) embedding because all the schemes considered are noetherian.

proof of 2.0.1. Let $C := \Gamma(X, \mathcal{B})$. We have a homogeneous map $\alpha = \varphi' \circ \psi' : E = R[t_{i,k}, z_{ij}] \to \Gamma(X, \mathcal{B}) = C$ giving the map ρ as above. Now using 2.2.1 we have that T_i' are relevant elements in E and $\alpha(T_i') = T_i$ for any $i = 1, ..., n$. We can then use 2.3.11 to get the following map that is also a closed embedding as we show below

$$
f : X \lhook\joinrel\longrightarrow \bigcup_{i=1}^{n} D_+(T_i') =: W \subseteq \operatorname{Proj}(E)
$$

. In fact f is closed because of the construction (the ring maps $E_{(T_i')} \twoheadrightarrow A_i$ are in fact surjective: use again $\Gamma(X_{T_i}, \mathcal{B}) \cong \Gamma(X, \mathcal{B})_{T_i}$) and we already noted that $X = \bigcup_{i=1}^{n} X_{T_i}$ so we can apply 2.3.11. Moreover W is smooth (and noetherian since it is quasi-compact) over R since the $D_+(T_i')$ are smooth over R because of Lemma S 2.2.1. In addition, $\operatorname{Proj}(E)$ is divisorial because of 2.3.10, and W is a noetherian open subscheme of it (hence it embeds into $\operatorname{Proj}(E)$ via a quasi-compact open embedding) so by 2.1.9 it is divisorial. The theorem is now fully proved. $\square$

Remark 2.4.6. • We note that the morphism ρ obtained in the proof can be seen as an equivariant morphism of schemes for the action of $G = \operatorname{Spec}(R[\mathbb{Z}^n])$

induced by the grading. Hence one might check that the embedding we built can be seen as arising from ρ after taking the GIT quotient. This is in fact the strategy of the argument contained in [Hau02]. Both our proof and the proof of Theorem 4.4 in [BS03] have in fact to be regarded as a generalisation of the proof of Theorem 3.2 contained in [Hau02].

- If we start with $X = \mathbb{P}_r^n$ and we consider the ample family of line bundles $\{\mathcal{O}(1), x_i\}_{i=1}^n$ we can run the construction of our smooth embedding choosing in the lifting process $d = 0$ and we end with a $W \subseteq \mathrm{Proj}(S)$ such that $W \cong \mathbb{P}_R^n$, by checking the gluing data.

- The scheme $\mathrm{Proj}(E)$ obtained in the proof is a simplicial torus embedding (with affine diagonal) as defined in [BS03] page 220.

Observation 2.4.7. We observe that the philosophy behind the previous proof is to find embeddings of $\mathrm{Spec}(A_i)$ in suitable affine spaces of the form $\mathbb{A}_R^{n_i}$ that can be patched together to a morphism from X to a suitable multihomogeneous projective space that contains those affine spaces as open subschemes. This is similar to solving the problem of finding local embeddings of a scheme into affine planes that glue together to give an embedding into the smooth locus of a suitable weighted projective space.

In fact we have the following corollary of our proof

Corollary 2.4.8. *Suppose to have a gluing datum*

$$\mathcal{G} = (\mathrm{Spec}(A_i), U_{ij} \subseteq \mathrm{Spec}(A_i), \varphi_{ji} : U_{ij} \xrightarrow{\cong} U_{ji})$$

$i = 1, ..., n$ where A_i are finitely generated R-algebras $A_i[e_{il}]/J$ for some noetherian ring R. Then if the scheme X resulting from $\mathcal{G}$ is divisorial, it admits a closed embedding to a divisorial scheme W arising from a gluing datum of the form

$$((\mathbb{A}_R^m)_i, W_{ij} \subseteq (\mathbb{A}_R^m)_i, \psi_{ji} : W_{ij} \xrightarrow{\cong} W_{ji})$$

for some positive integer m.

2.5 Applications

Let $\mathcal{E}$ be a vector bundle over a scheme Y. Denote as $\mathrm{Grass}_{n,Y}(\mathcal{E})$ (or simply $\mathrm{Grass}_n(\mathcal{E})$ if the base scheme is clear) the Grassmannian functor which associates to a Y-scheme $f : X \to Y$ the set of locally free $\mathcal{O}_X$-modules of rank n quotients

of $f^*\mathcal{E}$ ([GD71] I 9.7.3). This functor is representable by a Y-scheme $\mathbf{Grass}_{n,Y}(\mathcal{E})$ ([GD71] I 9.7.4).

Proposition 2.5.1. *If Y is a quasi-separated scheme having an ample family of line bundles and $\mathcal{E}$ is a vector bundle on it, then for any $n \in \mathbb{N}$, $\mathbf{Grass}_{n,Y}(\mathcal{E})$ has an ample family of line bundles.*

Proof. For any n, [GD71] I 9.8.4 tells us that the Plucker embedding $\mathbf{Grass}_{n,Y}(\mathcal{E}) \to \mathbb{P}(\wedge^n(\mathcal{E}))$ is a closed embedding. Since $\mathbb{P}(\mathcal{E})$ has an ample family of line bundles by 2.1.10, we conclude using 2.1.9 that $\mathbf{Grass}_{n,Y}(\mathcal{E})$ has an ample family too. $\qquad\square$

Proposition 2.5.2. *Under the above assumptions, $\mathbf{Grass}_{n,Y}(\mathcal{E})$ is separated and smooth over Y.*

Proof. We have separatedness because of [GD71] I 9.7.7. Moreover, choose a trivializing open subset U of Y for $\mathcal{E}$, so that $\mathcal{E}_{|U} \cong \mathcal{O}_{X|U}^m$ for some m. Then $\mathbf{Grass}_{n,Y}(\mathcal{E}) \times_Y U \cong \mathbf{Grass}_{n,U}(\mathcal{O}_{X|U}^m)$ is an open subscheme of $\mathbf{Grass}_{n,Y}(\mathcal{E})$, see [GD71] I 9.7.6. Varying U, this schemes form a Zariski cover for $\mathbf{Grass}_{n,Y}(\mathcal{E})$. But now we know that $\mathbf{Grass}_{n,U}(\mathcal{O}_{X|U}^n)$ is smooth over U because it is the standard Grassmannian, see [GW10] Corollary 8.15 for example. So because U is smooth over Y, it is easily verified by choosing a trivializing cover of Y that $\mathbf{Grass}_{n,Y}(\mathcal{E})$ is smooth over Y. $\quad\square$

We can now prove the following Proposition, whose proof relies on Theorem 2.5.2.

Proposition 2.5.3. *Let X be a scheme over a noetherian base scheme S admitting a quasi-compact immersion $i : X \to Y$ into a smooth divisorial scheme Y over S. Then for every vector bundle $\mathcal{E} \in \mathrm{Vect}(X)$ there is a smooth divisorial scheme $Y_\mathcal{E}$ over both Y and S, with a vector bundle $\mathcal{E}_{Y_\mathcal{E}}$ over $Y_\mathcal{E}$ together with a morphism $\psi_\mathcal{E} : X \to Y_\mathcal{E}$ such that $\psi_\mathcal{E}^*(\mathcal{E}_{Y_\mathcal{E}}) \cong \mathcal{E}$.*

Proof. Assume X is connected as a topological space so that every vector bundle over it has constant rank. Denote by $L_1, ..., L_n$ an ample family of line bundles for Y, so that $i^*L_1, ..., i^*L_n$ are an ample family for X (2.1.9). Given a vector bundle $\mathcal{E}$ of rank m, we have, by the properties of the ample families of line bundles ([SGA71] II 2.2.3 or [TT90] 2.1-2.1.3), that there exists natural numbers $p, _j q_1, ..., _j q_n$ where $j = 1, ..., p$ and a surjective map

$$L' := \bigoplus_{j=1}^p (\otimes_{i=1}^n i^*L_i^{\otimes -(_j q_i)}) \twoheadrightarrow \mathcal{E}$$

is an epimorphism. Now note that L' is a vector bundle and defining $L :=$ $\bigoplus_{j=1}^{p}(\otimes_{i=1}^{n}L_i^{\otimes -(_jq_i)})$, L is a vector bundle, and $i^*L \cong L'$. Define now $Y_{\mathcal{E}} :=$ $\mathbf{Grass}_{m,Y}(L)$, and note that this scheme is smooth over Y (2.5.2), and, since Y is smooth over S, it is also smooth over S and it is also divisorial because of 2.5.1. By (1), we have that $\mathcal{E}$ is an element of $\mathrm{Grass}_{m,Y}(L)(X)$, so that we can use the universal property of the Grassmannian ([GD71] I 9.7.5) to find $\psi_{\mathcal{E}}$ and $\mathcal{E}_{Y_{\mathcal{E}}}$ as in the statement of the proposition (this last one will be the tautological vector bundle over $Y_{\mathcal{E}}$). Hence we proved the proposition for X connected. Assume now X not connected and consider $\mathcal{E}$ vector bundle on it. Use 2.1.1 to write X as a disjoint union of its finite (remember divisorial schemes are assumed to be quasi-compact) connected components $(X, \mathcal{O}_X) \cong \amalg_{\alpha \in I}(U_\alpha, \mathcal{O}_{X|U_\alpha})$ and let be $\mathcal{E}_\alpha := \mathcal{E}_{|U_\alpha}$. Every U_α is connected, so that we can apply the previous case to every U_α (note that U_α are noetherian so they embed with a quasi-compact open embedding in X, and so they embed in Y with a quasi-compact immersion because the composition of two immersions is still an immersion, [GD67] I 4.2.5) and find $Y_{\mathcal{E}_\alpha} := \mathbf{Grass}_{n_\alpha,Y}(L_\alpha)$, $\psi_{\mathcal{E}_\alpha}$ and $\mathcal{E}_{Y_{\mathcal{E}_\alpha}}$ as before, where n_α is the rank of $\mathcal{E}_\alpha$ on U_α. Defining now $Y_{\mathcal{E}} := \amalg_{\alpha \in I}Y_{\mathcal{E}_\alpha}$, $\mathcal{E}_{Y_{\mathcal{E}}} := \oplus_{\alpha \in I}\mathcal{E}_{Y_{\mathcal{E}_\alpha}}$ (which is a vector bundle over $Y_{\mathcal{E}}$) and gluing together the maps $\psi_{\mathcal{E}_\alpha}$ to a map $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ we see that we have obtained the elements requested in the statement of the proposition, whose proof is now complete. $\qquad\square$

We extend the previous result to a finite number of vector bundles

Proposition 2.5.4. *Let X be a scheme over a noetherian base S admitting a quasi-compact immersion $i : X \to Y$ into a smooth divisorial scheme Y over S. Then given a finite number vector bundles $\mathcal{E}_1, ..., \mathcal{E}_n \in \mathrm{Vect}(X)$ there is a smooth divisorial scheme $Y_{\mathcal{E}}$ over S and a vector bundles $\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{n,Y_{\mathcal{E}}}$ over it together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ such that $\psi_{\mathcal{E}}^*(\mathcal{E}_{i,Y_{\mathcal{E}}}) \cong \mathcal{E}_i$ for every $i = 1, ..., n$.*

Proof. We apply n times Proposition 2.5.3 to get n schemes $Y_{\mathcal{E}_1}, ..., Y_{\mathcal{E}_n}$, vector bundles $\mathcal{E}_{Y_{\mathcal{E}_1}}, ..., \mathcal{E}_{Y_{\mathcal{E}_n}}$ and morphisms $\psi_{\mathcal{E}_i} : X \to Y_{\mathcal{E}_i}$ for any $i = 1, ..., n$ such that $\psi_{\mathcal{E}_i}^*(\mathcal{E}_{Y_{\mathcal{E}_i}}) \cong \mathcal{E}_i$ for any i. Define

$$Y_{\mathcal{E}} := Y_{\mathcal{E}_1} \times_Y \cdots \times_Y Y_{\mathcal{E}_n}, \quad \psi_{\mathcal{E}} := \psi_{\mathcal{E}_1} \times \cdots \times \psi_{\mathcal{E}_n} : X \to Y_{\mathcal{E}}$$

$$\mathcal{E}_{i,Y_{\mathcal{E}}} := \mathrm{pr}_i^*(\mathcal{E}_{Y_{\mathcal{E}_i}})$$

Note $Y_{\mathcal{E}}$ is still smooth over Y and divisorial because the $Y_{\mathcal{E}_i}$ are quasi-projective (they are grassmannians, which are quasi-projective via the Plucker embedding) and quasi-compact quasi-separated over Y so since quasi-projective morphisms are

stable under base change by [GD67] II.5.3.4 (iii) we can use [TT90] 2.1.2 (h). Since $\psi_{\mathcal{E}_i} = \mathrm{pr}_i \circ \psi_{\mathcal{E}}$ we know $\psi_{\mathcal{E}_i}^* \cong \psi_{\mathcal{E}}^* \mathrm{pr}_i^*$ so that for every i,

$$\psi_{\mathcal{E}}^*(\mathcal{E}_{i,Y_{\mathcal{E}}}) \cong \psi_{\mathcal{E}}^*(\mathrm{pr}_i^*(\mathcal{E}_{Y_{\mathcal{E}_i}})) \cong \psi_{\mathcal{E}_i}^*(\mathcal{E}_{Y_{\mathcal{E}_i}}) \cong \mathcal{E}_i$$

$\square$

Merging Proposition 2.5.2 and Theorem 2.0.1 we immediately get the following

Proposition 2.5.5. *Let X be a scheme of finite type over a noetherian affine scheme $S = \mathrm{Spec}(R)$ having an ample family of line bundles. Then given a finite number vector bundles $\mathcal{E}_1, ..., \mathcal{E}_n \in \mathrm{Vect}(X)$ there is a smooth divisorial scheme $Y_{\mathcal{E}}$ over S and vector bundles $\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{n,Y_{\mathcal{E}}}$ over it together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ such that $\psi_{\mathcal{E}}^*(\mathcal{E}_{i,Y_{\mathcal{E}}}) \cong \mathcal{E}_i$ for every $i = 1, ..., n$.*

Chapter 3

Endomorphisms in Higher K-Theory: going to K_0

3.1 From the homotopic world to K_0

We shall stick to the notations and assumptions detailed in 0.1 unless othewise indicated. Thanks to the result of Riou A.3.14 detailed in the appendix and Theorem 1.8.8, we begin the path to extend the work of Riou. That theorem identifies the endomorphisms of K-theory in the unstable motivic homotopy category with genuine endomorphisms of a presheaf of sets, K_0, which are well studied. Via this link we can obtain the necessary information to use the operations at the level of K_0 to define operations on higher K-theory without too much efforts: one simply use the theorem to lift them! So one obtains a powerful tool to study operations on higher K-theory using what is known at level of K_0, and one obtains in an easier way important results such as higher Riemann-Roch theorems. This was done by Riou in [Rio10], obtaining new theorems and also new proofs of some results of Gillet ([Gil81]). Operations on K-theory were studied with other methods because of their importance (see [Lev97], [GS99], [Sou85], [HKT17] and many others), so this new tool to define and study them is very important because it allows to have these operations in an easier way and it allows to easily get many functorial properties. In addition, because of the generality of the argument involved, this method could be used to study also operations on other theories which are representable in the unstable category by nice geometric objects as for example Hermitian K-theory. Moreover, by the explicit calculations of the K_0 of the Grassmannians due to Berthelot ([SGA71]) one can also obtain a more explicit description of these

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operations, using the fact that Grassmannians represent K-theory in the unstable motivic homotopy category. More precisely, one gets

$$[K, K]_{\mathcal{H}(S)} \cong \operatorname{End}_{\mathcal{H}(S)}(\mathbb{Z} \times \operatorname{Gr}) \cong \operatorname{End}_{\mathbf{Pre}(\operatorname{Sm}/S)}(K_0(-))$$
$$\cong \prod_{i \in \mathbb{Z}} K_0(S)[[c_1, ..., c_n, ...]] \qquad (3.1)$$

were the c_i are the usual Chern classes as detailed in [SGA71] or [Rio06]. This result of Riou only works on the Nisnevich site on Sm/S. One would then investigate if such results are true if we change the site. In particular it would be nice to say something for sites containing singular schemes over some nice scheme (fields for example), since the classical results such as the Riemann-Roch theorems also apply to this cases. From now on we will stick to the notation introduced in 0.1.1. The work done in the previous sections goes in this direction. We have linked the endomorphisms of K-theory on a site larger than the smooth one with endomorphisms in the unstable category in Section 1.8, so that merging those results with the theorem of Riou we immediately get the following

Proposition 3.1.1. *Let S be a regular scheme. Hence we have*

$$[K, K]_{\mathcal{P}^l_{Zar}\operatorname{Sch}_S} \cong [K, K]_{\mathcal{H}^{\operatorname{Sch}_S}_{Zar}} \cong [K, K]_{\mathcal{H}(S)} \cong \operatorname{End}_{\mathcal{H}(S)}(\mathbb{Z} \times \operatorname{Gr})$$
$$\cong \operatorname{End}_{\mathbf{Pre}(\operatorname{Sm}/S)}(K_0(-)) \cong \prod_{i \in \mathbb{Z}} K_0(S)[[c_1, ..., c_n, ...]]$$

Proof. This follows under the assumptions detailed in the previous sections from 1.8.8, the theorem of Riou A.3.14 and the fact that the projective local model structure is homotopy equivalent to the injective local one. $\qquad \square$

This is yet interesting since it gives us an explicit description of the operations of K-theory in $\mathcal{H}^{\operatorname{Sch}_S}_{Zar}$. We would now like to strengthen this result by proving that $[K, K]_{\mathcal{H}^{\operatorname{Sch}_S}_{Zar}} \cong \operatorname{End}_{\mathbf{Pre}(\operatorname{Sch}_S)}(K_0(-))$. Before that we note that even at this stage we have as a corollary of 1.8.7, the following proposition

Proposition 3.1.2. *If S is a regular noetherian scheme*

$$[K, K]_{\mathcal{H}^{\operatorname{Sch}_S}_{Zar}} \cong [K, K]_{\mathcal{I}^l_{Nis}\operatorname{Sm}/S} \cong [\mathbb{Z} \times \operatorname{Gr}, K]_{\mathcal{H}(S)}$$

$$[(\mathbb{Z} \times \operatorname{Gr})^n, K]_{\mathcal{H}^{\operatorname{Sch}_S}_{Zar}} \cong [(\mathbb{Z} \times \operatorname{Gr})^n, K]_{\mathcal{I}^l_{Nis}\operatorname{Sm}/S} \cong [(\mathbb{Z} \times \operatorname{Gr})^n, K]_{\mathcal{H}(S)}$$

3.1.1 Endomorphisms of K-theory only depend on π_0: statement of the theorem

Fixed S regular noetherian scheme we now consider the Zariski sites Sch_S of quasi-compact quasi-separated schemes of finite type over S admitting an ample family of line bundles and the full subcategory Sm/S of smooth schemes over S. We denote their injective local model categories as $\mathcal{I}^l_{Zar}\mathrm{Sch}_S$ and $\mathcal{I}^l_{Zar}\mathrm{Sm}/S$. Recall we have an obvious restriction functor

$$\mathrm{res} : \mathbf{Pre}(\mathrm{Sch}_S) \to \mathbf{Pre}(\mathrm{Sm}/S)$$

which takes a presheaf over the category Sch_S to its restriction on Sm/S. This also leads to a functor at the level of simplicial presheaves,

$$\mathrm{sres} : \mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S) \to \mathrm{s}\mathbf{Pre}(\mathrm{Sm}/S)$$

and we explicitly noted in Remark 1.8.9 that this functor together with the identity functors induces the isomorphisms

$$[K,K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}} \cong [K,K]_{\mathcal{H}^{\mathrm{Sm}/S}_{Nis}} \cong [K,K]_{\mathcal{H}(S)}$$

Hence we can draw the following commutative diagram

$$
\begin{array}{ccc}
[K,K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}} & \xrightarrow{\;\cong\;} & [K,K]_{\mathcal{H}(S)} \\
{\scriptstyle \pi_0}\big\downarrow & & \cong\big\downarrow{\scriptstyle \pi_0} \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-),K_0(-)) & \xrightarrow{\;\mathrm{res}\;} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-),K_0(-))
\end{array}
$$

where for a simplicial presheaf F satisfying descent, we denote $\pi_0 F := [-,F]_{\mathcal{H}}$ with $\mathcal{H}$ the appropriate homotopy category. We could have argued similarly for the case of n variables in the first entry of the various Hom spaces considered so that we can also draw the following

$$
\begin{array}{ccc}
[K^n,K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}} & \xrightarrow{\;\cong\;} & [K^n,K]_{\mathcal{H}(S)} \\
{\scriptstyle \pi_0}\big\downarrow & & \cong\big\downarrow{\scriptstyle \pi_0} \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-)^n,K_0(-)) & \xrightarrow{\;\mathrm{res}\;} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-)^n,K_0(-))
\end{array}
$$

Suppose we are able to prove that the lower horizontal arrows are also injective. Then we would have

Conjecture 3.1.3 (conjecture). *Let S be a regular noetherian scheme. Then for any $n \in \mathbb{N}$*

$$\pi_0 : [K, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-), K_0(-))$$

$$\pi_0 : [K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-)^n, K_0(-))$$

are bijections.

We are going to prove the previous conjecture for S being affine

Theorem 3.1.4. *Let $S = \mathrm{Spec}(R)$ be a regular noetherian affine scheme of finite Kruul dimension and Sch_S be the category of divisorial schemes of finite type over S. Then the maps*

$$\pi_0 : [K, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-), K_0(-))$$

$$\pi_0 : [K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-)^n, K_0(-))$$

are bijections for any $n \in \mathbb{N}$.

Remark 3.1.5. From now on we will always suppose to point $K \in \mathcal{H}(S)$ $(\mathcal{I}_{Zar,Nis}^l(\mathrm{Sch}_S, \mathrm{Sm}/S))$ and $K_0 \in \mathbf{Pre}(\mathrm{Sch}_S, \mathrm{Sm}/S)$ with the same element of $K_0(S)$ whenever we consider these objects as pointed. Unless otherwise stated, from now on the default choice will be the one of $0 \in K_0(S)$.

Summing up we have, as a corollary

Corollary 3.1.6. *Under the assumptions of the previous theorem we have for any $n \in \mathbb{N}$*

$$[K, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-), K_0(-)) \cong [K, K]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-), K_0(-))$$

$$[K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-)^n, K_0(-)) \cong [K^n, K]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-)^n, K_0(-))$$

Moreover, the following pointed versions hold

$$[K, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}, \bullet} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)_\bullet}(K_0(-), K_0(-)) \cong [K, K]_{\mathcal{H}_\bullet(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)_\bullet}(K_0(-), K_0(-))$$

$$[K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}, \bullet} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)_\bullet}(K_0(-)^n, K_0(-)) \cong [K^n, K]_{\mathcal{H}_\bullet(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)_\bullet}(K_0(-)^n, K_0(-))$$

Proof. We only notice that the pointed version follows from A.3.10 (with $X = E = K$) and A.3.12 (with $F = G = K_0$). This because of the fact that S is final and that we are pointing K and K_0 coherently as remarked in Remark 3.1.5. $\qquad\square$

3.1.2 Proof of Theorem 3.1.4

We want to show that the lower horizontal maps res are injective. Denote as $\mathbf{Pre}(\mathrm{Sch}_S)$ the category of presheaves of sets over the category of divisorial schemes of finite type over a noetherian affine base scheme S as above, $\mathbf{Pre}(\mathrm{Sm}/S)$ the category of presheaves over the category of divisorial smooth schemes over S, and as $K_0(-)$ the algebraic K-theory presheaf. We can prove the following

Proposition 3.1.7. *Assume to have, for any given $n \in \mathbb{N}$, two natural transformations $K_0(-) \underset{g}{\overset{f}{\rightrightarrows}} K_0(-)$ on $\mathbf{Pre}(\mathrm{Sch}_S)$ which agree after restriction to Sm/S. Then $f = g$.*

Proof. For every $X \in \mathrm{Sch}_S$, by the very definition of $K_0(X)$, we will need to verify that for every element $\mathcal{E} \in K_0(X)$, $f_X(\mathcal{E}) = g_X(\mathcal{E})$. One first notices that (representatives of) elements $\mathcal{E} \in K_0(X)$ are of the form $\mathcal{E} = [E_0] - [E_1]$ where $E_0, E_1 \in \mathrm{Vect}(X)$. Now using 2.5.4, we can find for every such $E_0, E_1 \in \mathrm{Vect}(X)$ vector bundles over a scheme X, a divisorial smooth scheme $Y_{\mathcal{E}}$ over S and vector bundles E_0', E_1' over it together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ such that $\psi_{\mathcal{E}}^*(E_i') \cong E_i$ for $i = 1, 0$. One now notices, since pullback is a group homomorphism that this implies that the element $\mathcal{E}_{Y_{\mathcal{E}}} = ([E_0'] - [E_1']) \in K_0(Y_{\mathcal{E}})$ has the property that $\psi_{\mathcal{E}}^*(\mathcal{E}_{Y_{\mathcal{E}}}) = \mathcal{E}$. This means that for every $\mathcal{E} \in K_0(X)$ we can find a divisorial smooth scheme $Y_{\mathcal{E}}$ over S and $\mathcal{E}_{Y_{\mathcal{E}}} \in K_0(Y_{\mathcal{E}})$ together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ (over S) such that $\psi_{\mathcal{E}}^*(\mathcal{E}_{Y_{\mathcal{E}}}) = \mathcal{E}$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
K_0(Y_{\mathcal{E}}) & \overset{f_{Y_{\mathcal{E}}}}{\underset{g_{Y_{\mathcal{E}}}}{\rightrightarrows}} & K_0(Y_{\mathcal{E}}) \\
\psi_{\mathcal{E}}^* \downarrow & & \downarrow \psi_{\mathcal{E}}^* \\
K_0(X) & \overset{f_X}{\underset{g_X}{\rightrightarrows}} & K_0(X)
\end{array}
$$

which implies that $f_X(\mathcal{E}) = g_X(\mathcal{E})$. In fact we know that $f_{Y_{\mathcal{E}}} = g_{Y_{\mathcal{E}}}$ by assumption so that

$$f_X(\mathcal{E}) = (f_X \circ \psi_{\mathcal{E}}^*)(\mathcal{E}_{Y_{\mathcal{E}}}) = (\psi_{\mathcal{E}}^* \circ f_{Y_{\mathcal{E}}})(\mathcal{E}_{Y_{\mathcal{E}}}) = (\psi_{\mathcal{E}}^* \circ g_{Y_{\mathcal{E}}})(\mathcal{E}_{Y_{\mathcal{E}}}) = (g_X \circ \psi_{\mathcal{E}}^*)(\mathcal{E}_{Y_{\mathcal{E}}}) = g_X(\mathcal{E})$$

Iterating this for every $X \in \mathrm{Sch}_S$ and any $\mathcal{E} \in K_0(X)$ gives us the assert. $\qquad \square$

Now we come to the product case

Proposition 3.1.8. *Assume to have two natural transformations* $K_0(-)^n \underset{g}{\overset{f}{\rightrightarrows}} K_0(-)$ *on* $\mathbf{Pre}(\mathrm{Sch}_S)$ *which agree after restriction to* Sm/S. *Then* $f = g$.

Proof. As in the proof of the previous proposition, for every $X \in \mathrm{Sch}_S$, by the very definition of $K_0(X)$, we will need to verify that for every element $\mathcal{E} = (\mathcal{E}_1, ..., \mathcal{E}_n) \in K_0(X)^n$, $f_X((\mathcal{E}_1, ..., \mathcal{E}_n)) = g_X((\mathcal{E}_1, ..., \mathcal{E}_n))$. By 2.5.4, arguing as in the proof of the previous proposition, we can find for every such element $\mathcal{E}$, a smooth scheme $Y_{\mathcal{E}}$ over S and $\mathcal{E}_{Y_{\mathcal{E}}} = (\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{1,Y_{\mathcal{E}}}) \in K_0(Y_{\mathcal{E}})^n$ together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ such that $\psi_{\mathcal{E}}^*(\mathcal{E}_{i,Y_{\mathcal{E}}}) = \mathcal{E}_i$ for every $i = 1, ..., n$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
K_0(Y_{\mathcal{E}})^n & \xrightarrow[g_{Y_{\mathcal{E}}}]{f_{Y_{\mathcal{E}}}} & K_0(Y_{\mathcal{E}}) \\
{\scriptstyle \psi_{\mathcal{E}}^{*,n} := \psi_{\mathcal{E}}^* \times \cdots \times \psi_{\mathcal{E}}^*} \downarrow & & \downarrow {\scriptstyle \psi_{\mathcal{E}}^*} \\
K_0(X)^n & \xrightarrow[g_X]{f_X} & K_0(X)
\end{array}
$$

which implies that $f_X((\mathcal{E}_1, ..., \mathcal{E}_n)) = f_X(\mathcal{E}) = g_X(\mathcal{E}) = g_X((\mathcal{E}_1, ..., \mathcal{E}_n))$. In fact we know that $f_{Y_{\mathcal{E}}} = g_{Y_{\mathcal{E}}}$ by assumption so that

$$
f_X((\mathcal{E}_1, ..., \mathcal{E}_n)) = (f_X \circ \psi_{\mathcal{E}}^{*,n})((\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{n,Y_{\mathcal{E}}})) = (\psi_{\mathcal{E}}^* \circ f_{Y_{\mathcal{E}}})((\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{n,Y_{\mathcal{E}}})) =
$$
$$
(\psi_{\mathcal{E}}^* \circ g_{Y_{\mathcal{E}}})((\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{n,Y_{\mathcal{E}}})) = (g_X \circ \psi_{\mathcal{E}}^{*,n})((\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{n,Y_{\mathcal{E}}})) = g_X((\mathcal{E}_1, ..., \mathcal{E}_n))
$$

Iterating this for every $X \in \mathrm{Sch}_S$ and every $\mathcal{E} \in K_0(X)^n$ gives us the assert. $\square$

The previous propositions complete the proof of 3.1.4. We also notice the following

Proposition 3.1.9. *Assume to have, for a given presheaf* F *and a given* $n \in \mathbb{N}$, *two natural transformations* $K_0(-)^n \underset{g}{\overset{f}{\rightrightarrows}} F$ *on* $\mathbf{Pre}(\mathrm{Sch}_S)$ *which agree after restriction to* Sm/S. *Then* $f = g$.

Proof. Repeat verbatim the proof of the previous Proposition. $\square$

The structure of the previous proposition can be easily generalised as follows

Proposition 3.1.10. *Let $\mathcal{A} \subseteq \mathcal{C}$ a full subcategory of a given category $\mathcal{C}$ and* $\mathrm{Res} :$ $\mathbf{Pre}(\mathcal{C}) \to \mathbf{Pre}(\mathcal{A})$ *the restriction functor. Consider the map* $\mathrm{res} : \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(F, G) \to$ $\mathrm{Hom}_{\mathbf{Pre}(\mathcal{A})}(F, G)$ *induced by* Res *for two fixed* $F, G \in \mathbf{Pre}(\mathcal{C})$ *and assume that for every* $X \in \mathrm{Ob}(\mathcal{C})$ *and for every* $a \in F(X)$ *there exist* $Y_{X,a} \in \mathrm{Ob}(\mathcal{A})$, $\varphi : X \to Y_{X,a}$ *and* $b \in F(Y_{X,a})$ *so that* $\varphi_F^*(b) := F(\varphi)(b) = a$. *Then* res *is injective.*

Proof. Suppose we have two natural transformations $F \overset{f}{\underset{g}{\Longrightarrow}} G$ such that $\mathrm{res}(f) =$ $\mathrm{res}(g)$. To show that $f = g$ it suffices to show that for any $X \in \mathrm{Ob}(\mathcal{C})$,
$f_X = g_X : F(X) \Longrightarrow G(X)$ In order to do that, let us consider $a \in F(X)$, $Y_{X,a} \in$ $\mathcal{A}$, $b \in F(Y_{X,a})$ and $\varphi : X \to Y_{X,a}$ as in the statement. Then we have

$$f_X(a) = f_X(\varphi_F^*(b)) = \varphi_G^*(f_{Y_{X,a}}(b)) = \varphi_G^*(g_{Y_{X,a}}(b)) = g_X(\varphi_F^*(b)) = g_X(a)$$

Iterating this for any $a \in F(X)$ gives the result. $\qquad\qquad\square$

3.1.3 The case of S regular and quasi-projective over a noetherian affine scheme

Suppose S is a regular (in the absolute sense) quasi-projective (over R) scheme (hence of finite type) over a noetherian affine scheme R. We want to show, under the assumptions and the notations of 0.1 that, as in the previous cases, that all the arrows in the following diagram are isomorphisms

$$\begin{array}{ccc}
[K^n, K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}} & \overset{\cong}{\xrightarrow{\hspace{3cm}}} & [K^n, K]_{\mathcal{H}(S)} \\
{\scriptstyle \pi_0}\Big\uparrow & & \cong\Big\downarrow{\scriptstyle \pi_0} \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-)^n, K_0(-)) & \overset{\mathrm{res}}{\twoheadrightarrow} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-)^n, K_0(-))
\end{array} \qquad (3.2)$$

To do this we need the following lemma

Lemma 3.1.11. *Let X be a divisorial scheme of finite type over S quasi-projective scheme over a noetherian affine scheme R and let be $\mathcal{E}$ a vector bundle over X. Then there exists a scheme $Y_{\mathcal{E}}$ divisorial and smooth over S, a vector bundle $\mathcal{F}$ on $Y_{\mathcal{E}}$ and an arrow $\psi : X \to Y_{\mathcal{E}}$ over S such that $\psi^*\mathcal{F} \cong \mathcal{E}$.*

Proof. Because of the assumptions denoting $f : X \to S$ and $\varphi : S \to R$ the two structure morphisms, we have by Theorem 2.5.5 that there exists a divisorial scheme Z smooth over R and and an arrow $X \overset{\gamma}{\to} Z \overset{\alpha}{\to} R$ over R such that there exists

a vector bundle $\mathcal{G}$ on Z having the property that $\gamma^*\mathcal{G} \cong \mathcal{E}$. We now consider the following diagram

$$
\begin{array}{ccc}
X & & \\
& \searrow^{\exists\beta} \overset{f}{\searrow} & \\
\gamma \searrow & Z \times_R S \overset{\alpha'}{\longrightarrow} S & \\
& \downarrow{\varphi'} \qquad \downarrow{\varphi} & \\
& Z \overset{\alpha}{\longrightarrow} R &
\end{array}
$$

Where the inner square is a pullback, the outer square commutes because of our assumptions, β exists because of the universal property of the pullback and φ' and α' are of finite type and smooth respectively because of stability under base change of these two properties. If we denote $\psi := \beta$, $Y_\mathcal{E} := Z \times_R S$ and $\mathcal{F} := \varphi'^*\mathcal{G}$ the lemma is fully proved: indeed $Y_\mathcal{E}$ is divisorial because φ' is quasi-projective (quasi-projective maps are stable under base change) so that we can apply [TT90] 2.1.2 (h). $\qquad\square$

Remark 3.1.12. If in the previous lemma R is supposed to be regular and $\varphi : S \to R$ is a regular R-scheme, with affine diagonal and of finite type over R, then the proof goes through as well. Indeed the property of being regular, having affine diagonal and being of finite type is stable under base change so also φ' in the previous proof has all these properties but then $Z \times_R S$ is regular and with affine diagonal as well and so divisorial because of Lemma 2.1.5.

We also have the variant "with many variables" of the previous lemma, which one can prove as 2.5.4 and which holds also under the assumptions of the previous remark.

Proposition 3.1.13. *Let X be a divisorial scheme of finite type over S quasi-projective scheme over a noetherian affine scheme R. Then given a finite number of vector bundles $\mathcal{E}_1, ..., \mathcal{E}_n \in \mathrm{Vect}(X)$ there is a smooth divisorial scheme $Y_\mathcal{E}$ over S and vector bundles $\mathcal{E}_{1,Y_\mathcal{E}}, ..., \mathcal{E}_{n,Y_\mathcal{E}}$ over it together with a morphism $\psi_\mathcal{E} : X \to Y_\mathcal{E}$ such that $\psi_\mathcal{E}^*(\mathcal{E}_{i,Y_\mathcal{E}}) \cong \mathcal{E}_i$ for every $i = 1, ..., n$.*

We can now prove the following arguing as in Proposition 3.1.8

Proposition 3.1.14. *Assume to have two natural transformations $K_0(-)^n \xrightarrow[g]{f} K_0(-)$ on $\mathbf{Pre}(\mathrm{Sch}_S)$ which agree after restriction to Sm/S. Then $f = g$.*

From this it follows that the lower arrow of the diagram (3.2) is injective so that all the arrows in that diagram are isomorphisms. This means we have proved the following

Theorem 3.1.15. *Under the assumptions of this subsection (i.e. S regular quasi-projective scheme over a noetherian affine scheme R) we have for any $n \in \mathbb{N}$*

$$[K, K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-), K_0(-)) \cong [K, K]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-), K_0(-))$$

$$[K^n, K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-)^n, K_0(-)) \cong [K^n, K]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-)^n, K_0(-))$$

Moreover, the following pointed versions hold

$$[K, K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}, \bullet} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)_\bullet}(K_0(-), K_0(-)) \cong [K, K]_{\mathcal{H}_\bullet(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)_\bullet}(K_0(-), K_0(-))$$

$$[K^n, K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}, \bullet} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)_\bullet}(K_0(-)^n, K_0(-)) \cong [K^n, K]_{\mathcal{H}_\bullet(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)_\bullet}(K_0(-)^n, K_0(-))$$

The same holds true under the hypothesis of Remark 3.1.12.

3.2 Restriction to affine schemes

In this sections we will see that Theorem 3.1.4 only relies on what happens to affine schemes, in a sense we will make precise below. We will fix a Noetherian regular base scheme S. We will denote as Sch_S the category of divisorial schemes locally of finite type over S and with $\mathrm{Sm}/S \subseteq \mathrm{Sch}_S$ its full subcategory of (noetherian) smooth schemes over S. We let $\mathrm{Aff}/S \subseteq \mathrm{Sch}_S$ the full subcategory of Sch_S generated by the schemes of Sch_S which are affine (over $\mathrm{Spec}(\mathbb{Z})$). Finally, we denote as $\mathrm{SmAff}/S \subseteq \mathrm{Sm}/S$ the full subcategory of Sm/S generated by the schemes of Sm/S which are affine (over $\mathrm{Spec}(\mathbb{Z})$). Notice that SmAff/S is the full subcategory of Aff/S consisting of smooth affine schemes of finite type over S.

3.2.1 The case of smooth affine schemes

We now focus on smooth affine schemes. Assume in this subsection that S is a regular noetherian scheme of finite dimension. We first consider the inclusion $i : \mathrm{SmAff}/S \hookrightarrow \mathrm{Sm}/S$ which gives restriction functors $i^* : \mathbf{Pre}(\mathrm{Sm}/S) \to \mathbf{Pre}(\mathrm{SmAff}/S)$ and $i_s^* : \mathbf{sPre}(\mathrm{Sm}/S) \to \mathbf{sPre}(\mathrm{SmAff}/S)$

Definition 3.2.1. We denote as $\mathcal{H}^{aff}(S)$ the homotopy category of the model category $\mathbf{sPre}(\mathrm{SmAff}/S)$ having model structure determined by considering the injective local model structure relative to the affine Nisnevich topology on it and then by inverting $\mathbb{A}^1$-weak equivalences (see [AHW17]). The restriction functor i_s^* gives rise to a functor $\mathrm{sres} : \mathcal{H}(S) \to \mathcal{H}^{aff}(S)$ as follows

Remark 3.2.2. We have the following adjoint functors arising from the inclusion $\mathrm{SmAff}/S \subset \mathrm{Sm}/S$ (see [SGA72] I Proposition 5.1)

$$i_{\#,s}, i_{*,s} : \mathbf{sPre}(\mathrm{SmAff}/S) \xrightarrow{\longleftarrow} \mathbf{sPre}(\mathrm{Sm}/S) : i_s^*$$

where $i_{\#,s}$ and $i_{*,s}$ are respectively left and right adjoint of i_s^*. Recall that a weak Quillen adjunction is a pair of adjoint functors such that the left (right) adjoint is only required to preserve cofibrant (fibrant) objects and weak equivalences between them (indeed, this is enough to derive the adjunction). If we give to both categories the Nisnevich injective local model structure and we invert $\mathbb{A}^1$-weak equivalencesthen these adjunctions becomes Quillen adjunctions (weak in the case of $i_{\#,s} \dashv i_s^*$) and we can derive them. One notice that i_s^* preserves weak equivalences on the nose so we do not need to derive it. We call then $\text{sres} := i_s^* : \mathcal{H}(S) \to \mathcal{H}^{aff}(S)$ the functor we get in this way.

We notice that since all the schemes in Sm/S and SmAff/S have an ample family of line bundles and S is regular we have that the Quillen's K-theory presheaf satisfies descent in both $\mathcal{H}(S)$ and $\mathcal{H}^{aff}(S)$ and that $\text{sres}(K)$ represents K-theory in $\mathcal{H}^{aff}(S)$. Hence for every representable $X \in \text{SmAff}/S$ we have $[X, K]_{\mathcal{H}^{aff}(S)} \cong K_0(X)$. As a consequence it makes sense to study the following diagram

$$
\begin{array}{ccc}
[K, K]_{\mathcal{H}(S)} & \longrightarrow & [K, K]_{\mathcal{H}^{aff}(S)} \\
{\cong}\downarrow{\pi_0} & & \downarrow{\pi_0} \\
\text{Hom}_{\mathbf{Pre}(\text{Sm}/S)}(K_0, K_0) & \xrightarrow[i^*]{} & \text{Hom}_{\mathbf{Pre}(\text{SmAff}/S)}(K_0, K_0)
\end{array}
\tag{3.3}
$$

We would like to show that all the arrows in the previous diagram are isomorphisms giving two proofs of this result: one very simple relying on the work by Riou, the other a little bit more complicated but more "hands on".

First proof

For the first proof we will need the following theorem, asserting that the top horizontal map is bijective

Proposition 3.2.3. *The arrow* $[K, K]_{\mathcal{H}(S)} \to [K, K]_{\mathcal{H}^{aff}(S)}$ *is an isomorphism.*

Proof. We can do this in two ways. For the first one we notice that in $\mathcal{H}(S)$ we have a weak equivalence (so an isomorphism) $\pi : \mathbb{Z} \times \text{Gr}^{\text{aff}} \to \mathbb{Z} \times \text{Gr}$ because of Theorem 4.2.2. This weak equivalence restricts to a weak equivalence in $\mathcal{H}^{aff}(S)$ so that K-theory is represented in $\mathcal{H}^{aff}(S)$ by $\mathbb{Z} \times \text{Gr}^{\text{aff}}$ as well so that we can replace K with $\mathbb{Z} \times \text{Gr}^{\text{aff}}$ in both $\mathcal{H}(S)$ and $\mathcal{H}^{aff}(S)$. Moreover the K-theory presheaf satisfies descent also in $\mathcal{H}^{aff}(S)$ (one needs to use the affine BG property, as in [AHW17] 2.1.5) so we are left to prove that the map $[\mathbb{Z} \times \text{Gr}^{\text{aff}}, K]_{\mathcal{H}(S)} \to [\mathbb{Z} \times \text{Gr}^{\text{aff}}, K]_{\mathcal{H}^{aff}(S)}$ is an isomorphism. This can be seen reasoning exactly as in

Theorem 1.8.8 using Proposition 1.8.4. For the second proof we notice that the functor i_s^* because of Theorem 3.3.2 in [AHW17] induces an equivalence on $\mathcal{I}_{Nis}^l$-fibrant simplicial presheaves so that we can see directly that the arrow $[K, K]_{\mathcal{H}(S)} \to [K, K]_{\mathcal{H}^{aff}(S)}$ is an isomorphism. Strictly speaking, in *op.cit.* the do not assume their schemes to be divisorial but we can repeat their argument even in this case or use our Theorem 1.10.1 directly to conclude (see also Remark 6.3.7). $\square$

Remark 3.2.4. If S would have been an affine scheme R then we could have chosen in the first proof of the previous proposition $\mathbb{Z} \times \mathrm{BGL}$ instead of $\mathbb{Z} \times \mathrm{Gr}^{aff}$ to represent K-theory, the schemes $\mathrm{GL}_{n,R}$ being affine in this case.

To conclude that all the arrows in the diagram (3.3) are isomorphisms, we shall need the following proposition, proved by Riou in [Rio06] and [Rio10]. Before that we recall some terminology in *op.cit.*. We denote by $\mathcal{T}$ the collection of maps in Sm/S which are vector bundle torsors (we could even assume that these vector bundle torsors are affine but we do not need it) and as $\mathcal{T}_{aff}$ the collection of projection maps of the form $\mathbb{A}_X^1 \to X$ in SmAff/S. We then have the following fact

Proposition 3.2.5 ([Rio06] Proposition II.16)**.** *There is an equivalence of categories* $\Theta : \mathrm{SmAff}/S[\mathcal{T}_{aff}^{-1}] \xrightarrow{\simeq} \mathrm{Sm}/S[\mathcal{T}^{-1}]$

Using [Rio06] Proposition B.8 or Remark 1.2.8 of [Rio10] we then have

Corollary 3.2.6. *The equivalence Θ induces an equivalence of categories between* $\mathcal{T}^{-1}$*-invariant presheaves in* $\mathbf{Pre}(\mathrm{Sm}/S)$ *and the $\mathbb{A}^1$-invariant presheaves in* $\mathbf{Pre}(\mathrm{SmAff}/S)$

Because $K_0(-)$ is $\mathbb{A}^1$-invariant (remember in our case S is regular), we obtain putting together Corollary 3.2.6 and Proposition 3.2.3, the following

Theorem 3.2.7. *Let S be a noetherian regular base scheme, Sm/S the category of smooth schemes over S having an ample family of line bundles and SmAff/S its full subcategory of affine smooth schemes. Then all the arrows in the following diagram are isomorphisms*

$$
\begin{array}{ccc}
[K, K]_{\mathcal{H}(S)} & \longrightarrow & [K, K]_{\mathcal{H}^{aff}(S)} \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0, K_0) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(K_0, K_0)
\end{array}
$$

Second proof

To obtain an alternative proof of Theorem 3.2.7 we will need the following lemma

Lemma 3.2.8. *([Rio06] Lemme III.11, [MV99] Example 2.3 page 106) Let X be a noetherian scheme, and $\mathcal{E}$ a vector bundle over it. For every $\mathcal{E}$-torsor (in the sense of Appendix A.2) $\pi : T \to X$ over X, the arrow π induces an isomorphism $\pi : T \cong X$ in $\mathcal{H}(S)$*

Corollary 3.2.9. *The arrow $i^* : \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0, K_0) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(K_0, K_0)$ is injective.*

Proof. Assume that $f, g : K_0 \rightrightarrows K_0$ are two natural transformations between functors in $\mathbf{Pre}(\mathrm{Sm}/S)$ which agree once restricted on SmAff/S. Then we want to show that for every $X \in \mathrm{Sm}/S$, $f_X = g_X$. We use the Jouanolou's trick to find $\pi_X : T_X \to X$ affine vector bundle torsor on X. The previous lemma then implies that $\pi_X^* : K_0(X) \xrightarrow{\cong} K_0(T_X)$ is an isomorphism. The fact that f, g are natural transformations implies that the following diagram commutes

$$
\begin{array}{ccc}
K_0(X) & \xrightarrow[g_X]{f_X} & K_0(X) \\
\pi_X^* \downarrow \cong & & \pi_X^* \downarrow \cong \\
K_0(T_X) & \xrightarrow[g_{T_X}]{f_{T_X}} & K_0(T_X)
\end{array}
$$

so that the equalities $\pi_X^* \circ f_X = \pi_X^* \circ g_X$ and $f_{T_X} = g_{T_X}$ together with the fact that π_X^* is an isomorphism and so a mono, imply that $f_X = g_X$ for all $X \in \mathrm{Sm}/S$ and so that $f = g$. $\qquad\square$

To conclude that all the arrows in the diagram (3.3) are isomorphisms, we are then left to prove the following Proposition

Proposition 3.2.10. *The map $\pi_0 : [K, K]_{\mathcal{H}^{aff}(S)} \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(K_0, K_0)$ is an isomorphism.*

Proof. One uses the model $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ for K-theory in $\mathcal{H}^{aff}(S)$. Then taking $\mathrm{SmAff}/S = \mathcal{C} = \mathcal{A}$ in Theorem A.4.5 we have that $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ satisfies the property (ii) relative to $\mathcal{A}$ because of Corollary 4.3.2. It then follows that the arrow $\pi_0 : [\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}, K]_{\mathcal{H}^{aff}(S)} \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(K_0, K_0)$ is surjective so that chasing diagram (3.3) we have that it is an isomorphism. One can see that it is an isomorphism also by noticing that K theory satisfies the property (K) relative to the system $\mathcal{K}_\bullet^{\mathrm{aff}}$ which have colimit $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ because of Proposition 4.3.4. $\qquad\square$

Summarizing we have given another proof of Theorem 3.2.7

3.2.2 The case of singular affine schemes

We now want to extend the result of the previous section to the singular case. In order to do so we need to assume that S is an affine regular scheme R since we need to use the fact that BGL can be seen as a homotopy colimit (not indexed by a filtered category!) of affine (in the absolute sense) schemes. We can then choose $\mathbb{Z}\times\mathrm{BGL}^+$ as representative of K-theory in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$. As in the beginning of this section, we are interested in the categories Sm/S, Sch_S, SmAff/S and Aff/S of (affine) divisorial scheme of finite type over S and of (affine) smooth divisorial schemes over S. We denote as $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ the homotopy category of the model category of simplicial presheaves over Sch_S with the Jardine local model structure with respect to the Zariski topology and we will use the notation $\mathcal{H}_{Zar}^{\mathrm{Aff}/S}$ for the analogous homotopy category having underlying Grothendieck (affine, see [AHW17])Zariski site Aff/S. Even in this case as in the case of smooth affine case we have a functor $\widetilde{\mathrm{sres}} : \mathcal{H}_{Zar}^{\mathrm{Sch}_S} \to \mathcal{H}_{Zar}^{\mathrm{Aff}/S}$ arising from the adjunctions (as in Remark 3.2.2)

$$i_{\#,s}, i_{*,s} : \mathbf{sPre}(\mathrm{Aff}/S) \xrightarrow{\longleftarrow} \mathbf{sPre}(\mathrm{Sch}_S) : i_s^*$$

By what we know so far we have the following commutative cube

The only thing stated in the diagram that we haven't proven so far is that all the arrows of the upper square are isomorphisms but this is easily solved by the following lemma

Lemma 3.2.11. *The arrow* $\varphi : [K,K]_{\mathcal{H}_{Zar}^{\mathrm{Aff}/S}} \to [K,K]_{\mathcal{H}^{aff}(S)}$ *is an isomorphism.*

Proof. The proof follows arguing as in Proposition 1.6.1 and Theorem 1.8.10 using 1.8.4 because all the $\mathrm{GL}_{n,R}$ are affine schemes. $\square$

To show that all the arrow in the cube are isomorphism we only need to prove the following Proposition

Proposition 3.2.12. *The restriction map $[K_0, K_0]_{\mathbf{Pre}(\mathrm{Aff}/S)} \to [K_0, K_0]_{\mathbf{Pre}(\mathrm{SmAff}/S)}$ is injective.*

This proposition follows as Proposition 3.1.8 provided we show the following two facts

Lemma 3.2.13. *Let $U \in \mathrm{Aff}/S$ and P a vector bundle over it (i.e. a finitely generated projective module). Then there exists an arrow $f : U \to U_P$ over S such that $U_P \in \mathrm{SmAff}/S$ and there exists a vector bundle Q on U_P such that $f^*Q \cong P$.*

Proof. Every vector bundle (say of rank n for simplicity otherwise we can reason on the connected components or we can use 2.5.5 directly) on U is generated by global sections so there exists a grassmannian Grass_n over S together with a map $g : U \to \mathrm{Grass}_n$ in Sch_S such that $g^*\mathcal{T} \cong P$ where $\mathcal{T}$ is the universal vector bundle of the grasmannian. Since the Grasmannians are divisorial we can use the Jouanolou's device to build an affine vector bundle torsor $\pi : W \to \mathrm{Grass}_n$ over the Grassmannian, which is then an element of SmAff/S. Now consider the following pullback

$$
\begin{array}{ccc}
U \times_{\mathrm{Grass}_n} W & \xrightarrow{\mathrm{pr}_1} & W \\
{\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \pi} \\
U & \xrightarrow{\quad g \quad} & \mathrm{Grass}_n
\end{array}
$$

We then have that $\mathrm{pr}_2 : U \times_{\mathrm{Grass}_n} W \to U$ is a torsor under a vector bundle and it is affine (π is affine so it is pr_2) so that it is a vector bundle ([Wei89] page 475) so that there exists an arrow $i : U \to U \times_{\mathrm{Grass}_n} W$ which splits pr_2. If we set $U_P := W$, $Q := \pi^*\mathcal{T}$ and $f := \mathrm{pr}_1 \circ i$ we have a datum as the one wanted in the statement of the lemma. $\square$

We then have the following extension of the previous lemma in many variables, whose proof follows as in the previous lemma given 2.5.4.

Proposition 3.2.14. *Let U be a scheme in Aff/S. Then given a finite number vector bundles $\mathcal{E}_1, ..., \mathcal{E}_n \in \mathrm{Vect}(U)$ there is a scheme $Y_\mathcal{E} \in \mathrm{SmAff}/S$ and vector bundles $\mathcal{E}_{1,Y_\mathcal{E}}, ..., \mathcal{E}_{n,Y_\mathcal{E}}$ over it together with a morphism $\psi_\mathcal{E} : U \to Y_\mathcal{E}$ such that $\psi_\mathcal{E}^*(\mathcal{E}_{i,Y_\mathcal{E}}) \cong \mathcal{E}_i$ for every $i = 1, ..., n$.*

proof of Proposition 3.2.12. The proof now is formally the same than the proof of Theorem 3.1.4. $\square$

Remark 3.2.15. We point out that the proofs of 3.2.3, 3.2.7, 3.2.11, 3.2.12 and the conclusion concerning the arrows in the cube of page 63 can be repeated verbatim

replacing K and K_0 with K^n and K_0^n for $n \in \mathbb{N}$ in the first variable of the Hom sets considered.

Summing up we have

Theorem 3.2.16. *Let be S an affine regular noetherian base scheme. All the arrows in the following commutative cube are isomorphisms for any $n \in \mathbb{N}$*

$$
\begin{array}{ccccc}
& [K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Aff}/S}} & \xrightarrow{\hspace{4cm}} & [K^n, K]_{\mathcal{H}^{aff}(S)} & \\
\nearrow & & & \nearrow & \\
[K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} & \xrightarrow{\hspace{2cm}} & [K^n, K]_{\mathcal{H}(S)} & & \\
\downarrow^{\pi_0} & \downarrow^{\pi_0} & \downarrow^{\pi_0} & \downarrow^{\pi_0} & \\
& [K_0^n, K_0]_{\mathbf{Pre}(\mathrm{Aff}/S)} & \xrightarrow{\hspace{2cm}} & [K_0^n, K_0]_{\mathbf{Pre}(\mathrm{SmAff}/S)} & \\
\nearrow & & & \nearrow & \\
[K_0^n, K_0]_{\mathbf{Pre}(\mathrm{Sch}_S)} & \xrightarrow{\hspace{3cm}} & [K_0^n, K_0]_{\mathbf{Pre}(\mathrm{Sm}/S)} & &
\end{array}
$$

The pointed version of this theorem also holds.

3.3 Separated Schemes

Since Riou and many authors do not consider divisorial smooth schemes over a regular divisorial base S but rather separated smooth schemes over a separated regular base S, we show how to add the hypothesis of separatedness to the one of divisoriality on smooth schemes. For this section we will always consider a base scheme S which is regular and separated. We denote as Sm/S^{sep} be the category of separated (in the absolute sense) smooth S-schemes. Notice this is a full subcategory of the divisorial schemes over S as we have remarked in Section 1.9. If S is affine, for any natural n we have the following commutative diagram (use the theorem of Riou and Theorem 1.9.1)

$$
\begin{array}{ccc}
[K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} & \xrightarrow{\cong} & [K^n, K]_{\mathcal{H}(S)^{sep}} \\
\downarrow^{\pi_0} & & \downarrow^{\pi_0}_{\cong} \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0(-)^n, K_0(-)) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S^{sep})}(K_0(-)^n, K_0(-))
\end{array}
$$

to show that all the maps in the previous commutative diagram are isomorphisms we shall need the following

Proposition 3.3.1. *Let S be an affine regular noetherian scheme. Assume to have two natural transformations $K_0(-)^n \underset{g}{\overset{f}{\rightrightarrows}} K_0(-)$ on $\mathbf{Pre}(\mathrm{Sch}_S)$ which agree after restriction to Sm/S^{sep}. Then $f = g$.*

Proof. We assume that $n = 1$ for simplicity, the general case being the same. Following what we did for divisorial schemes, for every $X \in \mathrm{Sch}_S$, by the very definition of $K_0(X)$, we will need to verify that for every element $\mathcal{E} \in K_0(X)$, $f_X(\mathcal{E}) = g_X(\mathcal{E})$. As in the proof of Proposition 1.8.7 we find for every $\mathcal{E} \in K_0(X)$ a divisorial smooth scheme $Y_{\mathcal{E}}$ over S and $\mathcal{E}_{Y_{\mathcal{E}}} \in K_0(Y_{\mathcal{E}})$ together with a morphism $\psi_{\mathcal{E}} : X \to Y_{\mathcal{E}}$ (over S) such that $\psi_{\mathcal{E}}^*(\mathcal{E}_{Y_{\mathcal{E}}}) = \mathcal{E}$. Now we use the Jouanolou's trick to find an affine vector bundle torsor $\pi : T \to Y_{\mathcal{E}}$, here T will be affine in the absolute sense, so separated (in the absolute sense), divisorial, and smooth over $Y_{\mathcal{E}}$ hence smooth over S: henceforth it lies in Sm/S^{sep}. Moreover π induces an isomorphism on K_0 (because of Lemma 3.2.8) so we get the following commutative diagram

$$
\begin{array}{ccc}
K_0(T) & \overset{f_T}{\underset{g_T}{\rightrightarrows}} & K_0(T) \\
\cong \big\uparrow \pi^* & & \cong \big\uparrow \pi^* \\
K_0(Y_{\mathcal{E}}) & \overset{f_{Y_{\mathcal{E}}}}{\underset{g_{Y_{\mathcal{E}}}}{\rightrightarrows}} & K_0(Y_{\mathcal{E}}) \\
\psi_{\mathcal{E}}^* \big\downarrow & & \big\downarrow \psi_{\mathcal{E}}^* \\
K_0(X) & \overset{f_X}{\underset{g_X}{\rightrightarrows}} & K_0(X)
\end{array}
$$

and since $f_T = g_T$ by assumption, chasing we get $f_X(\mathcal{E}) = g_X(\mathcal{E})$. Iterating this for every $X \in \mathrm{Sch}_S$ and any $\mathcal{E} \in K_0(X)$ gives us the assert. $\qquad\square$

Remark 3.3.2. The proof actually allows us to see that if two natural transformations $K_0(-)^n \overset{f}{\underset{g}{\rightrightarrows}} K_0(-)$ on $\mathbf{Pre}(\mathrm{Sm}/S)$ agree after restriction to Sm/S^{sep}, Then $f = g$.

As a corollary we have the following theorem

Theorem 3.3.3. *Assume that S is an affine regular noetherian scheme. All the arrows in the following commutative cube are isomorphisms for any $n \in \mathbb{N}$*

$$
\begin{array}{ccccc}
 & [K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Aff}/S}} & \longrightarrow & & [K^n, K]_{\mathcal{H}^{aff}(S)} \\
\nearrow & & & \nearrow & \\
[K^n, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} & \longrightarrow & [K^n, K]_{\mathcal{H}(S)^{sep}} & & \\
\big\downarrow \pi_0 & \big\downarrow \pi_0 & & \big\downarrow \pi_0 & \big\downarrow \pi_0 \\
 & [K_0^n, K_0]_{\mathbf{Pre}(\mathrm{Aff}/S)} & \longrightarrow & & [K_0^n, K_0]_{\mathbf{Pre}(\mathrm{SmAff}/S)} \\
\nearrow & & & \nearrow & \\
[K_0^n, K_0]_{\mathbf{Pre}(\mathrm{Sch}_S)} & \longrightarrow & [K_0^n, K_0]_{\mathbf{Pre}(\mathrm{Sm}/S^{sep})} & &
\end{array}
$$

The pointed version of this theorem also holds.

Proof. The proof follows as the one of Theorem 3.2.16 mutatis mutandis. $\qquad\square$

3.4 Non-divisorial schemes

We fix a regular noetherian base S. For this section only we assume that the condition of being regular does not come with the additional divisorial hypothesis, as in section 1.10. The methods we have used to link the homotopic world to what happens at the level of π_0 relied on the Riou's result and on 2.5.5 which makes use of the fact that if a scheme has an ample family of line bundles, we can find an embedding of it into a smooth one. Now, if we remove the hypothesis of being divisorial from all the schemes in Sch_S and Sm/S, and we want to keep Nisnevich and Zariski descent, in addition to homotopy invariance for regular schemes, we need to define our K-theory with the Thomason-Trobaugh definition as we did in 1.10. Hence the problem changes completely because then the K_0 are possibly different from the usual ones (i.e. they are not the Grothendieck's K_0). But we notice that the rear face of the cube does not change since affine schemes trivially satisfy the resolutio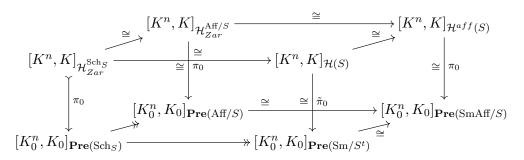n property and 3.2.14 does not change. Moreover, by work of Cisinski and Khan ([Kha16] 2.4.5 and 2.4.6 or [CK17]), or Theorem 3.3.2 in [AHW17], we still have that $[K^n, K]_{\mathcal{H}(S)} \cong [K^n, K]_{\mathcal{H}^{aff}(S)}$ and if we denote as Sm/S^t the full subcategory of smooth schemes having an affine vector bundle torsor we have an analogue of Corollary 3.2.6 that we can use. In addition, if S is affine, because of what we proved in Section 1.10, we can conclude the following

Theorem 3.4.1. *If S is an affine noetherian regular scheme, Sch_S and Sm/S the categories of (smooth) schemes of finite type over S, K is the Thomason's K-theory and n is any natural number, we have that the following commutative cube*

where $\tilde{\pi}_0$ is obtained as composition of the arrows

$$[K^n, K]_{\mathcal{H}(S)} \xrightarrow{\pi_0} [K_0^n, K_0]_{\mathbf{Pre}(\mathrm{Sm}/S)} \xrightarrow{\mathrm{res}} [K_0^n, K_0]_{\mathbf{Pre}(\mathrm{Sm}/S^t)}$$

with res *induced by the inclusion* $\mathrm{Sm}/S^t \subseteq \mathrm{Sm}/S$. *By diagram chase we also have the injectivity and the surjectivity of some maps as depicted in the diagram. Note that the arrows in the front face of the cube have the same properties even if S is not affine but any regular noetherian scheme. The pointed analogue also holds.*

Proof. The proof uses what we said at the beginning of this section in addition to Theorem 1.10.1 and Remark A.3.11. $\qquad\square$

Hopefully, a further study of the remaining maps will be addressed in a future work. Also, the fact that the K_0 of derived affine schemes only feels the K_0 of their ring of path components ([KST18] Theorem 2.16) and the above mentioned work of Cisinski and Khan suggests that maybe one can look also at the derived analogue for the cube, and get some interesting conclusions although this should be studied in a further work. If one wants to have homotopy invariance for non regular schemes, then one could employ homotopy invariant K-theory, but this will not be discussed here.

Affine representability of K-theory

In this section we prove that in $\mathcal{H}(S)$ (S regular noetherian base scheme, all the schemes are assumed to have an ample family of line bundles as detailed in 0.1), K-theory is representable not only by $\mathbb{Z} \times \mathrm{Gr}$ as shown in [MV99] but also by what we will call $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ where $\mathrm{Gr}^{\mathrm{aff}}$ is an ind-scheme obtained as filtered colimit of affine (in the absolute sense) schemes. This will imply that K-theory can be written in $\mathcal{H}(S)$ as the filtered colimit of *affine* representable schemes, in a way analogous to what Riou did. We start by discussing in full generality some consequences of the Jouanolou's trick.

4.1 Pulling back certain affine vector bundle torsors

As discussed in the seminal [Wei89], for schemes having an ample family of line bundles, the Jouanolou's trick does not give a construction of affine vector bundle torsors which is functorial for two reasons: the first is that the construction of these torsors over the divisorial schemes strongly depends on the choice of the ample family we consider and the second is that it is not true that affine vector bundle torsors pullback to affine vector bundle torsors. Ad hoc reasonings should then be made once one needs any kind of functoriality for such torsors. For example consider the following standard system of embeddings of projective spaces

$$\cdots \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1} \cdots$$

In this case every $\mathbb{P}^n$ has $(\mathcal{O}(1), x_0, ..., x_n)$ as ample family of line bundles and this family is the pullback of $(\mathcal{O}(1), x_0, ..., x_n, x_{n+1})$ on $\mathbb{P}^{n+1}$ after omitting the section which becomes $= 0$. Moreover all the morphisms involved in this system

are closed embeddings, that pullback affine vector bundle torsors. So in this very particular situation one might hope to find a system $\pi_n : W_n \to \mathbb{P}^n$ of affine vector bundle torsors (remind that as in Definition A.2.5 we require W_n to be affine in the absolute sense, i.e. over $\mathrm{Spec}(\mathbb{Z})$) such that the following diagram commutes

$$
\begin{array}{ccccccc}
\cdots & W_{n-1} & \hookrightarrow & W_n & \hookrightarrow & W_{n+1} & \cdots \\
& \pi_{n-1}\downarrow & & \pi_n\downarrow & & \pi_{n+1}\downarrow & \\
\cdots & \mathbb{P}^{n-1} & \hookrightarrow & \mathbb{P}^n & \hookrightarrow & \mathbb{P}^{n+1} & \cdots
\end{array}
$$

This is what can be done in the situations we would like to consider. We now lay down the general situation. Suppose X is a divisorial scheme with an ample family of line bundles $\{(L_i, s_i)\}_{i=0}^n$ as in 2.1.3. Assume also that we have a closed embedding $f : Y \to X$. Hence the family $\{(L_i' := f^*L_i, s_i' := f^*s_i)\}_{i=0}^n$ is an ample family on Y. We have the following

Proposition 4.1.1. *Let be X a divisorial scheme with an ample family of line bundles $\mathcal{L} = \{(L_i, s_i)\}_{i=0}^n$ and $f : Y \to X$ a closed embedding so that $\mathcal{L}' = \{(f^*L_i, f^*s_i)\}_{i=0}^n$ is an ample family on Y as above. Then the affine vector bundle torsor $\pi' : W' \to Y$ built on Y using the Jouanolou device on the family $\mathcal{L}'$ is the pullback along f of the one over X built using $\mathcal{L}$. The result does not change if we use the Jounalolou device on Y on an ample family $\mathcal{I}$ isomorphic to $\mathcal{L}'$ (by isomorphic we mean that the line bundles considered in the two families are isomorphic via some isomorphisms that map the sections being part of the datum of one family to the others).*

Proof. We need to follow the construction found in [Wei89] Proposition 4.4. Let $\mathcal{E} = \oplus_{i=0}^n L_i$ and $s = (s_0, ..., s_n) : \mathcal{O}_X \to \mathcal{E}$ the map induced by the sections s_i. If we set $\mathcal{F} := \mathrm{coker}(s)$ and we notice that s is a split mono on every X_{s_i} we have the following exact sequence of vector bundles

$$
\mathcal{O}_X \xrightarrow{s} \mathcal{E} \to \mathcal{F}
$$

[Wei89] 4.4 tells us that the open subscheme $W = \mathrm{Spec}(S(\mathcal{E})/(s-1)) = \mathbb{P}(\mathcal{E})\backslash\mathbb{P}(\mathcal{F}) \subseteq \mathbb{P}(\mathcal{E})$ is an affine vector bundle torsor $\pi : W \to X$ where π is obtained as the composition of the open embedding of W in $\mathbb{P}(\mathcal{E})$ and the projection $\mathbb{P}(\mathcal{E}) \to X$. Pulling

back along f, denoting $\mathcal{E}' := f^*\mathcal{E} \cong \oplus_{i=0}^n f^*L_i$, we have that the following is a pullback

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{E}') & \hookrightarrow & \mathbb{P}(\mathcal{E}) \\
\downarrow & & \downarrow \\
Y & \xhookrightarrow{\ f\ } & X
\end{array}
$$

and the map s pullbacks to a split mono $s' = (s'_0, ..., s'_n) : \mathcal{O}_Y \to \mathcal{E}'$ so that the following sequence of vector bundles is exact and it is the pullback of the previous one

$$\mathcal{O}_Y \xrightarrow{s'} \mathcal{E}' \to \mathcal{F}' := \operatorname{coker}(s')$$

Accordingly, we have that $W' = f^*W = \operatorname{Spec}(S(\mathcal{E}')/(s'-1)) = \mathbb{P}(\mathcal{E}')\backslash\mathbb{P}(\mathcal{F}')$ is an affine vector bundle torsor and is the pullback of W. In other words the following is a pullback

$$
\begin{array}{ccc}
W' & \hookrightarrow & W \\
\pi'\downarrow & & \downarrow\pi \\
Y & \xhookrightarrow{\ f\ } & X
\end{array}
$$

The last part of the statement is trivial. $\qquad\qquad\square$

We are then in the position to prove the following

Corollary 4.1.2. *Let X, Y, $\mathcal{L}'$ and W' as in the statement and in the proof of the previous Proposition. If we assume that $s'_n = f^*s_n = 0$ (so that $Y_{s'_n} = \emptyset$) then $\mathcal{L}'' = \{(L'_i, s'_i)\}_{i=0}^{n-1}$ is an ample family of line bundles for Y and the affine vector bundle torsor resulting from the Jouanolou device naturally embeds as a closed subscheme of W'. The result does not change if we start with an ample family $\mathcal{I}$ on Y isomorphic to $\mathcal{L}'$.*

Proof. First we notice that because of the hypothesis, $Y_{s'_n}$ is the empty scheme $(\emptyset, \mathcal{O}_\emptyset)$ affine and isomorphic to $\operatorname{Spec}((0))$ ($\mathcal{O}_\emptyset(\emptyset) = 0$). Hence $\mathcal{L}''$ is an ample family for Y. Using now the assumption that $s'_n = 0$ we have that the following commutative diagram of $\mathcal{O}_Y$-modules.

$$
\begin{array}{ccccc}
\mathcal{O}_Y & \xrightarrow{\ s'\ } & \mathcal{E}' = \oplus_{i=0}^n L'_i & \twoheadrightarrow & \mathcal{F}' \cong \mathcal{G} \oplus L'_n \\
id\downarrow & & \downarrow\mathrm{pr} & & \downarrow \\
\mathcal{O}_Y & \xrightarrow[g=(x'_0,...,x'_{n-1})]{} & \mathcal{E}'' := \oplus_{i=0}^{n-1} L'_i & \twoheadrightarrow & \mathcal{G} = \operatorname{coker}(g)
\end{array}
$$

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Using this one can see that the following induced diagram commutes

$$
\begin{array}{ccc}
\mathbb{P}(\mathcal{G}) & \lhook\joinrel\longrightarrow & \mathbb{P}(\mathcal{E}'') \\
\cup\big\downarrow & & \cup\big\downarrow \\
\mathbb{P}(\mathcal{F}') & \lhook\joinrel\longrightarrow & \mathbb{P}(\mathcal{E}')
\end{array}
$$

and with a little more effort, one can check that it is actually a pullback. It then follows from the construction of the affine vector bundle torsor on Y starting from the family $\mathcal{L}''$ that the affine torsor

$$
\pi'' : W'' := \mathbb{P}(\mathcal{E}'')\backslash\mathbb{P}(\mathcal{G}) = \mathrm{Spec}(S(\mathcal{E}'')/(g-1)) \to X
$$

is a closed subscheme of W' such that the following diagram commutes

$$
\begin{array}{ccc}
W'' \lhook\joinrel\longrightarrow & W' \lhook\joinrel\longrightarrow & W \\
\ \ \searrow\scriptstyle{\pi''} & \big\downarrow\scriptstyle{\pi'} & \big\downarrow\scriptstyle{\pi} \\
& Y \lhook\joinrel\xrightarrow{\ f\ } & X
\end{array}
$$

and the square is a pullback. The last assertion is easy. $\qquad\square$

The previous corollary allows us to consider the system $\cdots \mathbb{P}_S^{n-1} \hookrightarrow \mathbb{P}_S^n \hookrightarrow \mathbb{P}_S^{n+1} \cdots$ with S any base scheme and build affine vector bundle torsors $\pi_n : W_n \to \mathbb{P}_S^n$ for every n so that the following diagram commutes

$$
\begin{array}{ccccccc}
\cdots & W_{n-1} & \lhook\joinrel\longrightarrow & W_n & \lhook\joinrel\longrightarrow & W_{n+1} & \cdots \\
& \scriptstyle{\pi_{n-1}}\big\downarrow & & \scriptstyle{\pi_n}\big\downarrow & & \scriptstyle{\pi_{n+1}}\big\downarrow & \\
\cdots & \mathbb{P}^{n-1} & \lhook\joinrel\longrightarrow & \mathbb{P}^n & \lhook\joinrel\longrightarrow & \mathbb{P}^{n+1} & \cdots
\end{array}
$$

This means that if we see the system of embeddings of the projective spaces and of the torsors as functors $W_\bullet, P_\bullet : \mathbb{N} \to \mathrm{Sch}_S$ where by $\mathbb{N}$ we denote the natural numbers seen as a poset, then the projections π_n define a natural transformation $\pi_\bullet : W_\bullet \to P_\bullet$.

Definition 4.1.3. We define the *affine (infinite) projective space* $\mathbb{P}_{\mathrm{aff}}^\infty := \mathrm{colim}\, W_\bullet$. The natural transformation $\pi_\bullet$ just defined induces a map $\mathbb{P}_{\mathrm{aff}}^\infty \to \mathbb{P}^\infty$ in $\mathbf{Pre}(\mathrm{Sch}_S)$.

This point of view is natural once we try to do the same for the Grassmannians, where we seek an analogous result. First we need to recall some facts. We fix a base scheme S. With $\mathrm{Gr}_{d,r}$ and $\mathrm{Gr}_n(\mathcal{E})$ we shall stick to the notation of Morel

[Mor06] page 5 and of Grothendieck ([GD71] I 9.7) because for this section it is going to be more convenient. This means that for a scheme $X \in \mathrm{Sch}_S$ we have

$$\mathrm{Gr}_{d,r} = \{ \text{ locally free quotients of rank } d \text{ of } \mathcal{O}_X^{d+r} \}$$

Thus we differ from the notation used by Riou in [Rio06] and introduced in Appendix A.2 (i.e. d and r are swapped) but the reader will recognize that the two notions, although different, are interchangeable, in particular the embeddings $f_{(d,r)(d',r')}$: $\mathrm{Gr}_{d,r} \to \mathrm{Gr}_{d',r'}$ for $(d,r) \leq (d',r')$ are defined in the same way, i.e. following Morel-Grothendieck they are compositions of the maps $f_{(d,r)(d,r+1)} = l_{d,r} : \mathrm{Gr}_{d,r} \to \mathrm{Gr}_{d,r+1}$ defined for every scheme $X \in \mathrm{Sch}_S$ by the assignment

$$(\mathcal{O}_X^{d+r} \to P) \mapsto (\mathcal{O}_X^{d+r+1} \to \mathcal{O}_X^{d+r} \to P)$$

and the map $f_{(d,r)(d+1,r)}$: $\varphi_{d,r}$: $\mathrm{Gr}_{d,r} \to \mathrm{Gr}_{d+1,r}$ defined by the assignment, for every scheme X

$$(\mathcal{O}_X^{d+r} \to P) \mapsto (\mathcal{O}_X^{d+r+1} \to P \oplus \mathcal{O}_X)$$

We also have isomorphisms $\tau_{d,r} : \mathrm{Gr}_{d,r} \cong \mathrm{Gr}_{r,d}$ and equalities $\tau_{d,r+1} \circ l_{d,r} = \varphi_{r,d} \circ \tau_{d,r}$ and $\varphi_{d,r+1} \circ l_{d,r} = l_{d+1,r} \circ \varphi_{d,r}$. For a given vector bundle $\mathcal{E}$ there is an ambiguity very similar between the Grothendieck notation we follow and for example the notation of [GW10] 8.6. Given such a vector bundle over S, it is well known that we have the Plucker embedding $\varpi_{n,\mathcal{E}} : \mathrm{Gr}_n(\mathcal{E}) \hookrightarrow \mathbb{P}(\wedge^n \mathcal{E})$ given by the assignment, for any $X \in \mathrm{Sch}_S$

$$(\mathcal{E}_X \to P) \mapsto (\wedge^n \mathcal{E} \to \wedge^n P)$$

see [GD71] I 9.8.1.1 or [GW10] 8.10. In the case $\mathcal{E} = \mathcal{O}_S^{n+d}$ we have that $\mathbb{P}(\wedge^n \mathcal{O}_X^{n+r}) \cong \mathbb{P}_S^{\binom{n+r}{n}-1}$ so that we have our standard Plucker embeddings $\varpi_{d,r} : \mathrm{Gr}_{d,r} \hookrightarrow \mathbb{P}_S^{\binom{d+r}{d}-1}$. We notice that the ample families $\{\mathcal{O}(1), x_0, ..., x_{\binom{d+r}{d}-1}\}$ on $\mathbb{P}_S^{\binom{d+r}{d}-1}$ pullback to ample families

$$\mathcal{L}_{d,r} = \{L_{d,r} := \varpi_{d,r}^* \mathcal{O}(1), x_0' = \varpi_{d,r}^* x_0, ..., x_{\binom{d+r}{d}-1}' = \varpi_{d,r}^* x_{\binom{d+r}{d}-1}\}$$

Remark 4.1.4. By the definition of the Plucker embedding, it follows that for any (d,r), $L_{d,r} \cong \det \mathcal{T}_{d,r}^*$ where we have denoted by $\mathcal{T}_{d,r}^*$ the dual of the rank d universal bundle on $\mathrm{Gr}_{d,r}$. Moreover, because of [GD71] I 9.8.3 we have that $l_{d,r+1} \circ \varpi_{d,r} = \varpi_{d,r+1} \circ l_{d,r}$ so that the pullback of the ample family $\mathcal{L}_{d,r+1}$ along $l_{d,r}$ is actually $\mathcal{L}_{d,r}$. Using the maps $\tau_{d,r}$ and the identities stated before, we can also see that the pullback of the family $\mathcal{L}_{d+1,r}$ along the map $\varpi_{d,r}$ is isomorphic to $\mathcal{L}_{d,r}$

so that putting everything together we get that the pullback of the ample family $\mathcal{L}_{d',r'}$ on $\mathrm{Gr}_{d',r'}$ along the embedding $f_{(d,r)(d',r')}$ for $(d,r) \leq (d',r')$ is isomorphic to $\mathcal{L}_{d,r}$.

Now we consider $\mathbb{N}^2$ as a poset with the order given by $(d,r) \leq (d',r')$ if $d \leq d'$ and $r \leq r'$ and hence we see $\mathbb{N}^2$ as a category in this way. Using the Yoneda lemma, we define the functor $\mathrm{Gr}_\bullet : \mathbb{N}^2 \to \mathbf{Pre}(\mathrm{Sch}_S)$ as the usual filtered system, i.e.

$$\mathrm{Gr}_\bullet((d,r)) = \mathrm{Gr}_{d,r} \qquad \mathrm{Gr}_\bullet((d,r) \leq (d',r')) = f_{(d,r)(d',r')}$$

The colimit of this functor is the usual infinite Grassmannian Gr. Choosing on every Grasmannian $\mathrm{Gr}_{d,r}$ the ample family $\mathcal{L}_{d,r}$ we can use Remark 4.1.4 and Corollary 4.1.2 to construct, for every $(d,r) \in \mathbb{N}^2$ affine vector bundle torsors $\pi_{d,r} : \mathrm{Gr}_{d,r}^{\mathrm{aff}} \to \mathrm{Gr}_{d,r}$ which bundle together to give a functor

$$\mathrm{Gr}_\bullet^{\mathrm{aff}} : \mathbb{N}^2 \to \mathbf{Pre}(\mathrm{Sch}_S) \qquad (d,r) \mapsto \mathrm{Gr}_{d,r}^{\mathrm{aff}}$$

which takes an arrow $(d,r) \leq (d',r')$ to a closed embedding $f_{(d,r)(d',r')}^{\mathrm{aff}} : \mathrm{Gr}_{d,r}^{\mathrm{aff}} \hookrightarrow \mathrm{Gr}_{d',r'}^{\mathrm{aff}}$ built using Corollary 4.1.2.

Definition 4.1.5. We define the *affine (infinite) Grassmannian* as $\mathrm{Gr}^{\mathrm{aff}} := \mathrm{colim}\mathrm{Gr}_\bullet^{\mathrm{aff}}$.

Corollary 4.1.2 gives us the following Proposition, whose proof is immediate given 4.1.2 and 4.1.4

Proposition 4.1.6. *The projections $\pi_{d,r}$ bundle together to give a natural transformation $\pi_\bullet : \mathrm{Gr}_\bullet^{\mathrm{aff}} \to \mathrm{Gr}_\bullet$. This induces a map $\pi : \mathrm{Gr}^{\mathrm{aff}} \to \mathrm{Gr}$.*

We can now build the affine analogue of the systems $\mathcal{K}_\bullet$ and $\mathcal{P}_\bullet$ that Riou built in [Rio06], see Appendix A.2. We define the system $\mathcal{K}_\bullet^{\mathrm{aff}} : \mathbb{N}^2 \to \mathbf{Pre}(\mathrm{Sch}_S)$ by the assignment $\mathcal{K}_{(d,r)}^{\mathrm{aff}} = \bigsqcup_{2d+1} \mathrm{Gr}_{d,r}^{\mathrm{aff}}$ and using the arrows $f_{(d,r)(d',r')}^{\mathrm{aff}}$ to define the functor. Using the cofinal sequence $i : \mathbb{N} \to \mathbb{N}^2$, $n \mapsto (n,n)$ we define as in Appendix A.2 $\mathcal{P}_\bullet^{\mathrm{aff}} = \mathcal{K}_\bullet^{\mathrm{aff}} \circ i$. We then have the following proposition, which follows from 4.1.2 and 4.1.4

Proposition 4.1.7. *The arrows $\pi_{d,r}$ define a natural transformation $\pi_\bullet : \mathcal{K}_\bullet^{\mathrm{aff}} \to \mathcal{K}_\bullet$ which restricts to $\pi_\bullet : \mathcal{P}_\bullet^{\mathrm{aff}} \to \mathcal{P}_\bullet$. Moreover we have $\mathrm{colim}\mathcal{K}_\bullet^{\mathrm{aff}} \cong \mathrm{colim}\mathcal{P}_\bullet^{\mathrm{aff}} \cong \mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ and the natural transformations just defined induce a map $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}} \to \mathbb{Z} \times \mathrm{Gr}$.*

4.2 Proving the affine representability of K and Pic

We shall begin with a simple lemma in homotopical algebra

Lemma 4.2.1. *Suppose that a model category $\mathcal{M}$ is given and that we have two functors $F, G : \mathbb{N} \to \mathcal{M}$ and a natural transformation $\varphi : F \to G$ between them such that*

1) *$F(n)$ and $G(n)$ are cofibrant for every $n \in \mathbb{N}$ and all $F(a \leq b), G(a \leq b)$ are cofibrations.*

2) *All $\varphi_n : F(n) \to G(n)$ are weak equivalences.*

Then $\operatorname{hocolim} F \cong \operatorname{colim} F$, $\operatorname{hocolim} G \cong \operatorname{colim} G$ and the induced map

$$\operatorname{colim} F \to \operatorname{colim} G$$

is a weak equivalence.

Proof. Everything follows by basic homotopical algebra. The first assertion follows because of the construction of homotopy colimits, since in virtue of condition 1) F and G are cofibrant objects in the projective module structure of $\mathcal{M}^{\mathbb{N}}$. The second assertion follows because π, in virtue of our assumption 2), is a weak equivalence in $\mathcal{M}^{\mathbb{N}}$ with the projective model structure. For a reference see [Cis19] 2.3.13, 2.3.15. $\qquad\square$

We recall that we have defined in the previous section the affine grassmannian and the affine projective spaces $\operatorname{Gr}^{\mathrm{aff}}$ and $\mathbb{P}^{\infty}_{\mathrm{aff}}$. We then have

Theorem 4.2.2. *Let S be a regular noetherian scheme. The affine vector bundle torsors $\pi_n : W_n \to \mathbb{P}^n_S$ and $\pi_{d,r}, \operatorname{Gr}^{\mathrm{aff}}_{d,r} \to \operatorname{Gr}_{d,r}$ induce weak equivalences $\mathbb{Z} \times \operatorname{Gr}^{\mathrm{aff}} \xrightarrow{\sim} \mathbb{Z} \times \operatorname{Gr}$ and $\mathbb{P}^{\infty}_{\mathrm{aff}} \xrightarrow{\sim} \mathbb{P}^{\infty}$ in $\mathcal{H}(S)$.*

Proof. The assertion follows from the previous lemma using the construction we gave in the previous section. The closed embeddings in the systems $W_\bullet$ and $\operatorname{Gr}^{\mathrm{aff}}_\bullet$ are all cofibrations in $s\mathbf{Pre}(\mathrm{Sm}/S)$ and all the objects involved are cofibrant by definition. The arrows π_n and $\pi_{d,r}$ induce isomorphisms in $\mathcal{H}(S)$ because of Lemma 3.2.8 so that they are weak equivalences. Hence we can apply the previous lemma to the maps $\pi_\bullet : W_\bullet \to P_\bullet$ and $\pi_\bullet : \mathcal{P}^{\mathrm{aff}}_\bullet \to \mathcal{P}_\bullet$ to conclude. $\qquad\square$

Corollary 4.2.3. *Let be S a regular noetherian base scheme. Then $\mathbb{Z} \times \operatorname{Gr}^{\mathrm{aff}} \simeq K$ and $\operatorname{Pic} \simeq \mathbb{P}^{\infty}_{\mathrm{aff}}$ in $\mathcal{H}(S)$, i.e. the affine Grassmannian represents K-theory and the affine projective space represents the Picard functor in the unstable motivic homotopy category.*

Remark 4.2.4. We point out that we can repeat similar reasonings replacing $\operatorname{Gr}^{\mathrm{aff}}$, $\mathbb{P}^{\infty}_{\mathrm{aff}}$, $\mathcal{K}^{\mathrm{aff}}_\bullet$ and $W_\bullet$ with their finite nth products $(\operatorname{Gr}^{\mathrm{aff}})^n$, $(\mathbb{P}^{\infty}_{\mathrm{aff}})^n$, $(\mathcal{K}^{\mathrm{aff}}_\bullet)^n$ and $(W_\bullet)^n$ (notice that the schemes involved are affine because of [Sta18, Lemma 01SG]).

4.3 Affine Grassmannian and the properties (ii) and (K)

The aim of this section is to discuss the properties (ii) and (K) for the affine Grassmannian and the affine projective space. We will discuss in full detail only the case of the affine Grassmannian, the other being very similar. In this section by S we denote a regular noetherian base scheme and by Sm/S we denote the category of smooth schemes over S having an ample family of line bundles. We let SmAff/S to be its full subcategory of affine (over $\mathrm{Spec}(\mathbb{Z})$) schemes. First recall we defined the functor φ, and the properties (ii) and (K) in Appendix A.3 following the original definitions in [Rio06] and [Rio10]. We now want to show that the functor $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ satisfies the property (ii), i.e. that for every $U \in \mathrm{SmAff}/S$ the natural transformation (in $\mathbf{Pre}(\mathrm{Sm}/S)$)

$$\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}} \xrightarrow{\tau_{\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}}} \varphi(\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}) =: \pi_0(\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}) =: [-, \mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}]_{\mathcal{H}(S)}$$

is surjective, i.e. $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}(U) \xrightarrow{\tau_{\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}}(U)} \varphi(\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}})(U)$ is surjective. Remember we have a map $\pi : \mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}} \to \mathbb{Z} \times \mathrm{Gr}$ in $\mathbf{Pre}(\mathrm{Sm}/S)$ which induces a weak equivalence in $\mathcal{H}(S)$ because of Theorem 4.2.2. Because of this, the induced map $\varphi(\pi) : \varphi(\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}) \to \varphi(\mathbb{Z} \times \mathrm{Gr})$ is an isomorphism in $\mathbf{Pre}(\mathrm{Sm}/S)$. Moreover, for every $U \in \mathrm{SmAff}/S$ the following diagram commutes

$$
\begin{array}{ccc}
\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}(U) & \xrightarrow{\tau_{\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}}(U)} & \varphi(\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}})(U) \\
{\scriptstyle \pi(U)} \downarrow & & \downarrow {\scriptstyle \varphi(\pi)(U)} \cong \\
\mathbb{Z} \times \mathrm{Gr}(U) & \xrightarrow[\tau_{\mathbb{Z} \times \mathrm{Gr}}(U)]{} & \varphi(\mathbb{Z} \times \mathrm{Gr})(U)
\end{array}
$$

Here the arrow $\tau_{\mathbb{Z} \times \mathrm{Gr}}(U)$ is surjective because $\mathbb{Z} \times \mathrm{Gr}$ satisfies the property (ii).

Proposition 4.3.1. *For any $U \in \mathrm{SmAff}/S$, the arrow $\tau_{\mathbb{Z} \times \mathrm{Gr}}(U) \circ \pi(U)$ is surjective.*

Proof. We need to recall how Riou proved that $\tau_{\mathbb{Z} \times \mathrm{Gr}}(U)$ is surjective. For any $\gamma \in K_0(U)$ (we can see that there exists P, a finitely generated projective module on U, such that $\gamma = [P] - d + n$ in $K_0(U)$) Riou builds a map $f_\gamma : U \to \mathrm{Gr}_{d,r}$ such that the map $F_\gamma \in \mathbb{Z} \times \mathrm{Gr}(U)$ defined as the composition

$$U \xrightarrow{f_\gamma} \mathrm{Gr}_{d,r} \to \{n\} \times \mathrm{Gr} \to \mathbb{Z} \times \mathrm{Gr}$$

is mapped by $\tau_{\mathbb{Z} \times \mathrm{Gr}}(U)$ to γ. Consider now the following pullback in Sm/S

$$
\begin{array}{ccc}
U \times \mathrm{Gr}_{d,r}^{\mathrm{aff}} & \xrightarrow{\mathrm{pr}_1} & \mathrm{Gr}_{d,r}^{\mathrm{aff}} \\
{\scriptstyle \mathrm{pr}_2} \downarrow & & \downarrow {\scriptstyle \pi_{d,r}} \\
U & \xrightarrow{f_\gamma} & \mathrm{Gr}_{d,r}
\end{array}
$$

Now, $\pi_{d,r} : \mathrm{Gr}_{d,r}^{\mathrm{aff}} \to \mathrm{Gr}_{d,r}$ is an affine vector bundle torsors so its pullback $U \times \mathrm{Gr}_{d,r}^{\mathrm{aff}}$ is a vector bundle torsor over U and since the last is affine, as remarked in [Wei89] page 475 we have that $\mathrm{pr}_2 : U \times \mathrm{Gr}_{d,r}^{\mathrm{aff}} \to U$ is an affine vector bundle torsor and in particular a vector bundle, so that there exists a section of pr_2, i.e. a map (a closed embedding) $i : U \to U \times \mathrm{Gr}_{d,r}^{\mathrm{aff}}$ such that $\mathrm{pr}_2 \circ i = id_U$. We define $f'_\gamma = U \xrightarrow{i} U \times \mathrm{Gr}_{d,r}^{\mathrm{aff}} \xrightarrow{\mathrm{pr}_1} \mathrm{Gr}_{d,r}^{\mathrm{aff}}$ and we check that the following diagram commutes

$$
\begin{array}{ccccccc}
U & \xrightarrow{f'_\gamma} & \mathrm{Gr}_{d,r}^{\mathrm{aff}} & \longrightarrow & \{n\} \times \mathrm{Gr}_{d,r}^{\mathrm{aff}} & \longrightarrow & \mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}} \\
{\scriptstyle id} \downarrow & {\scriptstyle f_\gamma} \quad & \downarrow {\scriptstyle \pi_{d,r}} & & \downarrow {\scriptstyle id \times \pi_{d,r}} & & \downarrow {\scriptstyle \pi} \\
U & \xrightarrow{f_\gamma} & \mathrm{Gr}_{d,r} & \longrightarrow & \{n\} \times \mathrm{Gr}_{d,r} & \longrightarrow & \mathbb{Z} \times \mathrm{Gr}
\end{array}
$$

In fact

$$
\pi_{d,r} \circ f'_\gamma = \pi_{d,r} \circ \mathrm{pr}_1 \circ i = f_\gamma \circ \mathrm{pr}_2 \circ i = f_\gamma \circ id_U = f_\gamma
$$

where for the second and the third equality we have used the previous pullback and the fact that i is a section of pr_2. Defining the the composition of the top horizontal line of the previous diagram to be $F'_\gamma : U \to \mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ we have that $F'_\gamma \in \mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}(U)$ and finally the previous diagram tells us that $\pi(U)(F'_\gamma) = F_\gamma$ so that we have concluded the proof.

$\square$

Corollary 4.3.2. *The arrow $\tau_{U \times \mathrm{Gr}^{\mathrm{aff}}}(U)$ is surjective for every $U \in \mathrm{SmAff}/S$. In other words $\mathbb{Z} \times \mathrm{Gr}^{\mathrm{aff}}$ satisfies the property (ii).*

Proof. It follows immediately from the previous Proposition by diagram chase. $\square$

Proposition 4.3.3. $\mathbb{P}_{\mathrm{aff}}^\infty$ *satisfies the property (ii).*

Proof. One adapt the argument of Riou in a similar way to what we did for the affine Grassmannian. $\square$

Proposition 4.3.4. *The K-theory presheaf K satisfies the property (K) with respect to the system $\mathcal{K}_\bullet^{\mathrm{aff}}$. Pic satisfies the property (K) with respect to the system $W_\bullet$*

having colimit $\mathbb{P}^{\infty}_{\mathrm{aff}}$. *The same holds replacing* $\mathcal{K}^{\mathrm{aff}}_{\bullet}$ *and* $W_{\bullet}$ *with their finite nth products* $(\mathcal{K}^{\mathrm{aff}}_{\bullet})^n$ *and* $(W_{\bullet})^n$

Proof. It suffices to show that for any map $f^{\mathrm{aff}}_{(d,r)(d',r')} : \mathrm{Gr}^{\mathrm{aff}}_{d,r} \to \mathrm{Gr}^{\mathrm{aff}}_{d',r'}$ the induced map $(f^{\mathrm{aff}}_{(d,r)(d',r')})^* : K_1(\mathrm{Gr}^{\mathrm{aff}}_{d',r'}) \to K_1(\mathrm{Gr}^{\mathrm{aff}}_{d,r})$ is surjective, but this follows from Lemma 3.2.8 which gives us that the affine vector bundle torsors $\pi_{d,r} : \mathrm{Gr}^{\mathrm{aff}}_{d,r} \to \mathrm{Gr}_{d,r}$ induces isomorphisms $K_n(\mathrm{Gr}^{\mathrm{aff}}_{d,r}) \cong K_n(\mathrm{Gr}_{d,r})$ for any n and the fact that $\pi : \mathcal{K}^{\mathrm{aff}}_{\bullet} \to \mathcal{K}_{\bullet}$ is a natural transformation. The case of Pic is the same and the the last statement of the Proposition can be proved in an analogous way. $\square$

Part II

Applications and Hermitian K-theory

Applications to K-theory

5.1 Operations on higher K-theory

Using Theorem 3.1.15, we can define structures on K as an object of $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ and we can define operations on the higher K-theory groups for any scheme qcqs over a regular noetherian ring (for example) admitting an ample family of line bundles. The construction is the same than the one in [Rio10] 2.3 and uses Appendix A.5. The point is to use 3.1.15 together with the results of 3 to lift operations defined on K_0 to operations on K in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$. As an example we consider the structure of special λ-ring. We begin with some recollections on λ-rings.

5.1.1 Recollections on λ-rings

Let us start with the following definition (see [Wei13] Definition I 4.3.1 and [Yau10] Definition 1.10)

Definition 5.1.1. A special λ-ring, or simply a λ-ring, is the datum of a commutative unital ring R together with a family of sets maps $\lambda^k : R \to R$, $k \geq 0$ such that

1) $\lambda^0(x) = 1$, $\lambda^1(x) = x$ for every $x \in R$.

2) $\lambda^k(x + y) = \lambda^k(x) + \lambda^k(y) + \sum_{i=1}^{k-1} \lambda^i(x)\lambda^{k-i}(y)$ for every $x, y \in R$ for $k > 1$.

3) $\lambda^k(1) = 0$ for $k \geq 2$.

4) $\lambda^k(xy) = P_k(\lambda^1(x), ..., \lambda^k(x); \lambda^1(y), ..., \lambda^k(y))$ for all $x, y \in R$.

5) $\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), ..., \lambda^{kl}(x))$ for all $k, l \in \mathbb{N}$ and $x \in R$.

where P_k and $P_{k,l}$ are certain universal polynomial with coefficients in $\mathbb{Z}$. A pre-λ-homomorphism between λ-rings $(R, \{\lambda_R^r\})$ and $(S, \{\lambda_S^r\})$ is a ring homomorphism $f : R \to S$ such that $f \circ \lambda_R^r = \lambda_S^r \circ f$ for all $r \geq 0$ ([Yau10] Definition 1.25).

In literature one can find the name pre-λ-ring or simply λ-ring for a ring satisfying 1)-2) above and the name special λ-ring for rings satisfying 1)-5). Since we will be interested mainly in special λ-rings we will not make such a difference and we will stick to the notation introduced in the previous definition. Finally, note that for a λ-ring, a splitting principle is always satisfied, see [Yau10] Theorem 1.44 (a priori one could not require a so called positive structure or that our ring is λ-finite dimensional).

Definition 5.1.2. Suppose R is a λ-ring and A is an R-algebra (not necessarily unital) together with a family of sets maps $\lambda^k : A \to A$ for $k \geq 1$, we will say that A is an R-λ-algebra if $R \times A$ with the addition, the multiplication and the operations defined below is a λ-ring (see [Kra80] page 240).

1) For all $a, b \in R$ and $x, y \in A$ we set $(a, x) + (b, y) := (a + b, x + y)$.

2) For all $a, b \in R$ and $x, y \in A$ we set $(a, x)(b, y) := (ab, ay + bx + xy)$

3) For all $(a, x) \in R \times A$ we set $\lambda^k(a, x) := (\lambda^k(a), \sum_{i=0}^{k-1} \lambda^i(a)\lambda^{k-i}(x))$, $k \geq 1$.

Example 5.1.3. Suppose to have an $\mathbb{N}$-graded R-module $M_* = \oplus_{n \in \mathbb{N}} M_n$ where $M_0 = R$ is a λ-ring. Assume that we give to M_* the following product

$$(a_0, a_1, a_2, ...)(b_0, b_1, b_2, ...) := (a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + b_0 a_2, ...)$$

and that we define, for $k \geq 1$ (λ^0 being the map $(a_0, a_1, \cdots) \mapsto (1, 0, 0, \cdots)$)

$$\lambda^k(a_0, a_1, a_2, ...) := (\lambda_0^k(a_0), \sum_{i=0}^{k-1} \lambda_0^i(a_0)\lambda_1^{k-i}(a_1), \sum_{i=0}^{k-1} \lambda_0^i(a_0)\lambda_2^{k-i}(a_2), ...)$$

where $\lambda_n^i : M_n \to M_n$ are group homomorphisms for all $n \geq 1$ and all $i \geq 1$. Then the R-algebra M_* automatically satisfies 1)-3) of the definition of λ-ring. Moreover if M_n is an R-λ-algebra for $n \neq 0$, the product in M_n being trivial, then M_* satisfies also 4)-5) of 5.1.1 and so it is a λ-ring, see for example the proof of 7.1 and 8.13 in [HKT17].

Given a λ-ring, one can always define the so called Adams operations, which are very useful for many purposes.

Definition 5.1.4. Let R be a λ-ring. For each $n \geq 1$ we can define the nth *Adams operation* ψ^n by recursion as $\psi^1(x) = x$, $\psi^2(x) = x^2 - 2\lambda^2(x)$, $\psi^k(x) = \lambda^1(x)\psi^{k-1}(x) - \lambda^2(x)\psi^{k-2}(x) + ... + (-1)^k \lambda^{k-1}(x)\psi^1(x) + (-1)^{k+1} k\lambda^k(x)$ (these are called Newton formulas, see [Yau10] 3.10).

The Adams operations have the following properties

Proposition 5.1.5. *Let R be a λ-ring. Then the Adams operations satisfy the following properties*

1) *Each Adams operation ψ^n is a ring homomorphism.*

2) *For each $m, n \geq 1$, we have $\psi^m \psi^n = \psi^{nm} = \psi^n \psi^m$.*

3) *If p is a prime number and $x \in R$, then $\psi^p(x) \equiv x^p$ (Mod pR).*

For a proof see [Yau10] 3.6, 3.7. The Adams operations are very important on their own and they motivated the following definition, found for example in [Yau10] Definition 3.44 and probably having origin in [Knu73], page 49.

Remark 5.1.6. The Adams operations defined via the Newton formulas, if R is a pre-λ-ring, coincide with the ones defined in [SGA71] V 7.1 or in [Yau10] 3.1. This gives that ψ^n are group homomorphism even if R is only a pre-λ-ring.

Definition 5.1.7. A commutative ring R is called a ψ-*ring* if it is equipped with ring endomorphisms $\psi^k : R \to R$ for $k \geq 1$ such that $\psi^1 = id_R$ and for each $m, n \geq 1$ $\psi^m \psi^n = \psi^n \psi^m = \psi^{mn}$. If R is noncommutative, we say that it is a *noncommutative ψ-ring* if as in the commutative case, it is equipped with ring endomorphisms $\psi^k : R \to R$ for $k \geq 1$ such that $\psi^1 = id_R$ and for each $m, n \geq 1$ $\psi^m \psi^n = \psi^n \psi^m = \psi^{mn}$.

While in the previous Definition we considered also noncommutative rings for future use, we explicitly say that in this section, we only consider commutative rings (unless otherwise stated).

Remark 5.1.8. We note that as far as we know, there isn't a well developed theory (or even a notion) of lambda rings in the context of noncommutative rings. The problem, roughly speaking, is that the axioms of a λ-rings involves symmetric polynomials that do not easily fit in the context of noncommutative rings (the reader can try to make sense of axiom 2) in this context for example). However, the definition of ψ-ring easily extends to the noncommutative case. Indeed the only definition of "noncommutative λ-ring" we have been able to find in literature is the one

contained in [Pat95] (Definition I.1) that agrees with our definition of noncommutative ψ-ring (notice that in [Pat95] the only noncommutative rings considered are noncommutative R-algebras for some commutative ring R containing the rationals).

By the previous Proposition, it is clear that every λ-ring together with its Adams operations is a ψ-ring. If the ring R is $\mathbb{Z}$-torsion free, more can be said thanks to the following theorem of Knutson [Knu73], see also [Yau10] Theorem 3.49

Theorem 5.1.9. *Suppose we have a $\mathbb{Z}$-torsion free ψ-ring. Then we can define lambda operations $\lambda^k : R \otimes_{\mathbb{Z}} \mathbb{Q} =: R_{\mathbb{Q}} \to R_{\mathbb{Q}}$ for $k \geq 0$ which endows $R_{\mathbb{Q}}$ with the structure of a λ-ring.*

The structure that we have defined can be seen to be the unique on $R_{\mathbb{Q}}$ compatible with the $\psi^k_{\mathbb{Q}}$ in an obvious sense and under additional assumptions can also be seen to extend to a λ-ring structure on R. In fact we have the following important theorem of Wilkerson ([Wil82] or see [Yau10] Theorem 3.54)

Theorem 5.1.10 (Wilkerson). *Let R be a $\mathbb{Z}$-torsion free ψ-ring such that for every $x \in R$ and every prime integer p we have $\psi^p(x) \equiv x^p$ (Mod pR). Then the structure defined in 5.1.9 on $R_{\mathbb{Q}}$ descends to a λ-ring structure on R which is the unique λ-ring structure on R whose Adams operations coincide with the ones given by the ψ-ring datum on R.*

We also have the following highly related theorem, found in [Yau10] Theorem 3.15

Theorem 5.1.11. *Let R be a $\mathbb{Z}$-torsion free ring. Then every λ-structure on it is uniquely determined by its associated ψ-structure and vice versa.*

Remark 5.1.12. One might notice that the definition of λ-ring does make sense in every category $\mathcal{C}$ admitting finite products as it will be studied in the following section. Indeed this is well known for the structure of commutative unital ring, and adding to this datum a family of unary operators λ^k satisfying the relations listed in the definition of special λ-ring (notice this relations only involves universal polynomials in a finite number of variables and with coefficients in $\mathbb{Z}$ so this actually makes sense) actually allows us to define a language and then an abstract algebraic structure (both in the sense of Appendix A.5), $\mathfrak{S}^\lambda$, characterizing the structure of special λ-ring (one can do the same with the notion of pre λ-ring) so that a λ-ring in a category with finite products is, under the terminology introduced in the appendix, simply a $\mathfrak{S}^\lambda$-object (see also A.5).

Remind that $\mathbb{Z}$ ha a unique structure of λ-ring (lambda operations being defined by the formula $\lambda^r(n) = \binom{n}{r}$ for the naturals and $\binom{-n}{k}$ is set to be $(-1)^k \binom{n+k-1}{k}$), so that also $\mathbb{Z}^n$ will have an induced λ-ring structure for any $n \in \mathbb{N}_{\geq 1}$.

Definition 5.1.13. We say that a λ-ring A is *augmented* ([Yau10] Definition 1.30) if it admits a λ-ring homomorphism $\epsilon : A \to \mathbb{Z}$. If for some $n \in \mathbb{N}$, $A = \prod_{i=1}^n A_i$ is a lambda ring so that A_i are augmented (via maps ϵ_i) sub-λ-ring of it, we will say as well that A is augmented abusing the notation and we will call $\epsilon := \prod_i \epsilon_i : A \to \mathbb{Z}^n$ its induced augmentation. Now let R be a λ-ring. We can define for every $r \geq 0$ the γ operations $\gamma^r : R \to R$ via the formula $\gamma^r(x) = \lambda^r(x + r - 1)$, valid for every $x \in R$ (see [Yau10] Proposition 3.19).

Notice that our abuse of notation in the definition of augmented lambda ring is not really harmful since we can always reason "componentwise". We will stick to the classical definition of augmented until the end of this section leaving to the reader to notice that the following results are true even if we use the "abused" terminology.

Definition 5.1.14 ([Yau10] 3.26)**.** For every augmented λ-ring (R, ϵ) we define for every $n \geq 0$ the additive subgroup $F^n R$ of R generated by products $\gamma^{n_1}(a_1) \cdots \gamma^{n_d}(a_d)$ where every $a_i \in \ker(\epsilon)$ and $\sum_{i=1}^d n_i \geq n$. The sequence

$$F^0 R \supseteq F^1 R \supseteq F^2 R \cdots$$

is called the *Grothendieck γ-filtration*

Proposition 5.1.15 (essentially [Yau10] 3.27, 3.31, 3.41)**.** *Let (R, ϵ) be an augmented λ-ring. Then $F^0 R = R$, $F^1 R = \ker(\epsilon)$, every $F^n R$ is a λ-ideal of R and it is closed under γ^r for every $r \geq 1$. Moreover for every $x \in F^n R$, $r, n \geq 1$ we have $\psi^r(x) - r^n x \in F^{n+1} R$.*

Definition 5.1.16. Given any augmented λ-ring (R, ϵ), we can associate a graded ring

$$\mathrm{Gr}(R) := \oplus_{k=0}^\infty F^k R / F^{k+1} R$$

Moreover, for any $a \in R$, we say that a has γ-dimension n ([Yau10] 3.17) if it exists an integer n such that $\gamma^m(a) = 0$ for all $m > n$. If there is a natural number n such that every element of R has γ-dimension smaller than it, we will say that R has finite γ-dimension.

Given any augmented λ-ring (R, ϵ), we denote $R_\mathbb{Q} = R \otimes_\mathbb{Z} \mathbb{Q}$ and we will denote, for any natural numbers i, j, as $R_\mathbb{Q}^{(i,j)}$ the eigenspace of $\psi^i : R_\mathbb{Q} \to R_\mathbb{Q}$

relative to the eigenvalue i^j. Noticing that the proof of [FL85] III 3.2 only relies on Proposition 5.1.15 (and the extra hypothesis below), we have the following

Theorem 5.1.17 ([Wei13] Theorem II 4.10, [FL85] proposition III 3.2). *Assume (R, ϵ) is an augmented λ-ring such that there exists an natural number n such that $F^{n+1}R = 0$, then the following facts are true*

1) *The eigenspaces $R_{\mathbb{Q}}^{(i,j)}$ are independent of i, so that we can denote them simply as $R_{\mathbb{Q}}^{(j)}$.*

2) *We have*

$$R_{\mathbb{Q}} = \oplus_{i=0}^n R_{\mathbb{Q}}^{(i)} \cong \oplus_{i=0}^n F^i R_{\mathbb{Q}}/F^{i+1}R_{\mathbb{Q}} \cong \mathrm{Gr}(R)_{\mathbb{Q}} \cong \mathrm{Gr}(R_{\mathbb{Q}})$$

In particular $R_{\mathbb{Q}}^{(i)} \cong F^i R_{\mathbb{Q}}/F^{i+1}R_{\mathbb{Q}}$ for every $i \in \mathbb{N}$.

5.1.2 Lambda ring objects in a category

The previous section allows us to make sense of the notion of lambda ring in any category with finite products and a terminal object (sometimes the terminal object is referred as the empty product). We fix in this section such a category $\mathcal{C}$. Since we do not know any explicit reference for the notion of lambda ring object is such category besides the one that can be given using the reasoning of the previous section, we think it is worthwhile to spell out its structure here. We will denote by $*$ the terminal object of $\mathcal{C}$. Suppose we are given a commutative unital ring K in $\mathcal{C}$ (we define this notion using the machinery of Appendix A.5, see also [Bor94] Section 3.2 page 125). Thus a commutative ring object in $\mathcal{C}$ is a datum $(K, +, -, \cdot, 0, 1)$ where K is an object of $\mathcal{C}$, $+ : K \times K \to K$ and $\cdot : K \times K \to K$ represent the additive and the multiplicative laws of K, with a map $- : K \to K$ denoting the inverse for the group structure and with two maps $0, 1 : * \to K$ representing the additive and the multiplicative neutral elements. These maps satisfy the usual axioms required from the definition of commutative ring object, i.e. $(K, +, -, 0)$ is an abelian group object in $\mathcal{C}$, $(K, \cdot, 1)$ is a commutative monoid object in $\mathcal{C}$ and we require the obvious diagram expressing the right and left distributivity of the multiplication with respect of the addition to commute. We will write a polynomial of degree m in n variables with integer coefficients as

$$P = \sum_{|J| \leq m} a_J x^J, \quad J = (j_1, \ldots, j_n), \quad |J| = \sum_{i=1}^n j_i \leq m, \quad x^J = x_1^{j_1} \cdots x_n^{j_n} \quad (5.1)$$

here the x_i are the variables and $a_J \in \mathbb{Z}$ for every J. Now, given an integer $q \in \mathbb{Z}$ we define the multiplication by q as a map $\cdot q : K \to K$ to be the zero map if $q = 0$, and as the following composition if $q > 0$

$$K \xrightarrow{\text{diagonal}} K^q := K \underbrace{\times \cdots \times}_{q-1 \text{ times}} K \xrightarrow{+} K$$

which is well defined because of the associativity of the group law. If $q < 0$ we define the map in the same way but we postcompose with the map $- : K \to K$. We can do something analogue with the operation "raising to the power of j" for any $j \in \mathbb{N}$. If $j = 0$ we define this map as $1 : K \to K$. Otherwise we define the map $\cdot^j : K \to K$ as the following composition

$$K \xrightarrow{\text{diagonal}} K^j \xrightarrow{\cdot} K$$

also here this map is well defined because of the associativity of the multiplicative law. With the same process, we can define for every multivariable x^J of length n and degree m as above a map

$$x^J : K^n \to K$$

by considering the composition

$$K^n \xrightarrow{\cdot^{j_1} \times \cdots \times \cdot^{j_n}} K^n \xrightarrow{\cdot} K$$

which is well defined because of the axioms of commutative ring. Post composing the previous map with $a \cdot$ for any integer a gives us maps

$$ax^J : K^n \to K$$

Now, suppose to have a polynomial $P = \sum_{|J| \leq m} a_J x^J$ in n variables and of degree m as in (5.1). Denote by q_P the number of summands in P. We can define a map $P : K^n \to K$ as follows

$$K^n \xrightarrow{\times_{|J| \leq m} a_J x^J} K^{q_P} \xrightarrow{+} K$$

which is well defined because of the ring axioms (associativity, distributivity etc.). The last step to make sense of the λ-ring axioms is then to consider a family of maps $\lambda^r : K \to K$ with $r \geq 0$. Then we can write expressions using these operations as variables. For example suppose we want to formalize the axiom $\lambda^r(x + y) =$

$\sum_{i+j=r} \lambda^i(x)\lambda^j(y)$ as the equality between two maps $K \times K \to K$. We then interpret the left hand side as the composition

$$K \times K \xrightarrow{+} K \xrightarrow{\lambda^r} K$$

For the right hand side, we see it as the composition

$$K \times K \xrightarrow{(\lambda^0 \times \cdots \times \lambda^r) \times (\lambda^0 \times \cdots \times \lambda^r)} K^{r+1} \times K^{r+1} \xrightarrow{P} K$$

where P is the polynomial of degree 2 involved in the right hand side. If a map $K \times K \to K$ is built in this way, we will denote it as P^λ. Asking if the axiom 2) of lambda ring holds then amount to ask if those two maps are equal. The same can be done for the remaining axioms of special lambda ring: they all involve polynomials with coefficients in $\mathbb{Z}$. We can then give the following

Definition 5.1.18. A λ-ring object in a category $(\mathcal{C}, \times)$ with finite products is the datum of a commutative ring object $(K, +, -, \cdot, 0, 1)$ in it together with a family of morphisms $\{\lambda^n : K \to K\}_{n \in \mathbb{N}}$ in $\mathcal{C}$ such that the axioms 1)-5) of definition 5.1.1 hold, provided we make sense of the terms involved as we explained above. Remark that this definition coincides with the one we would get by using the machinery of Appendix A.5, see also Remark 5.1.12.

Notice that all the arrows induced by the polynomials involved in the definition of a λ-ring are naturally pointed, i.e. they are pointed maps $(K, 0) \to (K, 0)$.

5.1.3 Lambda rings in homotopy categories of simplicial presheaves

In this section we study the lambda ring structure that naturally arises on the homotopy groups of a simplicial presheaf, provided it is a lambda ring in a suitable homotopy category and that it satisfies certain properties. We fix a Grothendieck site $\mathcal{C}$, and we consider the model category $s\mathbf{Pre}(\mathcal{C})$ of simplicial presheaves with the Jardine local model structure localised at some class of maps S. This covers all the situations covered into this thesis. We denote as $\mathcal{H}$ the homotopy category $\mathrm{Ho}(s\mathbf{Pre}(\mathcal{C}))$ defined above. We let $\mathcal{H}_\bullet$ to be its pointed version. Notice that by this we mean that we consider the pointed category of simplicial presheaves, we give to it the pointed model category structure induced by the one we are considering on the unpointed one and we take the homotopy category, as customary (so we are not considering the homotopy category pointed). We suppose to have a (special) lambda ring $(K, +, \bullet, -, 0, 1)$ in $\mathcal{H}$ (which is then pointed) where all the maps $\lambda^r : K \to K$ are pointed for $r > 0$ and where the ring structure comes from a ring structure in $\mathcal{H}_\bullet$

(in this last category we are only looking at the non unital ring structure because we have to take care of the point). We also assume that the product $\bullet : K \times K \to K$ factors through the smash product i.e. that there exists a map $\wedge_\bullet$ so that the following diagram commutes

$$
\begin{array}{ccc}
K \times K & & \\
\varphi \downarrow & \searrow^{\bullet} & \\
K \wedge K & \xrightarrow[\wedge_\bullet]{} & K
\end{array}
$$

Remark that K is in particular an H-group, hence for every simplicial presheaf F, the set

$$\pi_n K(F) := [\mathbb{S}^n \wedge F_+, K]_{\mathcal{H}_\bullet} =: K_n(F)$$

has a group structure inherited from the H-group structure of K. Now, we assume that K satisfies descent (i.e. it admits a weakly equivalent sectionwise fibrant replacement) so that for every element $X \in \mathcal{C}$, $K_n(X)$ is really the nth homotopy group of the simplicial set $K(X)$. Moreover we assume that the H-group structure on K is compatible with the homotopy groups, i.e. that the group structure on $K_n(F)$ for $n \geq 0$ induced from the H-group structure coincides with the standard one (the one defined as in topology using the co-group structure on $\mathbb{S}^1$). Loopspaces are of this form, for example. We could relax these assumptions but we do not have a reason to do that since they allow the discussion to be simpler and all the examples we have in mind fall in this description. Now, for any simplicial presheaf F we immediately notice that by applying the functor π_0 we obtain a λ-ring structure on the set $K_0(F)$. This is true because π_0 preserves finite products so that taking π_0 of the datum of maps and compatibilities we have for K in $\mathcal{H}$ gives us what we want.

Remark 5.1.19. Notice that the product induced by $\bullet$ on any $K_n(F)$ is trivial if $n \geq 1$ because of Lemme 5.2 in [Kra80]. Indeed that lemma says that if we are given an H-space E together with a distributive multiplication over its H-space structure that factors through the smash product and a co-H-space X having the comultiplication factoring through the join (for example any suspension) in $\mathrm{Ho}(\mathrm{Top}_\bullet)$ then the monoid structure induced on $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}_\bullet)}(X, E)$ by the structure of E is trivial and in our case the same argument holds.

Since the multiplication of K factors through the smash product, we can define pairings $K_0(F) \times K_j(F) \to K_j(F)$ as for K-theory in Theorem 5.1.28.

Remark 5.1.20. We remind that for elements $x \in K_0(F)$ and $y \in K_j(F)$, the product $x \cdot y$, using the fact that in $\mathcal{H}_\bullet$ we have $\bullet = \wedge_\bullet \circ \varphi : K \times K \to K$, is defined to be the image of the couple (x, y) via the following composition of maps

$$\text{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^0 \wedge F_+ = F_+, K) \times \text{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^j \wedge F_+, K)$$
$$\wedge \downarrow$$
$$\text{Hom}_{\mathcal{H}_\bullet}(F_+ \wedge (\mathbb{S}^j \wedge F_+), K \wedge K)$$
$$-\circ\Delta_F \downarrow$$
$$\text{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^j \wedge F_+, K \wedge K)$$
$$\wedge_\bullet \circ - \downarrow$$
$$\text{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^j \wedge F_+, K)$$

Here with Δ_F we have denoted the map induced by the diagonal $F \to F \times F$. This coincides with the construction given in Theorem 5.1.28 in this case. If we now define the product of two elements $a, b \in K_n(F)$ to be the one induced on K_n by $\bullet : K \times K \to K$ we see that we can make sense of Axiom 4) of the definition of lambda ring using polynomials P_r built as in the previous section. The only caveat here is that in the construction of these polynomial maps, we build the monomial maps x^J using smash products instead of products, i.e. we get maps $x^J : K \wedge^n K \to K$. We can do this since under our assumptions, the multiplicative product we have factors through the smash. Now, reminding ourselves the notation of the previous section and the caveat just specified, the LHS (RHS) of the equation $\lambda^r(xy) = P_r(\lambda^1(x), ..., \lambda^r(x); \lambda^1(y), ..., \lambda^r(y)) = P_r^\lambda(x, y)$ representing axiom 4) can be read as the image of a map

$$\text{Hom}_{\mathcal{H}_\bullet}(F_+, K) \times \text{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^j \wedge F_+, K) \to \text{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^j \wedge F_+, K)$$

obtained as in the previous diagram using $\lambda^r \circ \wedge_\bullet \circ -$ (respectively $P_r^\lambda \circ -$) instead of $\wedge_\bullet \circ -$ in the last step.

We define the graded group

$$K_*(F) := \oplus_{n \geq 0} K_n(F)$$

where the $K_n(F)$ are $K_0(F)$-modules using the pairings of the previous Remark. Consider the maps $\lambda_n^r : \pi_n(\lambda^r) : K_n(F) \to K_n(F)$ for $r, n \geq 0$ and $r > 0$ if $n \geq 1$. Notice that for $n \geq 1$ these maps are group homomorphisms. Henceforth, it is

possible to give to $K_*(F)$ the structure of pre-λ-ring $(K_*(F), \cdot)$ as in Example 5.1.3 (i.e. axioms 1-3 of the definition of λ ring are satisfied) and we call the lambda operations we have λ_*^r. We want to check that this is indeed a lambda ring.

Proposition 5.1.21. *For any simplicial presheaf F, the ring $(K_*(F), \cdot, \lambda_*^r)$ is a λ-ring.*

Proof. We have to check that the axioms 4) and 5) of Definition 5.1.1 are satisfied. For elements in $K_0(F)$ this has already been done. Then we notice that because of the definitions we made in 5.1.3, we only need to verify that for any $n \geq 1$, the groups $K_n(F)$ are $K_0(F)$-λ-algebras. Since we already know that they are pre-$K_0(F)$-λ-algebras this really amounts to check axioms 4) and 5) for elements $x \in K_n(F)$ and $y \in K_0(F)$ using the notation of Definition 5.1.1 (see also the proof ot Theorems 7.1 and 8.18 in [HKT17] for more details about why it suffices to check this). We start with axiom 5). Using our dictionary, $x \in K_n(F)$ is a map in $[\mathbb{S}^n \wedge F_+, K]_{\mathcal{H}_\bullet}$. Now the verification of the axiom can be done in two steps. As a first step one has from the fact that K is a λ-ring that $\lambda^r \circ \lambda^s : K \to K$ and $P_{r,s}^\lambda : K \to K$ in $\mathcal{H}_\bullet$ given from the polynomial $P_{r,s}$ using the techniques of the previous Section are equal. As a second step one see that the left hand side of the equality prescribed by axiom 5) equals the map obtained from $[\mathbb{S}^n \wedge F_+, K]_{\mathcal{H}_\bullet}$ by postcomposition with the pointed map $\lambda^r \circ \lambda^s$ with $r, s \geq 1$. Then using Remark 5.1.19 one sees that the polynomilal maps $(P_{r,s}^\lambda)_n : K_n(X) \to K_n(X)$ involved in the right hand side defined using $(K_*(F), \cdot, \lambda_*^r)$ equals the ones obtained from $[\mathbb{S}^n \wedge F_+, K]_{\mathcal{H}_\bullet}$ by postcomposition with the map $P_{r,s}^\lambda : K \to K$ in $\mathcal{H}_\bullet$. So axiom 5) is verified. Notice, because of our definitions, that since $n \geq 1$, many of the products on the RHS are equal so that it will be really a multiple of $\lambda^{rs}(x)$ as noted in the proof of Theorem 8.18 in [HKT17]. The verification of axiom 4) can be done in a similar way using Remark 5.1.20. $\square$

Remark 5.1.22. Suppose that $\mathcal{C}$ is a point with the chaotic topology and that we consider only the Jardine injective model structure on $s\mathbf{Pre}(\mathcal{C})$. Then the homotopy category we obtain is the classical homotopy category of topological spaces Ho(Top). This means, as a corollary of the previous definition, that if we have a lambda ring X in Ho(Top) satisfying the above properties, then to the direct sum of its homotopy groups $\pi_*(X) := \oplus_{n \geq 0} \pi_n(X)$ can be given a structure of special λ-ring reasoning as above.

5.1.4 Lambda operations on higher K-theory groups

In this subsection we shall assume that our base scheme S satisfies the assumptions of Theorem 3.1.15, i.e. that it is regular (remind we assume that regular schemes

are also divisorial, unless otherwise stated) quasi-projective (hence of finite type) over a noetherian ring R. We still denote as Sch_S the category of divisorial schemes of finite type over S as in 0.1. To have a simple example in mind, the reader could think as S being a regular noetherian affine scheme using Theorem 3.1.6 in place of Theorem 3.1.15 throughout this subsection. Using Theorem 3.1.15, we can define structures on K as an object of $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ and we can define operations on the higher K-theory groups for any scheme qcqs of finite type over S admitting an ample family of line bundles. The construction is the same than the one in [Rio10] 2.3 and uses Appendix A.5. The point is to use 3.1.15 together with the results of 3 to lift operations defined on K_0 to operations on K in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$. As an example we consider the structure of special λ-ring. Accordingly, in this section we will discuss how one can use the operations defined on K_0 to define lambda operations on the higher K-theory groups of a scheme. We will study thiese operations also on the homotopy categories considered so far. Recall that by [SGA71] VI, 3.2 we have a special lambda ring structure on $K_0(X)$ for every quasi-compact scheme given by a family of unary operators $(\lambda^n : K_0(X) \to K_0(X))_{n \in \mathbb{N}}$.

Theorem 5.1.23. *Let S be a regular quasi-projective scheme over a noetherian ring R and Sch_S the category of schemes of finite type over S admitting an ample family of line bundles. Then there exists a unique structure of special lambda ring on K in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ such that for every $X \in \mathrm{Sch}_S$, the induced structure of special lambda ring on $\pi_0(K)(X) \cong K_0(X)$ is the usual one.*

Proof. We already have an H-group structure on K given by the co-group structure of the simplicial circle, but here we start from nothing: the group structure defined will be the usual one. We follow the argument of Riou. First of all we have to define a ring and a group structure on K. Before using 3.1.15 to lift the group and the ring multiplication, one has to define 0 and 1 as two morphisms $0 : \bullet \to K$ and $1 : \bullet \to K$ ($\bullet$ being the terminal object S). We define those two morphisms as the morphisms associated via the Yoneda lemma to the elements $0, 1 \in K_0(S)$. Then one can apply 3.1.15 to define the group and the ring structure on K and also to check that the group structure coming from this process is actually the canonical H-group structure we have on K. Hence we have obtained a commutative ring $(K, +, \times, 0, 1)$. Note the multiplication map is pointed. Passing to others operations, one first notice that the lambda operations defining the lambda ring structures on $K_0(X)$ are stable under base change, i.e. they are compatible with the morphisms $f^* : K_0(Y) \to K_0(X)$ induced by morphisms $f : X \to Y$ for any $X, Y \in \mathrm{Sch}_S$. This means that as a presheaf of sets, $K_0(-)$ has a special lambda ring structure in the category of presheaves in the sense of Definition 5.1.18. Now one combines

3.1.15 and A.5.6 (using Remark 5.1.12) to lift (uniquely) this structure to a special lambda ring structure on K in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$. This means that we obtain a family of maps $\lambda^n : K \to K$ in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ (these are pointed for $n \geq 1$) that satisfies the axioms of Definition 5.1.18 by direct application of Corollary 3.1.15. $\qquad\square$

Remark 5.1.24. The same construction applies to Adams and γ-operations. This means that we obtain, in particular, maps $\psi^j : K \to K$ in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ (pointed) by lifting the Adams operations we have on K_0. We remind that for any lambda ring, we defined the Adams operations using the Newton formulas, see Section 5.1.1. Now, in virtue of Corollary 3.1.15 this allows us to see that the for K seen as an element of $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$, the Newton formulas that comes from the lambda ring structure of K in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ that we have defined in the previous theorem lift the Newton formulas we have for K_0 in the category of presheaves of sets. This means that the Adams operations on K defined using the Newton formulas in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ or obtained lifting the Adams operations on K_0 with 3.1.15 coincide, then there is no ambiguity in our definitions.

Corollary 5.1.25. *The endomorphisms $\psi^k : K \to K$ given by the previous theorem by lifting the Adams operations are ring morphisms for every $k \geq 1$. Moreover for every $m, n \geq 1$ we have $\psi^m \psi^n = \psi^{nm} = \psi^n \psi^m$.*

Proof. One simply verifies the analogous statement on the level of K_0. $\qquad\square$

We now turn to the problem of defining operations on higher K-theory. From now on we will assume that the hypothesis of Theorem 5.1.23 hold. We define operations on $K_n(\mathcal{X})$ for every simplicial presheaf $\mathcal{X} \in \mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)$. We start by setting (recall that K, being a loopspace, is naturally pointed by 0 and an H-space as remarked in the proof of the previous theorem this group structure agrees with with the group structure induced by 3.1.15)

$$K_n(\mathcal{X}) := [\mathbb{S}^n \wedge \mathcal{X}_+, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S},\bullet}$$

where we have denoted with $\mathbb{S}^n$ the simplicial n-sphere as customary. Explicitly note that in the particular case where $\mathcal{X} = X \in \mathrm{Sch}_S$ is a scheme (seen as a simplicial presheaf), these groups agree with the ordinary Quillen's higher algebraic K-theory groups of vector bundles because all our schemes are assumed to be divisorial (notice that in this case Quillen's and Thomason's K-theories agree) and K-theory satisfies Zariski descent (see also Appendix B). The operations we define on $K \in \mathcal{H}_{Zar}^{\mathrm{Sch}_S},\bullet$ give us operations on $K_n(\mathcal{X})$. But now operations on $K \in \mathcal{H}_{Zar}^{\mathrm{Sch}_S},\bullet$ come from *pointed* maps of presheaves of sets for $n > 0$ because of 3.1.15 so we have to be careful. In

favourable case such as lambda, Adams or γ-operations, we can then use Corollary 3.1.15 to lift these natural operations from the unpointed case to the pointed case. In general we have

Theorem 5.1.26. *Unary unpointed operations* $\tau : K_0(-) \to K_0(-)$ *such that* $\tau_X(0) = 0$ *for every* $X \in \text{Sch}_S$ *(so that they become pointed) induce maps* $\widetilde{\tau} : K \to K$ *in* $\mathcal{H}^{\text{Sch}_S}_{Zar},_\bullet$ *that are uniquely determined by maps* $\widetilde{\tau} : K \to K$ *in* $\mathcal{H}^{\text{Sch}_S}_{Zar}$ *lifting* τ. *As a consequence such maps induce endomorphisms* $\pi_n\widetilde{\tau} : K_n(\mathcal{X}) \to K_n(\mathcal{X})$ *for every* $\mathcal{X} \in \text{s}\mathbf{Pre}(\text{Sch}_S)$.

Proof. One uses A.3.12 to see these operations as pointed operations and so lifting them to $K \in \mathcal{H}^{\text{Sch}_S}_{Zar},_\bullet$ using 3.1.15 (or Lemma A.3.10). $\qquad\square$

As a consequence we give the following

Definition 5.1.27. For every simplicial presheaf $\mathcal{X} \in \text{s}\mathbf{Pre}(\text{Sch}_S)$ we define the lambda and the Adams operations $\lambda_n^r, \psi_n^j : K_n(\mathcal{X}) \to K_n(\mathcal{X})$ by postcomposition with the maps $\lambda^r, \psi^j : K \to K$ defined in Theorem 5.1.23, i.e. as the maps

$$\lambda^r, \psi^j \circ - : [\mathbb{S}^n \wedge \mathcal{X}_+, K]_{\mathcal{H}^{\text{Sch}_S}_{Zar},_\bullet} \to [\mathbb{S}^n \wedge \mathcal{X}_+, K]_{\mathcal{H}^{\text{Sch}_S}_{Zar},_\bullet} \quad f \mapsto \lambda^r, \psi^j \circ f$$

We have then the following theorem.

Theorem 5.1.28. *Lambda, Adams and* γ-*operations* $K_0(-) \to K_0(-)$ *naturally induce maps on* $K_n(\mathcal{X})$ *for every* $\mathcal{X} \in \text{s}\mathbf{Pre}(\text{Sch}_S)$ *and the relations that hold at level of* K_0 *such as the Newton's formulas are true even in this setting. In particular this is true for the usual higher* K-*theory groups* $K_n(X)$ *for every* $X \in \text{Sch}_S$. *Moreover the multiplication law* $- \times - : K \times K \to K$ *induces a graded ring structure on the graded* $K_0(X)$-*module*

$$K_*(X) := \bigoplus_{n \in \mathbb{N}} K_n(X)$$

for any scheme $X \in \text{Sch}_S$. *Denote this ring together with its multiplication as* $(K_*(X), \cup)$. *The same holds true replacing* $X \in \text{Sch}_S$ *with* $\mathcal{X} \in \text{s}\mathbf{Pre}(\text{Sch}_S)$ *everywhere.*

Proof. The first part of the statement follows immediately from the previous theorem. For the second part, we simplify the notation and we set $\mathcal{C} = \mathcal{H}^{\text{Sch}_S}_{Zar}, \bullet$. Now, as we said before, the multiplication of K is actually a map $\times : K \times K \to K$ in $\mathcal{C}$. Using the argument contained in [Rio06] pag.96 or directly using 3.1.15 and Lemme III.33 of [Rio06] we get that there is an injective map $\alpha : \text{Hom}_\mathcal{C}(K \wedge K, K) \to \text{Hom}_\mathcal{C}(K \times K, K)$ induced by $K \times K \to K \wedge K$ such that $\times \in \text{Hom}_\mathcal{C}(K \times K, K)$ is

the image under α of a map $\times_\bullet \in \mathrm{Hom}_\mathcal{C}(K \wedge K, K)$. In particular, $\times_\bullet$ is the unique morphism which makes the following diagram to commute

$$
\begin{array}{ccc}
K \times K & & \\
\downarrow & \searrow^{\times} & \\
K \wedge K & \xrightarrow{\times_\bullet} & K
\end{array}
$$

Because of this, the fact that $\times$ is symmetric implies that $\times_\bullet$ is symmetric too, i.e. we have $\times_\bullet = \times_\bullet \circ \tau$ where τ is the usual switch map of $\wedge$. Now if we denote as $\Delta_X : \mathbb{S}^{i+j} \wedge X_+ \to (\mathbb{S}^i \wedge X_+) \wedge (\mathbb{S}^j \wedge X_+)$ the map in $\mathcal{C}$ induced by the diagonal map $X \to X \times X$ for any scheme $X \in \mathrm{Sch}_S$ we get a multiplication

$$
- \cup - : K_i(X) \times K_j(X) \to K_{i+j}(X)
$$

induced by the map

$$
\mathrm{Hom}_\mathcal{C}(\mathbb{S}^i \wedge X_+, K) \times \mathrm{Hom}_\mathcal{C}(\mathbb{S}^j \wedge X_+, K)
$$
$$
\downarrow^{\wedge}
$$
$$
\mathrm{Hom}_\mathcal{C}((\mathbb{S}^i \wedge X_+) \wedge (\mathbb{S}^j \wedge X_+), K^{\wedge 2})
$$
$$
\downarrow^{\times_\bullet \circ - \circ \Delta_X}
$$
$$
\mathrm{Hom}_\mathcal{C}(\mathbb{S}^{i+j} \wedge X_+, K)
$$

This multiplication induces the desired graded ring structure and the naturality is clear. Replacing $X \mathrm{Sch}_S$ with $\mathcal{X} \in \mathbf{sPre}(\mathrm{Sch}_S)$ we can proceed verbatim to prove the last statement.

$\square$

We now want to discuss the structure we can put on $K_*(X)$ for every scheme $X \in \mathrm{Sch}_S$. This abelian group in principle can have two multiplicative structure as a $K_0(X)$-algebra. The first one is the one given in Example 5.1.3, where the product of two homogeneous elements of positive degree is set to be 0. We will refer to this ring simply as $K_*(X)$ or $(K_*(X), \cdot)$ if confusion might arise. The second one is the noncommutative structure induced on it by the previous Theorem. In this case, we will denote the resulting noncommutative $K_0(X)$-algebra as $(K_*(X), \cup)$. Theorem 5.1.28 gives us families of operations $\lambda_n^k : K_n(X) \to K_n(X)$ and $\psi_n^k : K_n(X) \to K_n(X)$ which bundle to maps $\lambda^k, \psi^k : K_*(X) \to K_*(X)$ defined as follows.

Definition 5.1.29. We define for every $k \in \mathbb{N}$ natural transformations of presheaves of sets $\lambda^k : K_*(-) \to K_*(-)$ in $\mathbf{Pre}(\mathrm{Sch}_S)$ using the method of Example 5.1.3 and

for $j \geq 1$ natural transformations of presheaves of groups $\psi^k = \oplus_n \psi_n^k : K_*(-), \to K_*(-)$.

We notice the following

Proposition 5.1.30. *For every $X \in \mathrm{Sch}_S$ and every $a \in K_n(X)$, $b \in K_m(X)$, we have for every $k \geq 1$ that $\psi_*^k(a \cup b) = \psi_*^k(a) \cup \psi_*^k(b)$ where the product is induced by the pairing defined in Theorem 5.1.28 (this is also trivially true for the product $\cdot$). The same conclusion holds replacing $X \in \mathrm{Sch}_S$ with $\mathcal{X} \in \mathrm{sPre}(\mathrm{Sch}_S)$ everywhere.*

Proof. One follows Riou [Rio06] page 99. In fact as a consequence of Corollary 5.1.25 we have that in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}, \bullet$, for every $k \geq 1$, the equality $\psi^k \circ \times_\bullet = \times_\bullet \circ (\psi^k \wedge \psi^k)$ holds. This concludes the proof. $\square$

Remark 5.1.31. If we have a ring R which is noncommutative, then there isn't a well defined notion of noncommutative λ-ring. However, one notices that the notion of ψ-ring makes perfect sense even in the noncommutative case, see Definition 5.1.7. We believe that for noncommutative rings the notion of ψ-ring should be regarded as the best analogue of the notion of λ-ring. See also Remark 5.1.8.

We then have the following theorem

Theorem 5.1.32. *Consider $X \in \mathrm{Sch}_S$. Then the datum $(K_*(X), \cdot, \lambda^k)$ is a lambda ring with associated ψ-ring $(K_*(X), \cdot, \psi^k)$. Moreover, $(K_*(X), \cup, \psi^k)$ is a noncommutative ψ-ring and the maps $\psi^k : (K_*(X), \cup) \to (K_*(X), \cup)$ are morphisms of noncommutative ψ-rings. These structures are functorial.*

Proof. The noncommutative assertions follows simply from Proposition 5.1.30 and Corollary 5.1.25. For the first part, to check 4) and 5) (1)-3) follow from the very definition, see 5.1.3) we use Proposition 5.1.21 in 5.1.3. To check that the Adams operations we defined before agree with the ones induced by λ^k using the Newton formulas we notice that they are both additive so that we only need to check for elements of the form $x \in K_n(X)$ but this follows from the Newton formulas we have for $K \in \mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ and the definitions we have given. Indeed as a first step one notices that the Newton formulas we have for $K \in \mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ restrict on $K_n(X)$ to the usual Newton formulas for $K_0(X)$ in the case $n = 0$ and to $\psi_n^k = (-1)^{k+1} k \lambda^k$ for $n \neq 0$ in virtue of Remark 5.1.19 so they are the same formulas we get starting from $(K_*(X), \cdot, \lambda^k)$ because the product of two positive homogeneous elements are set to be trivial. This comparison only requires the pre-λ-ring structure because of Remark 5.1.6 in Section 5.1.1. The fact that $(K_*(X), \cdot, \psi^k)$ is a ψ-ring follows from this comparison or can be proved independently using 5.1.30. If $K_*(X)$ is

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$\mathbb{Z}$-torsion free, then we could use the fact that $(K_*(X), \cdot, \psi^k)$ is a ψ-ring together with Theorem 5.1.11 to show that $(K_*(X), \cdot, \lambda^k)$ is a lambda ring. $\qquad \square$

Corollary 5.1.33. *For every scheme* $X \in \mathrm{Sch}_S$, *the ring* $(K_*(X)_{\mathbb{Q}}, \cdot)$, *if* $(K_*(X), \cdot)$ *is* $\mathbb{Z}$-*torsion free, admits a (unique)* λ-*ring structure induced from* $(K_*(X), \cdot, \lambda^k)$ *defined before. Moreover, all* $K_n(X)$ *are* $K_0(X)$-λ-*algebras, the product of the elements in* $K_n(X)$ *being trivial for* $n \geq 1$.

Proof. For the first part, one simply uses Theorem 5.1.9. The second part follows from Theorem 5.1.32 from the very definition of $K_0(X)$-λ-algebra. $\qquad \square$

Remark 5.1.34. One might wonder if changing our base scheme S, we change the structures induced by the operations on $K_n(X)$. Indeed, a scheme can be seen as a scheme over many bases. However, Riou showed ([Rio10] Proposition 2.3.2) that the operations we get on K-theory in $\mathcal{H}(S)$ do not depend on the choice of S, as long as S is regular. Since we can reduce to the smooth schemes, we have that the operations that we define for any divisorial scheme of finite type over our allowed bases S are the unique we can define using this method.

Remark 5.1.35. Notice that because of Theorem 1.8.10, using as a starting point Riou's theorem, we can define lambda operations even if we assume our base scheme S to be regular (remind that we assume that regular schemes are also divisorial unless otherwise stated) and noetherian. Indeed all we need to define such operations is that we are allowed to use 5.1.21, i.e. that the assumptions of Subsection 5.1.3 are satisfied. This is the point of view taken in the following Subsection.

5.1.5 Lambda operations for higher K-theory groups of non divisorial schemes

In this section we study the lambda ring structures that naturally arise on the higher K-theory groups of schemes which are possibly non divisorial. We fix S a regular noetherian base scheme. And we consider the categories Sm/S and Sch_S of (smooth) noetherian schemes of finite type over S. Because of Theorems 1.10.1, 3.4.1 and 3.3.3, if we consider K as the Thomason's K-theory simplicial presheaf, we have that

$$[K^n, K]_{\mathcal{I}^l_{Zar,Nis}\mathrm{Sch}_S} \cong [K^n, K]_{\mathcal{H}(S)}$$

and that

$$[K^n, K]_{\mathcal{I}^l_{Zar,Nis}\mathrm{Sch}_{S,\bullet}} \cong [K^n, K]_{\mathcal{H}(S)_{\bullet}}$$

so that, because of Proposition A.5.6 we can repeat verbatim almost all the considerations we made in the previous Section, in particular we have

Theorem 5.1.36. *Let S be a regular noetherian scheme and Sch_S the category of schemes of finite type over S. Then there exists a unique structure of special lambda ring on K in $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ and for every $X \in \mathrm{Sch}_S$, $\pi_0(K)(X) \cong K_0(X)$ (this is the K-theory of perfect complexes) is a special lambda ring.*

Proof. The proof uses the facts recalled before and the fact that π_0 commutes with finite products. $\square$

Also, all the facts of the previous section are true even in this settings since we just used formal arguments and Proposition 5.1.21, hence we content ourselves to state the results, whose proofs is mutatis mutandis the one of the analogue result in the previous section.

Definition 5.1.37. For every simplicial presheaf $\mathcal{X} \in s\mathbf{Pre}(\mathrm{Sch}_S)$ we define the lambda and the Adams operations $\lambda_n^r, \psi_n^j : K_n(\mathcal{X}) \to K_n(\mathcal{X})$ by postcomposition with the maps $\lambda^r, \psi^j : K \to K$ defined in Theorem 5.1.36, i.e. as the maps

$$\lambda^r, \psi^j \circ - : [\mathbb{S}^n \wedge \mathcal{X}_+, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}, \bullet} \to [\mathbb{S}^n \wedge \mathcal{X}_+, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}, \bullet} \quad f \mapsto \lambda^r, \psi^j \circ f$$

Theorem 5.1.38. *Lambda, Adams and γ-operations $K \to K$ in $\mathcal{H}(S)$ naturally induce maps on $K_n(\mathcal{X})$ for every $\mathcal{X} \in s\mathbf{Pre}(\mathrm{Sch}_S)$ and the relations that hold in $\mathcal{H}(S)$ such as the Newton's formulas are true even in this setting. In particular this is true for the usual higher K-theory groups $K_n(X)$ for every $X \in \mathrm{Sch}_S$. Moreover the multiplication law $- \times - : K \times K \to K$ induces a graded ring structure on the graded $K_0(X)$-module*

$$K_*(X) := \bigoplus_{n \in \mathbb{N}} K_n(X)$$

for any scheme $X \in \mathrm{Sch}_S$. Denote this ring together with its multiplication as $(K_*(X), \cup)$. The same holds true replacing $X \in \mathrm{Sch}_S$ with $\mathcal{X} \in s\mathbf{Pre}(\mathrm{Sch}_S)$ everywhere.

Definition 5.1.39. We define for every $k \in \mathbb{N}$ natural transformations of presheaves of sets $\lambda^k : K_*(-) \to K_*(-)$ in $\mathbf{Pre}(\mathrm{Sch}_S)$ using the method of Example 5.1.3 and for $j \geq 1$ natural transformations of presheaves of groups $\psi^k = \oplus_n \psi_n^k : K_*(-), \to K_*(-)$.

Proposition 5.1.40. *For every $X \in \mathrm{Sch}_S$ and every $a \in K_n(X)$, $b \in K_m(X)$, we have for every $k \geq 1$ that $\psi_*^k(a \cup b) = \psi_*^k(a) \cup \psi_*^k(b)$ where the product is induced by the pairing defined in Theorem 5.1.38 (this is also trivially true for the product $\cdot$).*

Theorem 5.1.41. *Consider $X \in \mathrm{Sch}_S$. Then the datum $(K_*(X), \cdot, \lambda^k)$ is a lambda ring with associated ψ-ring $(K_*(X), \cdot, \psi^k)$. Moreover, $(K_*(X), \cup, \psi^k)$ is a noncommutative ψ-ring and the maps $\psi^k : (K_*(X), \cup) \to (K_*(X), \cup)$ are morphisms of noncommutative ψ-rings. These structures are functorial.*

Corollary 5.1.42. *For every scheme $X \in \mathrm{Sch}_S$, the ring $(K_*(X)_\mathbb{Q}, \cdot)$, if $(K_*(X), \cdot)$ is $\mathbb{Z}$-torsion free, admits a (unique) λ-ring structure induced from $(K_*(X), \cdot, \lambda^k)$ defined before. Moreover, all $K_n(X)$ are $K_0(X)$-λ-algebras, the product of the elements in $K_n(X)$ being trivial for $n \geq 1$.*

Remark 5.1.43. These operations restrict to the ones defined in the previous section for divisorial schemes.

5.2 Comparison with the structures defined before

As done in the thesis of Riou [Rio06], we can compare the structures we have just defined with some structures defined in literature. In places the methods we use are a simple extension of what was done by Riou, hence we will use his results to skip some details.

5.2.1 Comparisons between products

We fix a regular quasi-projective divisorial base scheme S over a noetherian ring R. We remind that we have 4 relevant homotopy categories, $\mathcal{H}_{Zar}^{\mathrm{Aff}/S}$, $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$, $\mathcal{H}^{aff}(S)$ and $\mathcal{H}(S)$ (see 0.1). We also have their discrete counterparts $\mathbf{Pre}(\mathrm{Aff}/S)$, $\mathbf{Pre}(\mathrm{Sch}_S)$, $\mathbf{Pre}(\mathrm{SmAff}/S)$ and $\mathbf{Pre}(\mathrm{Sm}/S)$. In virtue of Theorem 3.2.16, the endomorphisms of K and K_0 respectively on all these categories are in bijections. This means that the product we have defined by extending the product defined by Riou, agrees at least with the ones equivalent to the one defined by Riou in the smooth case. We then only need to say something concerning the product defined for possibly singular schemes. The main pairings we are interested in are the ones defined by Waldhausen ([Wal85], [Wal78]) and by Loday ([Lod76]), later generalized by May ([May80]). Comparisons between these constructions were made for K-theory of rings in [Wei81] and in [Shi88] at the homotopical level. The construction which more easily fits into our discussion is the one of Waldhausen. As recalled in the references provided, for every biexact functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are exact categories Waldhausen defines (functorially for exact functors) a map of pointed topological spaces

$$BQ(\mathcal{A}) \times BQ(\mathcal{B}) \to BQQ(\mathcal{C})$$

where by QQ we have denoted the bicategory construction of [Wal78] p.194. In *op.cit.* it is shown that for every exact category $\mathcal{A}$ we have an homotopy equivalence $BQ(\mathcal{A}) \simeq \Omega BQQ(\mathcal{A})$. Notice that these pairings can be refined to pairings $\Omega BQ(\mathcal{A}) \wedge \Omega BQ(\mathcal{B}) \to \Omega BQ(\mathcal{C})$ that descend from maps $\Omega BQ(\mathcal{A}) \times \Omega BQ(\mathcal{B}) \to \Omega BQ(\mathcal{C})$, the details of this can be found in [Wei13] page 362 or in a more detailed form in [Wei81] pages 503, 504. Now one can show that the operation of taking tensor product of vector bundles can be made functorial to give a biexact natural transformation between presheaves of exact categories

$$- \otimes - : \mathrm{Vect}(-) \times \mathrm{Vect}(-) \to \mathrm{Vect}(-)$$

on the categories of schemes we are considering. Applying then the Waldhausen machinery one gets a map

$$- \otimes - : K \times K \to K$$

in all $\mathcal{H}_{Zar}^{\mathrm{Aff}/S}$, $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$, $\mathcal{H}^{aff}(S)$ and $\mathcal{H}(S)$ which descends to a pointed map

$$- \otimes - : K \wedge K \to K$$

giving rise to to the Waldhausen pairings $K_i \times K_j \to K_{i+j}$.

Theorem 5.2.1. *The Waldhausen pairings induced by $-\otimes-$ on every $X \in \mathrm{Sch}_S$ in higher K-theory agrees with the ones given by Theorem 5.1.28.*

Proof. It is a straighforward application of Theorem 3.2.16: checking at the K_0 level reveals that the map $\times$ considered in 5.1.28 and the map induced by $-\otimes-$ are the same. $\qquad\square$

Corollary 5.2.2. *The pairings defined by Loday an Waldhausen for affine schemes in Aff/S agrees with the one defined in Theorem 5.1.28 and the latter agrees with the products of Soulé ([Sou85]) Quillen (in [Qui73]) and Riou for smooth schemes.*

Proof. The second part follows from the fact that the products defined by Riou were shown by him to agree with the others in [Rio06], and our products agree with the ones of Riou when restricted to regular schemes by definition. For the first part, one uses Theorem 5.2.1 to link our pairings with the Waldhausen's ones and then refers to [Wei81], [Shi88] for the remaining agreements. $\qquad\square$

Remark 5.2.3. Notice that for affine schemes other products can be defined but they have been shown to agree with the products considered so far. See for example [Wei81] or [Ina95] Chapter III.4 for a more extensive discussion.

5.2.2 Comparisons between the lambda structures

In the previous subsections we have shown that for a large class of schemes, the ring structures that we can usually put on $K_*(-)$ agree. We shall then keep the assumption and the notation we used in 5.1.4. We fix a base scheme S satisfying the hypothesis of Subsection 5.1.4, i.e. we shall assume S to be regular (remind we assume that regular schemes are also divisorial, unless otherwise stated) quasi-projective over a noetherian ring R. In this section we are concerned about the lambda and Adams operations that we have built. First we recall the following Theorem by Riou

Theorem 5.2.4. *The operations $\lambda_n^k, \psi_n^k : K_n(X) \to K_n(X)$ for $X \in \mathrm{Sm}/S$ defined by 5.1.26 agrees with the ones defined by Soulé for regular schemes in [Sou85] and [GS99].*

We now explain some results and constructions needed in the proof of the previous theorem, which will allow us to extend this result to the generality of our work. The lambda operations defined by Gillet and Soulé in [GS99] and [Sou85] came as a generalization of the operations define by Kratzer in [Kra80] for rings. We start by explaining the argument of Gillet and Soulé of [GS99], Section 4. We translate their construction in our language so that further comparisons will be easier. However notice that their language and in some cases also their arguments are a little bit different: what follows is then our reconstruction. For every natural number r, we can consider the Grothendieck group of linear representations of $\mathrm{GL}_{r,\mathbb{Z}}$, denoted $\mathrm{R}_{\mathbb{Z}}\mathrm{GL}_r$ ([Wei13] Ex. II.4.2 for example). By the seminal work of Serre [Ser68], it is a special λ-ring (with involution) where the lambda ring structure is induced by tensor products and exterior powers of representations. It can be shown that as a λ-ring it is isomorphic to the polynomial ring generated by exterior powers of the identity representation $id_{\mathrm{GL}_{r,\mathbb{Z}}} =: id_r$ with the determinant representation inverted, i.e. $\mathrm{R}_{\mathbb{Z}}\mathrm{GL}_r \cong \mathbb{Z}[\lambda^0(id_r), ..., \lambda^r(id_r), \lambda^r(id_r)^{-1}]$ as lambda rings (see [GS99] 4.1). There are evident maps $\mathrm{R}_{\mathbb{Z}}\mathrm{GL}_{r+1} \to \mathrm{R}_{\mathbb{Z}}\mathrm{GL}_r$ given by $id_{r+1} \mapsto id_r$, which corresponds to the standard embeddings $\mathrm{GL}_r \to \mathrm{GL}_{r+1}$. We have a canonical map $\varphi_r : \mathrm{R}_{\mathbb{Z}}\mathrm{GL}_r \to K_0(\mathrm{BGL}_r) \cong [\mathrm{BGL}_r, K]_{\mathcal{H}} \cong [\mathrm{BGL}_r^+, K]_{\mathcal{H}}$ (can use [GS99] Lemma 20 or [Rio06] Section II.8.3) where $\mathcal{H}$ can be both $\mathcal{H}_{Zar}^{\mathrm{Sch}S}$ or $\mathcal{H}(S)$ and the isomorphisms follows by the content of Section 1.8. Reasoning as in [GS99] 4.2 (or see [Sou85] page 511) we can extend this map to a map $\varphi_r : \mathrm{R}_{\mathbb{Z}}\mathrm{GL}_r \to [\mathbb{Z}\times\mathrm{BGL}_r^+, \mathbb{Z}\times\mathrm{BGL}^+]$. Taking inverse limit on both sides one gets a map $\varphi : \mathrm{R}_{\mathbb{Z}}\mathrm{GL} \to [\mathbb{Z} \times \mathrm{BGL}^+, \mathbb{Z} \times \mathrm{BGL}^+]_{\mathcal{H}} \cong [K, K]_{\mathcal{H}}$ so that the lambda operations we had on $\mathrm{R}_{\mathbb{Z}}\mathrm{GL}$ are mapped to operations in $[K, K]_{\mathcal{H}}$ as in [GS99] page 45 (or as in [Sou85] pages 492 and 512, [Kra80] pages

240-241) which can be seen to act on the higher K-theory groups of schemes in Sch_S. The expert reader familiar with the work of Gillet and Soulé will have noticed that we have simplified a little their argument, and this is due to some simplifications that are possible after our work and the work of Riou. Indeed, from [Rio06] pag. 118 we have that

$$\varprojlim_{r\in\mathbb{N}}[\mathrm{BGL}_r^+, K]_{\mathcal{H}(S)} \cong \varprojlim_{r\in\mathbb{N}} K_0(\mathrm{BGL}_r) \cong \varprojlim_{r\in\mathbb{N}} K_0(\mathrm{Gr}_{r,\infty}) \cong [\mathrm{Gr}, K]_{\mathcal{H}(S)} \cong [\mathrm{BGL}^+, K]_{\mathcal{H}(S)}$$

which implies, because of our results, that $\varprojlim_{r\in\mathbb{N}}[\mathrm{BGL}_r^+, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \cong [\mathrm{BGL}^+, K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}}$. The construction in [GS99] applies to a larger class of schemes: we compare here only their construction with the one we got in Theorem 5.1.32, leaving a more general comparison to forthcoming work. Riou was able to show ([Rio06] III.95 and III.96), in our notation, that if we take $\mathcal{H} = \mathcal{H}(S)$ then the map φ is injective and maps the lambda operations to the lambda operations we have from lifting the operations we had at the level of K_0.

Remark 5.2.5. Another proof of this can be given by looking at K_0 and noticing that the the lambda operations induced from φ restricts to the usual lambda operations on K_0 and so they have to agree with ours. Notice also that this make the map φ a λ-ring homomorphism.

Since it does not matter if we take $\mathcal{H}$ equal to $\mathcal{H}(S)$ or to $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ we have proved the following

Theorem 5.2.6. *For every scheme $X \in \mathrm{Sch}_S$, the $K_0(X)$-λ-algebra structure induced using the method of [GS99] on every $K_n(X)$ agrees with the one defined in Theorem 5.1.32.*

Remark 5.2.7. As a consequence, our operations on $K_n(A)$ for every ring noetherian ring A of finite type over a regular ring R agree with the ones defined by all the previous authors, for example Kratzer and Hiller ([Kra80] and [Hil81]). It would be nice to compare with the operations defined in [HKT17] following the purely algebraic definition of higher K-groups based on binary bicomplexes given by Grayson in [Gra12].

We now compare our operations with the ones defined by Levine in [Lev97]. It is stated in many places that they agree with the ones of Gillet and Soulé at least for regular schemes and it is tacitly assumed in many others that they are indeed the same for any scheme having an ample family of line bundles. So they agree with

ours. However, since we do not know a reference for such comparison, we think to make a good service to the mathematical community by giving an argument here proving this folklore result. Let be S a regular base scheme so that Theorem 3.1.15 holds. Now, suppose $G \in \mathcal{H}^{\mathrm{Sch}_S}_{Zar}$ satisfies descent and that it is an H-group. Then $\varphi \in [G, G]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}}$ defines a map $\varphi' : G \to G_f$ of simplicial presheaves unique up to simplicial homotopy, where $G \xrightarrow{\sim} G_f$ is a sectionwise fibrant replacement. This can be defined by considering $\varphi \in [G, G]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}} \cong \pi_0 \mathfrak{Map}(G, G_f)$ and then by choosing a representative of it in the last set of path components. Remark that if the starting φ happended to be pointed, then all the maps deduced from it can be pointed as well, so that the pointed analogue also hold. Notice that if G is sectionwise fibrant, this map also defines a map $\varphi' : G \to G$ in $\mathrm{Ho}_{BK}(\mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S))$ where by the last one is the homotopy category of simplicial presheaves over Sch_S having the global projective model structure. Now, if φ was pointed, letting $\mathrm{Ho}(\mathrm{Top}) =: h\mathcal{T}$, we have that φ' defines a map $\tilde{\varphi} \in \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S, h\mathcal{T})}(G, G))$which depends only on φ and not on the choice of φ' and such that all the presheaves of groups $\pi_n(\varphi)$, $\pi_n(\varphi')$ and $\pi_n(\tilde{\varphi})$ are isomorphic. Now let be $G = K$ the Quillen algebraic K-theory presheaf. We have defined a family of maps $\lambda^i \in [K, K]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}}$ $(i \geq 1)$ which give us maps $\lambda^{i'}$ and $\tilde{\lambda}^i$ as before which in turn defines pointed maps $\lambda^i(X) : K(X) \to K(X)$ in $h\mathcal{T}$ whose induced maps on the homotopy groups are the lambda operations $\lambda^i_n :$ $K_n(X) \to K_n(X)$ that we considered so far. Now, since these maps are pointed, we can further restrict the map $\lambda^{i'}$ to maps $\lambda^i : K(-)_0 \to K(-)_0$ in $\mathrm{Ho}_{BK}(\mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S))$ where by $K(-)_0$ we are denoting the simplicial presheaf associating to a scheme X the conected component of the distinguished point of $K(X)$. This map will in turn induce taking π_n for $n \geq 1$ the same maps than $\lambda^{i'}$. These are the maps we are going to use for our comparison. We fix a scheme $X \in \mathrm{Sch}_S$ and we denote as $\mathcal{U} = \{U_i \cong \mathrm{Spec}(A_i) \xrightarrow{f_i} X\}_{i \in I, |I| = q \in \mathbb{N}}$ a finite affine open cover of X. Denote by $< \mathcal{U} >$ the poset of nonempty subsets of I ordered by inclusion, as $\mathcal{R}_{\mathcal{U}} :< \mathcal{U} >\to \mathrm{Rng}$ the functor $J \mapsto \Gamma(\cap_{i \in J} U_i, \mathcal{O}_X)$ and as $< \mathcal{R}_{\mathcal{U}} >$ its image (which we can regard as a category). Now we know that for any ring R we have a canonical natural (in R) weak (and hence homotopy) equivalence $\varphi_R : \mathrm{BGL}(R)^+ \xrightarrow{\sim} \Omega BQ\mathcal{P}(R)_0 = K(R)_0$. These maps can be used to get natural maps $K(X)_0 \xrightarrow{f_i^*} K(U_i)_0 \xrightarrow{\varphi_{A_i}^{-1}}$ $\mathrm{BGL}(A_i)^+$ which in turn give a map $\varphi_{<\mathcal{U}>} : K(X)_0 \xrightarrow{\sim} (\underset{<\mathcal{U}>}{\mathrm{holim}} \mathrm{BGL}(\mathcal{R}_{\mathcal{U}})^+)_0$ which can be seen to be a weak equivalence as in the proof of Theorem 5.3 in [Lev97], for example. Now, the lambda ring structure used for higher algebraic K-theory in *op.cit.* is induced from the usual lambda ring structure on $K_0(X)$ for any scheme X and for $n \geq 1$ are induced by taking the nth homotopy group of the homotopy

limit of maps $\lambda^i : \mathrm{BGL}(\mathcal{R}_\mathcal{U})^+ \to \mathrm{BGL}(\mathcal{R}_\mathcal{U})^+$ in $\mathrm{Ho}_{BK}(\mathbf{sPre}(<\mathcal{U}>^{op}))$ (given by restriction of the pointed maps $\lambda^i : K_0(\mathcal{R}_\mathcal{U}/X) \times \mathrm{BGL}(\mathcal{R}_\mathcal{U})^+ \to K_0(\mathcal{R}_\mathcal{U}/X) \times \mathrm{BGL}(\mathcal{R}_\mathcal{U})^+$ in $\mathrm{Ho}_{BK}(\mathbf{sPre}(<\mathcal{U}>))$ as in *op.cit.* Theorem 5.3 to the connected component of the distinguished point). Indeed, following Levine, from the previous maps, using the functor $\underset{<\mathcal{U}>}{\mathrm{holim}} : \mathrm{Ho}_{BK}(\mathbf{sPre}(<\mathcal{U}>^{op})) \to \mathrm{Ho}(\mathrm{Top})$ we get maps $\lambda^i_{<\mathcal{U}>} : (\underset{<\mathcal{U}>}{\mathrm{holim}}\mathrm{BGL}(\mathcal{R}_\mathcal{U})^+)_0 \to (\underset{<\mathcal{U}>}{\mathrm{holim}}\mathrm{BGL}(\mathcal{R}_\mathcal{U})^+)_0$ and using the fact that the map $\varphi_{<\mathcal{U}>}$ is an isomorphism in $\mathrm{Ho}(\mathrm{Top})$ we get maps $\lambda^i_X : K(X)_0 \to K(X)_0$ in $\mathrm{Ho}(\mathrm{Top})$ by considering the following commutative diagram

$$
\begin{array}{ccc}
K(X)_0 & \xrightarrow[\sim]{\varphi_{<\mathcal{U}>}} & (\underset{<\mathcal{U}>}{\mathrm{holim}}\mathrm{BGL}(\mathcal{R}_\mathcal{U})^+)_0 \\
\exists\lambda^i_X \downarrow & & \downarrow \lambda^i_{<\mathcal{U}>} \\
K(X)_0 & \xrightarrow[\sim]{\varphi_{<\mathcal{U}>}} & (\underset{<\mathcal{U}>}{\mathrm{holim}}\mathrm{BGL}(\mathcal{R}_\mathcal{U})^+)_0
\end{array}
$$

Now, the homomorpisms $\pi_n(\lambda^i_X)$ are the ones which give the lambda structures of Levine. We now consider the maps $\lambda^i : K(-)_0 \to K(-)_0$ in $\mathrm{Ho}_{BK}(\mathbf{sPre}(\mathrm{Sch}_S))$ we obtained via our method: they restrict to maps $\lambda^i : K(-)_0 \to K(-)_0$ in $\mathrm{Ho}_{BK}(\mathbf{sPre}(<\mathcal{U}>))$ so that we get the following commutative diagram in $\mathrm{Ho}(\mathrm{Top})$ since for all $J \in <\mathcal{U}>$, $\cap_{i \in J} U_i =: U_J = \mathrm{Spec}(A_J := \Gamma(U_J, \mathcal{O}_X))$ are affine because X is divisorial (hence it has affine diagonal so we can assume that our affine cover is semi-separating)

$$
\begin{array}{ccccc}
K(X)_0 & \xrightarrow{f_J^*} & K(U_J)_0 & \xrightarrow{\varphi_{A_J}^{-1}} & \mathrm{BGL}(A_J)^+ \\
\lambda^i_X \downarrow & & \lambda^i_{U_J} \downarrow & & \downarrow \exists\lambda^i_{A_J} \\
K(X)_0 & \xrightarrow{f_J^*} & K(U_J)_0 & \xrightarrow{\varphi_{A_J}^{-1}} & \mathrm{BGL}(A_J)^+
\end{array}
$$

It follows, because of the universal property of the homotopy limits (all the simplicial sets in our diagrams here are fibrant) that we get maps $\lambda^r_{<\mathcal{U}>} : (\underset{<\mathcal{U}>}{\mathrm{holim}}\mathrm{BGL}(\mathcal{R}_\mathcal{U})^+)_0 \to (\underset{<\mathcal{U}>}{\mathrm{holim}}\mathrm{BGL}(\mathcal{R}_\mathcal{U})^+)_0$ so that for any $J \in <\mathcal{U}>$ the following diagram commutes in $\mathrm{Ho}(\mathrm{Top})$

$$\begin{array}{ccc}
(\mathrm{holim}\mathrm{BGL}(\mathcal{R}_{\mathcal{U}})^+)_0 & \xrightarrow{\quad\lambda^r_{<\mathcal{U}>}\quad} & (\mathrm{holim}\mathrm{BGL}(\mathcal{R}_{\mathcal{U}})^+)_0
\end{array}$$

We now have, because of how Levine defines them, that the maps $\lambda^r_{U_J}$ and $\lambda^r_{A_J}$ in the previous diagram using the construction introduced in this text and that agree with the ones of Gillet, Soulé, Hiller and Kratzer, agree up to homotopy with the maps inducing the lambda operations of Levine. We then see that up to homotopy the maps λ^i_X defined by Levine have to agree with ours and so they give to all $K_n(X)$, $n \geq 1$, the same $K_0(X)$-λ-algebra structure since all the pairings $K_0(X) \times K_n(X) \to K_n(X)$ considered are always the Waldhausen's ones. This shows the agreement of the lambda operations defined by Levine with ours, in the case they are both defined.

5.2.3 Consequence of the comparison: the γ-filtration for higher K-theory of singular schemes

For every divisorial scheme of finite type over a regular noetherian ring R we have defined a λ-ring $K_*(X)$. If we denote by $\pi_0(X)$ the finite set of connected components of X, we can define an augmentation $\epsilon : K_*(X) \to \mathbb{Z}^{\pi_0(X)}$ by setting $\epsilon(a) = 0$ if $a \in K_n(X)$ for $n \geq 1$ and $\epsilon(a) = \mathrm{rank}(a)$ otherwise. Hence $(K_*(X), \epsilon)$ is an augmented λ-ring. From now one we will suppose that the schemes we consider are of finite type over a base scheme which is a field k. Gillet and Soulé in [GS99] 5.4 prove that the hypothesis of Theorem 5.1.17 are satisfied for $(K_*(X), \epsilon)$ so that we have the following

Theorem 5.2.8. *Let be X a divisorial scheme of finite type over a field k of dimension d. Then for the ring $K_*(X)$ provided with the λ-ring structure described in 5.1.32 and the augmentation described before it holds, for $n = 2d + 1$*

$$K_*(X)_{\mathbb{Q}} = \oplus_{i=0}^n K_*(X)_{\mathbb{Q}}^{(i)} \cong \oplus_{i=0}^n F^i K_*(X)_{\mathbb{Q}}/F^{i+1} K_*(X)_{\mathbb{Q}} \cong \mathrm{Gr}(K_*(X))_{\mathbb{Q}} \cong \mathrm{Gr}(K_*(X)_{\mathbb{Q}})$$

In particular $K_*(X)_{\mathbb{Q}}^{(i)} \cong F^i K_*(X)_{\mathbb{Q}}/F^{i+1}K_*(X)_{\mathbb{Q}}$ for every $i \in \mathbb{N}$. This decomposition is functorial.

Remark 5.2.9. Notice that the only novelty is that the lambda structure we are considering can be actually built without the use of representation rings and, more important, these decompositions are functorial (which one could have proved differently with what was already known in literature, anyway). If we had a good notion of Chern charachter we could be able to repeat as we will do below the formal machinery of [FL85] to prove higher Riemann-Roch theorems for general divisorial schemes in the generality reached for K_0 in *op. cit.*. The main obstruction was found from the author in proving the analogue of [FL85] 3.5: without a powerful splitting principle we can not prove that the Chern charachter is a ring homomorphism. The author will address these points in forthcoming work.

Remark 5.2.10. The argument in [GS99] suggests that the previous theorem should be true for every divisorial scheme of finite type over a noetherian ring R of global dimension d, although this is not clear.

5.3 Computing group natural endomorphisms of higher K-theory

So far we have been involved into the study of endomorphisms of K-theory as a presheaf of *sets*. But K-theory is more naturally a group so it makes sense to study endomorphisms of K-theory as presheaf of *groups*. This has been done by Riou for smooth schemes and he used his result to draw some stable considerations on algebraic K-theory, including a very general version of the Grothendieck-Riemann-Roch theorem. The aim of this section is then to extend some results of Riou from the smooth to the singular world. To this end, we denote as Ω_f^i the right derived functor of Ω^i in the simplicial model categories we will consider (i.e. take a fibrant replacement of the presheaf considered and then apply Ω^i). Denote by K the K-theory presheaf and assume all the Grothendieck sites we consider are formed by divisorial schemes so that we can use equivalently both the Quillen's and the Thomason's K-theory. We let S to be a regular noetherian scheme (Riou assumes S to be also separated, but the hypothesis of having affine diagonal suffices to prove his main theorems as pointed out in Appendix A). Riou proves in [Rio06] Theoreme III.32 and [Rio10] 1.2.10 that

$$\pi_0 : \mathrm{Hom}_{\mathcal{H}(S)}(K, \Omega_f^i K) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0, K_i)$$

is a bijection. He then considers the map $\delta : \mathrm{Pic} \to K_0$ in $\mathbf{Pre}(\mathrm{Sm}/S)$, given for any scheme $X \in \mathrm{Sm}/S$ by the assignment $[L] \mapsto [L]$ for any L line bundle on X, and uses this map to show that the map

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S),\mathrm{Ab}}(K_0, K_i) \xrightarrow{\delta^*} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S),\mathrm{Sets}}(\mathrm{Pic}, K_i)$$

is a bijection ([Rio10] Proposition 5.1.1). We want to prove the singular case. From now on, S will be a regular quasi-projective scheme of finite type over a noetherian affine scheme. We start with the following Proposition

Proposition 5.3.1. *All the arrows in the following diagram are isomorphisms.*

$$
\begin{array}{ccc}
[K, \Omega_f^i K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} & \longrightarrow & [K, \Omega_f^i K]_{\mathcal{H}(S)} \\
\pi_0 \downarrow & & \pi_0 \downarrow \cong \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(K_0, K_i) & \xrightarrow[\mathrm{res}]{} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0, K_i)
\end{array}
$$

Moreover, the arrow induced by restriction

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S),\mathrm{Ab}}(K_0, K_i) \xrightarrow{\mathrm{res}} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S),\mathrm{Ab}}(K_0, K_i)$$

is injective.

Proof. One needs again to show first that the upper horizontal arrow is an isomorphism and then that the arrow res is injective to conclude. The first assertion is proved exactly as in 3.1.6, i.e. one uses 1.8.7. For the second assertion one can use Proposition 3.1.9. The last statement in the proposition follows from the first or follows analogously. $\qquad\square$

     Now we want to replace K_0 with Pic in the previous Proposition. We start with the following lemma

Lemma 5.3.2. *The map*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(\mathrm{Pic}, K_i) \xrightarrow{\mathrm{res}} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{Pic}, K_i)$$

is injective for any i.

Proof. One chases as in Proposition 3.1.9 (or one can employ Proposition 3.1.10) using Proposition 3.1.13. In fact if in the statement of Proposition 3.1.13 we start with a family of *line bundles* $\mathcal{E}_1, ..., \mathcal{E}_n \in \mathrm{Pic}(X)$ we get a smooth scheme $Y_{\mathcal{E}}$ over

S and *line bundles* $\mathcal{E}_{1,Y_{\mathcal{E}}}, ..., \mathcal{E}_{n,Y_{\mathcal{E}}}$ on it such that the same conclusion holds. This allows our usual argument to apply. $\qquad\qquad\square$

Now we remind that in both $\mathcal{H}(S)$ and $\mathcal{H}_{Zar}^{\mathrm{Sch}_S}$ the functor Pic is represented by $B\mathbb{G}_m$ Moreover, in $\mathcal{H}(S)$, Pic is also represented by $\mathbb{P}^\infty$. We can then prove the following

Theorem 5.3.3. *All the arrows in the following diagram are isomorphimsms*

$$
\begin{array}{ccc}
[B\mathbb{G}_m, \Omega_f^i K]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} & \longrightarrow & [B\mathbb{G}_m, \Omega_f^i K]_{\mathcal{H}(S)} \\
\pi_0 \downarrow & & \pi_0 \downarrow \cong \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(\mathrm{Pic}, K_i) & \xrightarrow[\mathrm{res}]{} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{Pic}, K_i)
\end{array}
$$

Moreover, also all the arrows in the following commutative diagram are isomorphisms

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S),\mathrm{Ab}}(K_0, K_i) & \xrightarrow{\delta_{\mathrm{Sch}}^*} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(\mathrm{Pic}, K_i) \\
\beta \downarrow & & \downarrow \cong \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S),\mathrm{Ab}}(K_0, K_i) \xrightarrow[\cong]{\delta_{\mathrm{Sm}}^*} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{Pic}, K_i) & \cong \lim_n K_i(\mathbb{P}^n) & \cong K_i(S)[[U]]
\end{array}
$$

where the maps δ_{Sm}^* and δ_{Sch}^* are induced from the presheaves maps $\delta_{\mathrm{Sm},\mathrm{Sch}} : \mathrm{Pic} \to K_0$ and $U = [\mathcal{O}(1)] - 1$ is the compatible family in $\lim_n K_0(\mathbb{P}^n)$.

Proof. The fact that all the maps in the first commutative diagram are isomorphisms follows in the usual way thanks to the previous lemma. In particular the top horizontal map is an isomorphism because of Proposition 1.8.7, and the injectivity of the lower horizontal map closes the argument since the right vertical π_0 map is an isomorphism because of Proposition 5.1.1 in [Rio10]. For the second diagram on has that the right vertical arrow is an isomorphism because of what we just proved. The isomorphisms $\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{Pic}, K_i) \cong \lim_n K_i(\mathbb{P}^n) \cong K_i(S)[[U]]$ are proved in [Rio10] 5.1.1, who proves also that the bottom horizontal line is an isomorphism. The arrow β is injective because of the previous lemma and so also the arrow δ_{Sch}^* is 1-1 by diagram chase. We are then left to prove that β is surjective. Let us study the map

$$
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S),\mathrm{Ab}}(K_0, K_i) \xrightarrow[\beta]{} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S),\mathrm{Ab}}(K_0, K_i) \xrightarrow[\cong]{\varphi} K_i(S)[[U]]
$$

arising from the diagram. Denote as $\psi^k : K_0 \to K_0$ the kth Adams operation. Riou shows in [Rio10] that denoting as $x \cdot \psi^k \in \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S),\mathrm{Ab}}(K_0, K_i)$ the map given by $y \mapsto x \cdot \psi^k(y)$ for $x \in K_i(S)$, this is mapped via φ to $x(1+U)^k$ in $K_i(S)[[U]]$ and these elements generates the image of φ by [Rio06] IV.15 or [Rio10] page 10. So if we show that all the $x \cdot \psi^k$ are in the image β, we can conclude. But this is true because the Adams operations on K_0 over smooth schemes comes, because of our theorems, as restriction of the operations we have built on K_0 for singular schemes. Hence the theorem is fully proved. $\qquad\square$

Corollary 5.3.4. *Under the assumptions of the previous theorem, we have*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S),\mathrm{Ab}}(K_0, K_i) \cong K_i(S)[[U]]$$

for any i.

5.4 Some revisited Riemann-Roch algebra

In this Section we follow the arguments contained in [FL85] generalizing a little the results contained in *op.cit.* in order to develop a formal machinery that we utilize to prove our version of the Adams-Riemann-Roch theorem. The expert will recognize that there is substantially nothing new here besides of the fact that we have removed some hypothesis and introduced new terminology more convenient to our aims. On a first reading, the reader who is familiar with the work of [FL85] or with their methods can safely skip this section and can prove the Adams-Riemann-Roch theorem simply by going through the proof of the Adams-Riemann-Roch theorem for K_0 as in [FL85]. Indeed we have put a functorial lambda ring structure (strong enough to prove the Adams-Riemann-Roch theorem) on $K_*(X)$ for any divisorial scheme X of finite type over a regular ring R and the other ingredients that are used in [FL85] to prove this theorem are the resolution property, the projective bundle theorem and the projection formula (that are all true in the context of higher K-theory of divisorial schemes). This is for example the approach taken in [K̈98]. However, since [FL85] contains some inaccuracies, we think there is no harm to revisit some of its machinery here.

5.4.1 Riemann-Roch formalism and abstract Adams-Riemann-Roch

Denote as $\mathcal{A}$ the category having as objects unital possibly noncommutative graded rings of the form $A = \bigoplus_{i \in \mathbb{N}} A_i$, A_i abelian groups, such that for any such object A_0 is a commutative unital ring which makes A into an A_0-graded algebra and such that

the multiplication in A satisfies $ab = (-1)^{ij}ba$ if $a \in A_i$ and $b \in A_j$. Morphisms in $\mathcal{A}$ are graded ring homomorphisms, i.e. ring maps $f : A = \bigoplus_{i \in \mathbb{N}} A_i \to B = \bigoplus_{i \in \mathbb{N}} B_i$ such that $\text{Im}(f|A_i) \subseteq B_i$ for every $i \in \mathbb{N}$. Note we always have $1 \in A_0$. Assume we have a contravariant functor $K : \mathcal{C} \to \mathcal{A}$ with $\mathcal{C}$ any category. Then we denote, given an arrow $f : X \to Y$ in $\mathcal{C}$, as $f^* : K(Y) \to K(X)$ the map $F(f)$. Let be $\rho : K \to K$ a natural transformation.

Definition 5.4.1. A Riemann-Roch datum is a triple (K, ρ, g) with K and ρ as above and $g : X \to Y \in \mathcal{C}$ such that there exists a abelian graded group homomorphism $g_* : K(X) \to K(Y)$ which satisfies the projection formula, i.e.

$$g_*(x \cdot g^*(y)) = g_*(x) \cdot y \qquad \forall x \in K(X),\ y \in K(Y)$$

Remark 5.4.2. The projection formula implies $g_*(g^*(y)) = g_*(1) \cdot y$, $\forall y \in K(Y)$.

Definition 5.4.3. We say that Riemann-Roch holds with respect to the RR datum (K, ρ, g) if for some $\tau_g \in K_0(X)$ the following diagram commutes

$$
\begin{array}{ccc}
K(X) & \xrightarrow{\tau_g \cdot \rho} & K(X) \\
{\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\
K(Y) & \xrightarrow{\rho} & K(Y)
\end{array}
$$

i.e. $\rho(g_*(x)) = g_*(\tau_g \cdot \rho(x))\ \forall x \in K(X)$.

With this formalism, we can repeat almost verbatim [FL85] II Theorems 1.1-1.2.

Theorem 5.4.4. *Assume that $f : X \to Y$ and $g : Y \to Z$ are two arrows in $\mathcal{C}$ such that (K, ρ, f) and (K, ρ, g) are RR data satisfying the property that RR holds with multipliers τ_f and τ_g, then $(K, \rho, g \circ f)$ is a RR datum with $(g \circ f)_* = g_* f_*$ and RR holds for it with multiplier $\tau_{g \circ f} = f^*(\tau_g) \cdot \tau_f$.*

Proof. The assertion that $(K, \rho, g \circ f)$ is a RR datum is simple to verify. Now the theorem follows because of the following

$$
\begin{aligned}
\rho(g_*(f_*(x))) &= g_*(\tau_g \cdot \rho(f_*(x))) && \text{RR holds for } g \\
&= g_*(\tau_g \cdot f_*(\tau_f \cdot \rho(x))) && \text{RR holds for } f \\
&= g_*(f_*(f^*(\tau_g) \cdot \tau_f \cdot \rho(x))) && \text{proj. formula}
\end{aligned}
$$

$\square$

Theorem 5.4.5. *If $(K, \rho, g : X \to Y)$ is a RR datum, $g^* : K(Y) \to K(X)$ is surjective and there exists $\tau \in K_0(Y)$ such that $\rho(g_*(1)) = g_*(1)\tau$, then RR holds with respect to the triple (K, ρ, g) with multiplier $\tau_g = g^*(\tau)$.*

Proof. Given any $x \in K(X)$, since g^* is surjective, choose $y \in K(Y)$ such that $x = g^*(y)$. Now

$$
\begin{aligned}
\rho(g_*(x)) &= \rho g_* g^*(y) \\
&= \rho(g_*(1) \cdot y) & \text{proj. formula} \\
&= \rho(g_*(1)) \cdot \rho(y) \\
&= g_*(1) \cdot \tau \cdot \rho(y) & \text{assumption} \\
&= g_*(g^*(\tau \cdot \rho(y)) & \text{proj. formula} \\
&= g_*(g^*(\tau) \cdot g^*(\rho(y))) \\
&= g_*(g^*(\tau) \cdot \rho(g^*(y)) & \rho \text{ is a natural transf.} \\
&= g_*(\tau_g \cdot \rho(x)) & x = g^*(y)
\end{aligned}
$$

$\square$

Definition 5.4.6. Given a RR datum $(K, \rho, f : X \to Y)$, we say that f admits a basic deformation to a morphism $f' : X \to Y' \in \mathcal{C}$ if there are morphisms as in the following diagram

$$
\begin{array}{ccc}
 & \overset{f'}{\longrightarrow} Y' \overset{g'}{\longrightarrow} & \\
X & & \overset{g}{\rightrightarrows} M \\
 & \overset{f}{\longrightarrow} Y \overset{\pi}{\longleftarrow} &
\end{array}
$$

and a finite number of morphisms $h_v : C_v \to M \in \mathcal{C}$ with integers $m_v \in \mathbb{Z}$ such that

F) There are maps $g_*, \pi_*, g'_*, f'_*, h_{v*}$ such that $(K, \rho, \{g \vee \pi \vee g' \vee f' \vee h_v\})$ are RR data and they are functorial (i.e. K is a covariant functor in groups on the above diagram)

BD1) For every $x \in K(X)$ there exists $z \in K(M)$ such that

$$
f_*(x) = g^*(z) \text{ and } f'_*(x) = g'^*(z)
$$

BD2)

$$g_*(1) = g'_*(1) + \sum m_v h_{v*}(1) \qquad (5.2)$$

Note that this equality takes place in degree 0 by our assumptions.

BD3) For every $z \in K(M)$ as in BD1, we have $h_v^*(z) = 0$ for all v.

BD4) g is a section of π and $\pi \circ g' \circ f' = f$.

What follows is almost verbatim a rewriting of [FL85] II Theorem 3.1

Theorem 5.4.7. *Assume that $(K, \rho, f : X \to Y)$ is a RR datum such that Riemann-Roch holds for a RR datum (K, ρ, f') with $f' : X \to Y'$ a basic deformation of f. Then RR holds with respect to (K, ρ, f) with multiplier $\tau_f = \tau_{f'}$.*

Proof. Let $x \in K(X)$. Consider $z \in K(M)$ given by BD1. So

$$
\begin{aligned}
g_*(\rho(f_*(x))) &= g_*(\rho(g^*(z))) & \text{BD1} \\
&= g_*(g^*(\rho(z))) & \text{naturality} \\
&= g_*(1)\rho(z) & \text{proj. formula} \\
&= g'_*(1)\rho(z) + \sum m_v h_{v*}(1)\rho(z) & \text{BD2} \\
&= g'_*(g'^*(\rho(z))) + \sum m_v h_{v*}(h_v^*(\rho(z))) & \text{proj. formula} \\
&= g'_*(\rho(g'^*(z))) + \sum m_v h_{v*}(\rho(h_v^*(z))) & \text{naturality} \\
&= g'_*(\rho(g'^*(z))) & \text{BD3} \\
&= g'_*(\rho(f'_*(x))) & \text{BD1}
\end{aligned}
$$

Now, since g is a section of π,

$$
\begin{aligned}
\rho(f_*(x)) &= \pi_*(g_*(\rho(f_*(x)))) & \\
&= \pi_*(g'_*(\rho(f'_*(x)))) & \text{passages above} \\
&= \pi_*(g'_*(f'_*(\tau_f \cdot \rho(x)))) & \text{RR for } f' \\
&= f_*(\tau_f \cdot \rho(x)) & \text{BD4}
\end{aligned}
$$

$\square$

Definition 5.4.8. A λ^*-functor is the datum $(K, (\lambda^k)_{k \in \mathbb{N}}, (\psi^j)_{j \geq 1})$ of a functor $K : \mathcal{C}^{op} \to \mathcal{A}$, natural transformations ψ^j of the form $\psi^j = \oplus_{i \in \mathbb{N}} \psi_i^j : \oplus_{i \in \mathbb{N}} K_i(-) \to \oplus_{i \in \mathbb{N}} K_i(-)$ and $\lambda^k : K \to K$ natural transformations of presheaves of sets that sends elements of degree 0 in elements of degree 0 with the additional property that their restriction $\lambda_0^k : K_0(-) \to K_0(-)$ define an augmented special lambda ring structure on $K_0(-)$ with an involution and a positive structure (as in [FL85] pages 3,4) in the category of functors $\mathcal{C}^{op} \to$ Sets. Notice this simply means that our λ_0^k makes K_0 into a functor taking values in the category of special lambda rings (with an involution in this case) satisfying all the hypothesis of [FL85]. We moreover assume that ψ_0^j coincide with the Adams operations induced from the lambda operations λ_0^k.

Notice that given $X \in \mathcal{C}$, we have $\psi^j(ab) = \psi_0^j(a)\psi_i^j(b)$ if $a \in K_0(X)$ and $b \in K_i(X)$. Moreover we say that $u \in K_0(X)$ is a line element, positive element, etc. if it is such for the λ-ring structure on $K_0(X)$ (note that if we consider the augmentation defined in 5.2.3 we have a different notion of positive elements and they are more than this ones but they play no role in our discussion about K-theory).

Definition 5.4.9. A morphism $f : X \to Y$ in $\mathcal{C}$ is called an **elementary imbedding** with respect to the λ^*-functor (K, λ^k, ψ^j) if $f^* : K(Y) \to K(X)$ is surjective, $(K, \psi^j, f)_{j \in \mathbb{N}_{>0}}$ are RR data and there is a positive element $e \in K_0(Y)$ such that $f_*(1) = \lambda_{-1}(e) \in K_0(Y)$ (λ_{-1} being defined in [FL85] as usual).

Before proving the formal Adams-Riemann-Roch theorem for regular imbeddings in this context, we need

Proposition 5.4.10. *Given a λ^*-functor (K, λ^k, ψ^j) and a positive element $e \in K_0(Y)$ for some $Y \in \mathcal{C}$, we have that*

$$\psi^j(\lambda_{-1}(e)) = \psi_0^j(\lambda_{-1}(e)) = \lambda_{-1}(e)\theta^j(e) \qquad \forall j \in \mathbb{N}$$

Where $\theta^j(e)$ are the bott cannibalistic classes as in [FL85] page 24.

Proof. The second inequality is the content of Proposition 6.2 in Chapter I of [FL85] (notice the typos there), while the first equality is simply the definition since $\lambda_{-1}(e) \in K_0(Y)$. $\square$

We can state and prove

Theorem 5.4.11. *Given a λ^*-functor (K, λ^k, ψ^j), if $f : X \to Y$ is an elementary imbedding with respect to it then Riemann-Roch holds with respect to the RR data $(K, \psi^j, f)_{j \in \mathbb{N}}$ with multiplier $f^*(\theta^j(e))$.*

Proof. Since f is an elementary imbedding, there exists $e \in K_0(Y)$ such that $f_*(1) = \lambda_{-1}(e)$. Then we can apply Theorem 5.4.5 combining Proposition 5.4.10 with the equality just stated. $\qquad \square$

We can now turn our attention to the analogue of the elementary projections. Given a λ^*-functor (K, λ^k, ψ^j), consider the ring $A = K(X)$ for some $X \in \mathcal{C}$. Then $K_0(X) = A_0$ is a special lambda ring with involution and given a positive element $e \in K_0(X)$ (of finite dimension d such that $\lambda^d(e)$ is invertible) we can define as in [FL85] Chapter 1 Section 2 a ring extension

$$K_0(X)_e = K_0(X)[T]/(p_e(T)) = K_0(X)[l]$$

where

$$p_e(T) = \sum_{i=0}^{r+1} (-1)^i \lambda^i(e) T^{r+1-i}, \qquad r+1 \text{ is the dimension of } e$$

and l is the image of T mod $p_e(T)$ (called canonical generator). It is possible to show that there is a canonical A_0-linear map $\eta_{0,e}(X) : K_0(X)_e \to K_0(X) = A_0$ such that $K_0(X)_e$ is still a λ-ring and other interesting properties are satisfied (see *op. cit.* I Proposition 2.2 for the details and the properties). Fulton and Lang call this the functional associated with the extension $K_0(X)_e$ of $K_0(X)$ (notice that l is invertible because $K_0(X)$ is an augmented λ-ring with a positive structure). We set

$$K_e(X) := K(X) \otimes_{K_0(X)} K_0(X)_e = K(X)[T]/(p_e(T))$$

$$\eta_e(X) = (\eta_{0,e})_{K(X)} : K_e(X) := K(X) \otimes_{K_0(X)} K_0(X)_e \to K(X)$$

where T is seen as an element of degree 0 and we will call $\eta_e(X)$, or simply η_e when no confusion is possible, **the functional associated with the positive element** e. Notice that this map is $K(X)$-linear and that $K_e(X)$ is a lambda ring in degree zero.

Definition 5.4.12. Given a λ^*-functor (K, λ^k, ψ^j), we say that an arrow $f : X \to Y$ in $\mathcal{C}$ is an **elementary projection** with respect to it if $(K, \psi^j, f)_{j>0}$ are RR data and the map $f_* : K(X) \to K(Y)$ is isomorphic (via an isomorphism in $\mathcal{A}$ of $K(Y)$-modules wich is a λ-homomorphism in degree zero) to the functional

$$\eta_e(Y) : K_e(Y) \to K(Y)$$

associated with some positive element $e \in K_0(Y)$. In this case we will call $\eta_e(Y)$ simply $f_e(Y)$. If from our isomorphism from f_* to $\eta_e(Y)$ we remove the hypothesis of being a λ-homomorphism on degree zero we will call f a *weak elementary projection.*

Remark 5.4.13. Notice that in the situation of the previous definition, if f is an elementary projection we have that $K(X) \cong K_e(Y)$

The following is the formal Adams-Riemann-Roch theorem for elementary projections.

Theorem 5.4.14. *Let be* (K, λ^k, ψ^j) *a λ^*-functor and $f : X \to Y$ an elementary projection with respect to it (so we have by definition a distinguished positive element e associated to it). If $j \in \mathbb{N}$ is invertible in $K(Y)$, then $\theta^j(el^{-1})$ is invertible in $K(X) \cong K_e(Y)$ and RR holds with respect to the datum (K, ψ^j, f) with multiplier $\tau_f = j\theta^j(el^{-1})^{-1}$.*

Proof. The invertibility of $\theta^j(el^{-1})$ follows from [FL85] II Theorem 3.2. Because of our assumptions it suffices now to show that the following diagram commutes, where the top horizontal map is induced by the given isomorphism between f_* and f_e

$$
\begin{array}{ccc}
K(X) \cong K(Y) \otimes_{K_0(Y)} K_0(Y)_e & \xrightarrow{j\theta^j(el^{-1})^{-1}\psi^j} & K(Y) \otimes_{K_0(Y)} K_0(Y)_e \\
\downarrow{f_e(X)} & & \downarrow{f_e(X)} \\
K(Y) & \xrightarrow{\psi^j} & K(Y)
\end{array}
$$

which follows (ψ^j being an arrow in $\mathcal{A}$) because at degree 0 level is true by [FL85] II Theorem 3.2. $\qquad\square$

Remark 5.4.15. For weak elementary projections, we have chasing as in the proof of the previous theorem that RR holds if and only if it holds in degree zero.

5.4.2 Remarks on some constructions in [FL85]

In this subsection we review some construction made in [FL85] showing that they fit in the framework we are considering in this work. This could certainly be considered folklore. From now on we assume that all our schemes are noetherian. Let $f : X \to Y$ be a closed imbedding with $\mathcal{I}$ ideal sheaf defining X as subscheme of Y. We denote as $C_{X/Y}$ the conormal sheaf on X (which is coherent) given by $C_{X/Y} = \mathcal{I}/\mathcal{I}^2$ (recall the abuse of notation, see [Sta18, Section 01R1]). We recall that a closed embedding as above is called *regular* (of codimension d) if every point $p \in X \subseteq Y$ has an affine neighbourhood $\mathrm{Spec}(A)$ in Y such that $\mathcal{I}(A)$ is generated by a regular sequence (of length d).

Proposition 5.4.16 ([FL85] IV 3.2-3.3)**.** *If $f : X \to Y$ is a regular embedding, then $C_{X/Y}$ is a locally free sheaf on X. Moreover if f realizes X as the zero scheme of a regular section of a locally free sheaf $\mathcal{E}$ on Y then $C_{X/Y} \cong f^*\mathcal{E}^\vee$ ($\vee$ denoting the dual). In addition if $\mathcal{E}$ is a vector bundle on X, then the zero section $X \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$ is a regular embedding with conormal sheaf $\mathcal{E}$.*
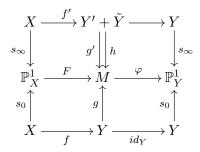
We denote as $\mathrm{Bl}_X Y$ the blow-up of Y along a closed subscheme X.

Proposition 5.4.17. *[FL85] IV 4.3] Assume X and Y noetherian schemes. Then*

a) *If $f : X \to Y$ is a regular embedding then $E \cong \mathbb{P}(C_{X/Y})$, where E is the exceptional divisor $\varphi^{-1}(X)$, $\varphi : \mathrm{Bl}_X Y \to Y$ standard projection.*

b) *If f realizes X as the zero scheme of a section of a vector bundle $\mathcal{E}$ on Y, then there is a canonical embedding of $\mathrm{Bl}_X Y$ into $\mathbb{P}(\mathcal{E}^\vee)$ over Y which is a regular embedding if f is a regular embedding.*

c) *If Y has an ample family of line bundles and $f : X \to Y$ is a regular embedding, then $\mathrm{Bl}_X Y \to Y$ is a projective local complete intersection morphism and $\mathrm{Bl}_X Y$ has an ample family of line bundles.*

Proof. The only novelty from the proof of [FL85] IV 4.3 is that we make explicit that in c), the embedding $\mathrm{Bl}_X Y \to \mathbb{P}(\mathcal{E}^\vee)$ built in that proof gives us that $\mathrm{Bl}_X Y$ has an ample family of line bundles because of 2.1.9 and 2.1.10. $\square$

We now assume that $f : X \to Y$ is a regular embedding of codimension d with conormal sheaf $C_{X/Y} =: C$ and we denote as $f' : X \to Y' = \mathbb{P}(C_{X/Y} \oplus \mathcal{O}_X)$ the zero section embedding. We want to recall the construction of the deformation of the normal bundle as in [FL85] IV 5, for example. We notice we have two canonical sections $s_0, s_\infty : Y \to \mathbb{P}^1_Y = \mathrm{Proj}(O_Y[T_0, T_1])$. We denote $X(\infty) = s_\infty(f(X))$ and $M = \mathrm{Bl}_{X(\infty)} \mathbb{P}^1_Y$ is the so called deformation of the normal cone. Let $\varphi : M \to \mathbb{P}^1_Y$ the canonical morphism, π the composite $p \circ \varphi : M \to Y$ (with $p : \mathbb{P}^1_Y \to Y$ canonical projection), $g : Y \to M$ a section of π determined by s_0 because $s_0(Y)$ is disjoint from $s_\infty(Y)$ which makes Y a Cartier divisor on M and $h : \tilde{Y} = \mathrm{Bl}_X Y \to M$ the regular embedding of codimension 1 exhibiting $\tilde{Y}$ as a Cartier divisor of M (see *op.cit.* for more details). One also get a closed regular embedding $F : \mathbb{P}^1_X \to M$ and one is able to see that $\mathbb{P}(C \oplus \mathcal{O}_X) = Y'$ intersects regularly $\tilde{Y}$ as a Cartier divisor in $\mathbb{P}(C)$. Moreover $\mathcal{O}(Y) \cong \mathcal{O}(\tilde{Y} + Y')$ where $\tilde{Y} + Y' = \varphi^*(Y(\infty))$. This is summarized by the following deformation diagram ([FL85] page 99)

$$\begin{array}{ccccc}
X & \xrightarrow{\ f'\ } & Y' + \tilde{Y} & \longrightarrow & Y \\
{\scriptstyle s_\infty}\downarrow & & {\scriptstyle g'}\Vert\ {\scriptstyle h}\Downarrow & & \downarrow{\scriptstyle s_\infty} \\
\mathbb{P}^1_X & \xrightarrow{\ F\ } & M & \xrightarrow{\ \varphi\ } & \mathbb{P}^1_Y \\
{\scriptstyle s_0}\uparrow & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle s_0} \\
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ id_Y\ } & Y
\end{array}$$

where every square is a pullback and every vertical arrow is an embedding of a Cartier divisor. $+$ here denote the scheme theoretic union ([Sta18, Tag 0C4H] and [Sta18, Tag 01WQ]). This diagram can be refined to a diagram

$$\begin{array}{ccc}
 & \xrightarrow{\ f'\ } Y' \xrightarrow{\ g'\ } & \\
X & \underset{\ f\ }{\searrow}\ \ \underset{g}{\nearrow}\ \underset{\pi}{\gtrless} & M \\
 & Y &
\end{array}$$

with $\pi \circ g = id_Y$ and $\pi \circ g' \circ f' = f$

Remark 5.4.18. Notice that if Y has an ample family of line bundles, then all the schemes in the previous diagram have an ample family of line bundles beacuse of 2.1.10 and 5.4.17.

We now recall two useful facts from [FL85]. For the two following proposisitions if $f : X \to Y$ is a map we denote (if it is defined) as f_*, f^* the pullback and the pushforward map induced at level of K_0.

Proposition 5.4.19. *[[FL85] V Proposition 4.4] Let A, B, C effective Cartier divisors on a scheme M and assume that $\mathcal{O}(A) \cong \mathcal{O}(B + C)$ and that B and C meet regularly in M. Then letting $D = B \cap C$ and a, b, c, d the embeddings of A, B, C, D in M we have that*

$$a_*(1) = b_*(1) + c_*(1) - d_*(1) \text{ in } K_0(M) \tag{5.3}$$

Proposition 5.4.20. *[[FL85] V Proposition 4.5] Assume that $F : P \to M$ is a regular embedding and that $\varphi : Y \to M$ is a morphism. Then we can draw the pullback diagram*

$$\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \varphi} \\
P & \xrightarrow{\ F\ } & M
\end{array}$$

If φ is a regular embedding and P, Y meet regularly in M then

$$\varphi^* F_* = f_* \psi^*$$

and if Z is a subscheme of Y disjoint from $f(X)$ and $h : Z \to M$ is the morphism induced by φ, then

$$h^* F_* = 0$$

The previous result can be generalized using the work of Thomason.

Proposition 5.4.21. *Proposition 5.4.20 holds if we consider the pullback and the pushforward maps in the higher K-groups K_i for any i.*

Proof. This follows (and the 5.4.20 too) from [TT90] 3.18. $\qquad\qquad\square$

5.4.3 On the connected hypothesis

In section 5.5 we will assume that our schemes were connected, so that we could use the results of [FL85] in our theorems and the definition of special lambda ring given there. As noted in [FL85] Appendix to Chapter V, this hypothesis can be removed as far as our schemes are noetherian. In fact the connected components of a noetherian scheme are open (see the beginning of Section 2.1.1) so that we can write any such scheme as disjoint union of its connected components and reason componentwise (K-theory is in fact additive). The main difference will be in the definition of the augmentation of the lambda ring $K_0(X)$ for X noetherian scheme. In fact if X is connected we have the rank map $\varepsilon : K_0(X) \to \mathbb{Z}$ taking an element of the Grothendieck group to its rank. If X is not connected, reasoning componentwise, one has to consider an augmentation of the form $\varepsilon : K_0(X) \to \mathbb{Z}^{\pi_0(X)}$ in the definition of special lambda ring, and the rank function is adequate even in this case (it is a locally constant function in fact) as discussed in Section 5.1.1. Therefore by simply modifying the definition of the augmentation in the definition of special lamba ring we can consider noetherian schemes which are non-connected and not only connected. This is in fact the point of view of Riou in [Rio10] and [Rio06].

5.5 Higher Adams-Riemann Roch

In this section we prove the Adams-Riemann-Roch theorem for divisorial schemes of finite type over a regular noetherian affine scheme R and a particular class of morphisms. We will use the machinery available from [FL85] and recalled in Section 5.4. We let $\mathcal{C} = \mathrm{Sch}_S$ be the category of schemes of finite type over a regular

noetherian base scheme $S = \mathrm{Spec}(R)$ having an ample family of line bundles (over this category 3.1.3 and 5.1.28 apply). For simplicity, we also assume that our schemes are connected but all the facts contained in this section are valid even if X is not connected, with the same proof provided one gives the correct definition of augmented λ-ring, as discussed before in Section 5.1.1. See also [FL85] Chapter V Appendix concerning this point. Our construction of lambda and Adams operations gives trivially the following

Corollary 5.5.1. *The contravariant functor $K_* : \mathrm{Sch}_S \to \mathcal{A}$ where for any scheme $X \in \mathrm{Sch}_S$ $K_*(X)$ equals $(K_*, \cdot)$ or $(K_*, \cup)$ (see the following Remark) and $\mathcal{A}$ is the category described in Appendix 5.4 describes a λ^*-functor (K_*, λ^k, ψ^j)*

Proof. This follows from Theorem 5.1.32, the functorialities requested coming from the construction of the operations. $\qquad\square$

Remark 5.5.2. One can consider both the ring structures on $K_*(X)$, i.e. the anticommutative one induced by Theorem 5.1.28 and the one where the product of two homogeneous elements of positive degree is set to be zero. It is simple to verify that this defines a λ^*-functor $(K_*, \cup, \lambda^k, \psi^j)$ as in the previous Corollary. From now on we will not distinguish between $(K_*, \cdot)$ and $(K_*, \cup)$ since they both define λ^*-functors and the projection formula and all the constructions we will use can be performed in both cases. Moreover, composing K_* with the functor $\mathcal{A} \to \mathrm{Rng}$ taking every element of $\mathcal{A}$ to its zero degree part and every morphism to its 0 degree part, we see that K_* becomes the usual K_0 functor taking values in the category of λ-rings.

We now use the terminology introduced in the beforementioned sections to see that with our machinery available, the formal arguments used in [FL85] to prove the Adams-Riemann-Roch theorems apply even in this setting. We follow then their argument. We start with the analogue of [FL85] V 6.1.

Remark 5.5.3. Because of [TT90] 3.16, if a map f is of the form [TT90] 3.16.4-3.16.7, we have that pushforwards are well defined and we have the projection formula and all the functorialities requested ([TT90] 3.16 and Proposition 3.17 for the projection formula) so that (K_*, ψ^j, f) is in fact a RR datum.

Lemma 5.5.4. *Let $f : X \to Y$ be a regular embedding in Sch_S (which is part of a RR datum (K_*, ψ^j, f)), $Y' = \mathbb{P}(C_{X/Y} \oplus \mathcal{O}_X)$ and $f' : X \to Y'$ the zero section embedding. Then using the deformation of the normal bundle revised in Section 5.4.2 we have that f' is a basic deformation with respect to f.*

Proof. First of all notice that all the schemes involved in the deformation are in Sch_S as remarked in 5.4.18. We have to verify that the axioms of Definition 5.4.6 are satisfied. Because of Remark 5.5.3, since all the maps considered are of the form [TT90] 3.16.4-3.16.7, we have that pushforwards are well defined and we have all the conditions requested by axiom F). BD1-BD4 now follow in the same way of [FL85] V 6.1, we recall their argument here briefly. BD4 is valid by the construction of the deformation, see Section 5.4.2, and BD2 follows from Proposition 5.4.19 and (5.3) in that Proposition. Now for BD1 and BD3 for a given homogeneous $x \in K_*(X)$ we define $\tilde{x} = \mathrm{pr}^*(x) \in K_*(\mathbb{P}^1_X)$ with $\mathrm{pr} : \mathbb{P}^1_X \to X$ the standard projection and now we consider $z = F_*(\tilde{x})$ where $F : \mathbb{P}^1_X \to M$ is defined in Section 5.4.2. With this definitions, BD1 follows from the construction of the deformation together with Proposition 5.4.21 and BD3 follows from Proposition 5.4.21. $\qquad\square$

We can now mimick [FL85] V 6.2.

Lemma 5.5.5. *Let $\mathcal{E}$ be a vector bundle over a scheme $X \in \mathrm{Sch}_S$ and consider the zero section embedding $f : X \to \mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$. Then f is an elementary imbedding with respect to the λ^*-functor (K_*, λ^k, ψ^j)(see Definition 5.4.9).*

Proof. If $\mathcal{H}$ is the universal hyperplane sheaf of $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)$ (see [FL85] IV Section 1) and $q = [\mathcal{H}] \in K_0(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X))$, then $f_*(1) = \lambda_{-1}(q)$ because of [FL85] V 6.2. The fact that (K_*, ψ^j, f) are RR data for every j follows from Remark 5.5.3 and the surjectivity of f^* comes from the fact that f is a section. $\qquad\square$

We can now state our version of the Adams-Riemann-Roch theorem for regular embeddings.

Theorem 5.5.6. *Assume $f : X \to Y$ is a regular embedding in Sch_S. Then RR holds with respect to the datum (K_*, ψ^j, f) for every $j \in \mathbb{N}$ with multiplier $\theta^j(C_{X/Y})$ where θ^j denotes the Bott cannibalistic class and $C_{X/Y}$ is the conormal bundle (see Section 5.4.2). This means that for any j the following diagram commutes*

$$
\begin{array}{ccc}
K_*(X) & \xrightarrow{\theta^j(C_{X/Y}) \cdot \psi^j} & K_*(X) \\
\left\downarrow{\scriptstyle f_*}\right. & & \left\downarrow{\scriptstyle f_*}\right. \\
K_*(Y) & \xrightarrow[\psi^j]{} & K_*(Y)
\end{array}
$$

Proof. The proof goes in the same way as the one of [FL85] V 6.3. In fact f admits a basic deformation to an arrow f' provided by 5.5.4. This means (because of Theorem 5.4.6) we only have to prove that RR holds with respect of the datum (K_*, ψ^j, f')

where f' is the zero section embedding provided by 5.5.4. But then Lemma 5.5.5 shows that f' is an elementary embedding in the sense of Definition 5.4.9 so the hypotheses of Theorem 5.4.11 are satisfied and we can conclude. $\qquad\square$

Corollary 5.5.7. *Assume* $f : X \to Y$ *is a regular embedding in* Sch_S. *Then for every* $i, j \in \mathbb{N}$ *the following commutes*

$$
\begin{array}{ccc}
K_i(X) & \xrightarrow{\theta^j(C_{X/Y}) \cdot \psi^j} & K_i(X) \\
f_* \downarrow & & \downarrow f_* \\
K_i(Y) & \xrightarrow{\quad \psi^j \quad} & K_i(Y)
\end{array}
$$

Proof. This follows from the previous theorem by considering only the graded part i of $K_*(X)$ and $K_*(Y)$. $\qquad\square$

Remark 5.5.8. For $i = 0$ the previous corollary takes the form of the usual Adams-Riemann-Roch theorem as stated in [FL85] for regular embeddings. Notice however that we do not assume our schemes to be quasi-projective as in *op.cit.*. In fact it seems that their assumption was inserted only to make sure all the schemes they are considering satisfy what they call property $(*)$, i.e. that all the schemes they consider have the resolution property. Schemes with an ample family of line bundles have such a property (and quasi-projective schemes too, of course) so that they would have certainly been able to state their result for the class of schemes having an ample family of line bundles.

We now turn our attention to the elementary projections. Let us consider $X \in \mathrm{Sch}_S$ and $\mathcal{E}$ vector bundle over it. We have a canonical projection $\pi : \mathbb{P}(\mathcal{E}) \to X$ (recall $\mathbb{P}(\mathcal{E}) \in \mathrm{Sch}_S$ by 2.1.9) such that, using the terminology of [FL85], the pushforward π_* (on K_0) is isomorphic to the functional $\pi_{0,e} : K_0(\mathbb{P}(\mathcal{E})) \cong K_0(X)_e \to K_0(X)$ as proved in [FL85] V Corollary 2.4 (e being the positive element $[\mathcal{E}]$). We refer to that book for a more detailed description of that isomorphism. Moreover, we notice that (K_*, ψ^j, π) is a Riemann-Roch datum. Indeed we can define a pushforward for π following [Wei13] page 406 (where it is done for quasi-projective schemes) or the same argument in [Kö98] page 427. Now we can use the Projective Bundle Theorem (see [Wei13] V 1.5 for the form that we are using, just notice as remarked there that there is no need to assume X to be quasi-projective because of [TT90] 4.1), which gives us the isomorphism

$$
K_*(X) \otimes_{K_0(X)} K_0(\mathbb{P}(\mathcal{E})) \to K_*(\mathbb{P}(\mathcal{E}))
$$

so that if we define the functional

$$\pi_e(X) : K_e(X) := K_*(X) \otimes_{K_0(X)} K_0(X)_e \to K_*(X)$$

as in section 5.4.1 we have that it is isomorphic to $\pi_* : K_*(\mathbb{P}(\mathcal{E})) \to K_*(X)$. We have then shown the following

Proposition 5.5.9. *Let us consider $X \in \text{Sch}_S$ and $\mathcal{E}$ vector bundle over it. Then the canonical projection $\pi : \mathbb{P}(\mathcal{E}) \to X$ is an elementary projection with respect to the λ^*-functor (K_*, λ^k, ψ^j).*

The following in then a corollary of Theorem 5.4.14 and could be seen as the higher Adams-Riemann-Roch theorem for projections from a projective bundle.

Theorem 5.5.10. *Let us consider $X \in \text{Sch}_S$ and $\mathcal{E}$ vector bundle over it. Let be $\pi : \mathbb{P}(\mathcal{E}) \to X$ the canonical projection. Then RR holds with respect to the RR datum $(\mathbb{Z}[1/j] \otimes K_*, \psi^j, \pi)$ with multiplier coming from Theorem 5.4.14.*

We now remind that a morphism between schemes $f : X \to Y$ is called a **projective local complete intersection (l.c.i.) morphism** if it factors as $f = \pi \circ i : X \to \mathbb{P}(\mathcal{E}) \to Y$ for some vector bundle $\mathcal{E}$ over Y where i is a regular embedding and $\pi : \mathbb{P}(\mathcal{E}) \to Y$ is the canonical projection. We can now state the final form of our higher Adams-Riemann-Roch theorem.

Theorem 5.5.11. *Let be $f : X \to Y$ a projective l.c.i. morphism in Sch_S. Then $(K_*, \cdot, \psi^j, f)$ $((K_*, \cup, \psi^j, f))$ is a RR datum and RR holds with respect to the datum $(\mathbb{Z}[1/j] \otimes K_*, \psi^j, f)$ for every j with multiplier $\tau_f \in K_0(X)$ given by Theorem 5.4.4. This means that the following diagram commutes*

$$
\begin{array}{ccc}
\mathbb{Z}[1/j] \otimes K_*(X) & \xrightarrow{\tau_f \cdot \psi^j} & \mathbb{Z}[1/j] \otimes K_*(X) \\
f_* \downarrow & & \downarrow f_* \\
\mathbb{Z}[1/j] \otimes K_*(Y) & \xrightarrow{\psi^j} & \mathbb{Z}[1/j] \otimes K_*(Y)
\end{array}
$$

Proof. We factor f as $\pi \circ i$ as in the definition of projective l.c.i. morphisms. Now since (K_*, ψ^j, i) and (K_*, ψ^j, π) are RR data and the pushforward is functorial as presheaf of groups, so also (K_*, ψ^j, f) is a RR datum. We can then use Theorem 5.4.4 together with Theorems 5.5.6 and 5.5.10 to conclude. $\square$

The following can be proved exactly in the same way than Corollary 5.5.7.

Corollary 5.5.12. *Assume $f : X \to Y$ is a projective l.c.i. morphism in* Sch_S. *Then for every $i, j \in \mathbb{N}$ the following commutes*

$$\begin{array}{ccc}
\mathbb{Z}[1/j] \otimes K_i(X) & \xrightarrow{\ \tau_f \cdot \psi^j\ } & \mathbb{Z}[1/j] \otimes K_i(X) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathbb{Z}[1/j] \otimes K_i(Y) & \xrightarrow[\ \psi^j\ ]{} & \mathbb{Z}[1/j] \otimes K_i(Y)
\end{array}$$

with τ_f given by Theorem 5.5.11.

Remark 5.5.13. The degree zero part is actually the usual Adams-Riemann-Roch theorem as stated in [FL85] Theorem 7.6, the only difference being that they assume the schemes involved to be quasi-projective, but we have already remarked in 5.5.8 that our generalization at the zero level is not a novelty. Moreover, from [FL85] Theorem 7.6 we can also get for free who is the misterious multiplier we have to insert in Theorem 5.5.11: it is $\theta^j(T_f^\vee)^{-1}$ where T_f is the tangent element in $K_0(X)$ defined in [FL85] V Section 7.

Remark 5.5.14. After this work was completed, the author discovered the existence of the article [K̈98]. In that paper it is shown that an Adams Riemann Roch theorem holds for the higher equivariant K-theory groups of a divisorial scheme, hence implying the non equivariant case. That is done following the method of [FL85] although not introducing our terminology. Moreover, the lambda operations defined in *op. cit.* are defined using the construction of Grayson (and not the ones we consider in this work) and the fact that they do give to higher K-theory the structure of a λ-ring is partially left as a conjecture, although it is claimed in *op. cit.* that the conjectural parts left out are only required for the Grothendieck-Riemann-Roch theorem and not for the Adams-Riemann-Roch theorem. It is believed that at least in the non equivariant case, the results of that paper should agree with ours. We will address the agreement between the operations defined in this text with the ones in [K̈98] in the non equivariant case in future work.

Operations on Hermitian K-theory

The results contained in the thesis of Riou provided a nice unified way to build operations on higher K-theory and we obtained a generalization of some of them form the world of smooth schemes to the world of singular ones. One might wonder if the same can be done for other cohomology theories, for example for Hermitian K-theory. First one would like to obtain the exact analogue of the result of Riou A.3.14 in the context of unstable motivic homotopy theory. Then one would like to extend the result from smooth schemes to divisorial ones. We have seen that the main ingredients of Riou's argument are a representability result which would allow us to represent our cohomology theory as a filtered colimit of representable schemes in $\mathcal{H}(S)$, the property (ii) (see A.3.4), which should follow easily from the representability result, and some explicit computations to solve a $\lim^1$ problem and conclude the argument. It turns out that thanks to work of Schlichting and Tripathi ([ST15]) and of Panin and Walter ([PW10a]) we have representability results and in the case of symplectic K-theory, we also have computations. We can then start to run the Riou's argument. We could hope in the future to lift to GW the lambda operations which have been recently defined by Zibrowius in [Zib18] at the level of GW_0. We will study separately Hermitian (or orthogonal) K-theory and symplectic K-theory. Section 6.6 standing as the only exception, in this chspter we will keep the usual assumptions and notations used in this thesis and detailed in 0.1, i.e. for a given noetherian base scheme S we shall denote as Sch_S the category of divisorial schemes of finite type over S and with Sm/S its full subcategory of smooth schemes. Also, unless otherwise indicated, we will always assume that in the base schemes S we consider, 2 is invertible in $\Gamma(S, \mathcal{O}_S)$

6.1 Bilinear Grassmannians and ϵ-symmetric spaces

In this section we introduce a common ground useful to study both symmetric and symplectic hermitian K-theory. The material here is basically contained in [ST15], although our presentation deviates from their (they do only consider symmetric forms, but the generalization is straightforward as they notice). Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. For $\epsilon = \pm 1$, an ϵ-symmetric bilinear form on $\mathcal{F}$ is a map $\varphi : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_X$ of $\mathcal{O}_X$-modules such that $\varphi \circ \tau = \epsilon\varphi$ where $\tau : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{G} \otimes_{\mathcal{O}_X} F$ is the twisting map. If $2 \in \Gamma(S, \mathcal{O}_S)^*$, -1 (skew-)symmetric forms $(\mathcal{F}, \varphi)$ are called symplectic and they are uniquely determined by $\varphi \circ \Delta = 0$ where $\Delta : \mathcal{F} \to \mathcal{F} \otimes \mathcal{F} \ x \mapsto x \otimes x$ is the diagonal map. A form φ is called *non-degenerate* and $(\mathcal{F}, \varphi)$ is called an *ϵ-inner product space* if $\mathcal{F}$ is a vector bundle on X and the adjoint morphism $\widehat{\varphi} : \mathcal{F} \to \mathcal{F}^* = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) : s \mapsto \varphi(- \otimes s)$ is an isomorphism. The form φ is ϵ-symmetric if and only if $\widehat{\varphi} = \epsilon\widehat{\varphi}^*\mathrm{can}_F$ where $\mathrm{can}_F : F \xrightarrow{\cong} F^{**}$ is the canonical isomorphism. One can see that if $g : \mathcal{G} \to \mathcal{F}$ is a map of $\mathcal{O}_X$-modules, we can define the restriction $\varphi_{|\mathcal{G}}$ of φ to $\mathcal{G}$ using adjoint map $\widehat{\varphi}_{|\mathcal{G}} = g^* \circ \widehat{\varphi} \circ g : \mathcal{G} \xrightarrow{g} \mathcal{F} \xrightarrow{\widehat{\varphi}} \mathcal{F}^* \xrightarrow{g^*} \mathcal{G}^*$. If $p : X \to S$ is a morphism of schemes and $\mathcal{F}$ is a sheaf on S, we denote $\mathcal{F}_X := p^*\mathcal{F}$. Fix a base scheme S with $2 \in \Gamma(S, \mathcal{O}_S)^*$. We assume that this condition holds until the end of this chapter.

Definition 6.1.1. For an ϵ-symmetric form $V = (\mathcal{F}, \varphi)$ with $\mathcal{F}$ a quasi-coherent sheaf on S we define the ϵ-bilinear grassmannian of non degenerate locally free subspaces of V to be the presheaf

$\mathrm{GrB}_S(V) : (\mathrm{Sch}_S)^{op} \to \mathrm{Sets}$

$\qquad (p : X \to S) \mapsto \{E \subset \mathcal{F}_X \mid E \text{ loc.free of finite rank s.t. } \varphi_{|E} \text{ is non degenerate}\}$

on the objects, and in the case of morphisms $f : X \to Y$ in Sch_S, $\mathrm{GrB}_S(Y) \to \mathrm{GrB}_S(X)$ is induced by f^* (remind pullback sends isomorphisms to isomorphisms so there are no problems with non degeneratedness). We define the ϵ-bilinear grassmannian of non degenerate locally free of rank n subspaces of V to as the subpresheaf of $\mathrm{GrB}_S(V)$ of the following form

$\mathrm{GrB}_{n,S}(V) : (\mathrm{Sch}_S)^{op} \to \mathrm{Sets}$

$\qquad (p : X \to S) \mapsto \{E \subset \mathcal{F}_X \mid E \text{ loc.free of rank n s.t. } \varphi_{|E} \text{ is non degenerate}\}$

We then have the following, which is a mere reformulation of [ST15] Lemma 2.2

Theorem 6.1.2. *Let $V = (\mathcal{F}, \varphi)$ be an ϵ-symmetric inner product space over S. Then for every n we have that $\mathrm{GrB}_{n,S}(V)$ is representable by a scheme which is smooth (divisorial) and affine over S (notice for $\epsilon = -1$ n has to be even), so in particular it is a sheaf. This is an open subscheme of the Grassmannian $\mathrm{Gr}_{n,S}(\mathcal{F})$ of rank n subbundles of $\mathcal{F}$. We explicitly spell out the universal property of this scheme. For every S-scheme X and every rank n ϵ-inner product space $B = (\mathcal{B}, \alpha)$ which comes as a restriction along a mono $\mathcal{B} \hookrightarrow \mathcal{F}_X$ there exists a unique map $f : X \to \mathrm{GrB}_{n,S}(V)$ over S such that $f^*(\mathcal{T} \subset \mathcal{F}_{\mathrm{Gr}_{n,S}(\mathcal{F})}) \cong \mathcal{B} \subset \mathcal{F}_X$ via the canonical isomorphism $\mathcal{F}_X \cong f^* \mathcal{F}_{\mathrm{Gr}_{n,S}(\mathcal{F})}$ and $B = f^*T$ where $T = (\mathcal{T}, \varphi_{|\mathcal{T}})$ is the ϵ-inner product space induced by $V_{\mathrm{GrB}_{n,S}(V)}$ on $\mathcal{T}$. Here $\mathcal{T}$ is the restriction to $\mathrm{GrB}_{n,S}(V)$ of the tautological rank n vector bundle on $\mathrm{Gr}_{n,S}(\mathcal{F})$.*

Proof. We just remind the argument of [ST15] Lemma 2.2 which works *mutatis mutandis* also in this context, as remarked in *op. cit.*. The proof goes by showing that $\mathrm{GrB}_n(V)$ is representable by an open subscheme of the usual grassmannian $\mathrm{Gr}_n(V)$. To this aim one considers the universal rank n bundle $\mathcal{T}$ over $\mathrm{Gr}_n(V)$. Now, $\varphi_{|\mathcal{T}}$ might be degenerate so we define $\mathrm{GrB}_n(V)$ as the open subscheme of $\mathrm{Gr}_n(V)$ where $\varphi_{|\mathcal{T}}$ is non degenerate. This scheme is easily seen to to represent our functor. One notices that denoting $\varphi' := p^* \varphi$ where $p : X := \mathrm{Gr}_n(V) \to S$ is the structure morphism, we have that $\mathrm{GrB}_n(V)$ is the non vanishing locus X_s of the section $s = \Lambda^n \widehat{\varphi'_{|\mathcal{T}}} \in \Gamma(X, \underline{\mathrm{Hom}}(\Lambda^n \mathcal{T}, \Lambda^n \mathcal{T}^*))$. This also settles the problem to determine whether $\mathrm{GrB}_n(V)$ is smooth. For the affineness one considers for any $X \in \mathrm{Sch}_S$ the bijection

$$\psi : \mathrm{GrB}(V)(X) \xrightarrow{\cong} \{f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_X, \mathcal{F}_X) \mid f = f^2, \quad f^* \circ \widehat{\varphi} = \widehat{\varphi} \circ f\}$$

with $\psi(i : M \subseteq \mathcal{F}_X) = i \circ (\widehat{\varphi}_{|M}^{-1}) \circ i^* \circ \widehat{\varphi}$. It is a bijection because $\sigma(f) = \mathrm{Im}(f) \subseteq \mathcal{F}_X$ defines an inverse. Iterating this for any $X \in \mathrm{Sch}_S$ allows us to see that $\mathrm{GrB}_n(V)$ is a closed subscheme of $\mathrm{GrB}(V)$ which is a closed subscheme of the vector bundle $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F})$ which is affine over S, hence, closed embeddings being affine maps, we conclude. $\qquad\square$

Remark 6.1.3. Notice that the above theorem is true even if $V = (\mathcal{F}, \varphi)$ is an ϵ-symmetric form with $\mathcal{F}$ coherent sheaf on S, but we won't need that (in that case it could be not smooth over S).

Definition 6.1.4. Suppose $V = (\mathcal{F}, \varphi)$ is an ϵ-symmetric form on a vector bundle over S. We then define the *split metabolic space* $M(\mathcal{F}, \varphi)$ or $M(V)$ as

$$M(V) = M(\mathcal{F}, \varphi) = \left(\mathcal{F} \oplus \mathcal{F}^*, \left(\begin{smallmatrix} \widehat{\varphi} & 1 \\ \epsilon\mathrm{can} & 0 \end{smallmatrix}\right) : F \oplus F^* \to F^* \oplus F^{**}\right)$$

where can is the canonical isomorphism $\mathcal{F} \xrightarrow{\cong} \mathcal{F}^{**}$. For a locally free sheaf $\mathcal{F}$ we define the hyperbolic space $\mathbb{H}_\epsilon(\mathcal{F})$ as $M(\mathcal{F}, 0)$, i.e.

$$\mathbb{H}_\epsilon(\mathcal{F}) = (\mathcal{F} \oplus \mathcal{F}^*, \left(\begin{smallmatrix} 0 & 1 \\ \epsilon\text{can} & 0 \end{smallmatrix} \right))$$

This is an ϵ-inner product space. Notice $\mathbb{H}_\epsilon(\mathcal{O}_X)$ for a scheme X are the hyperbolic spaces considered in the next sections.

We recall the following well known lemma (use [Knu91] page 19)

Lemma 6.1.5. *Let X be any quasi-compact scheme such that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. Then every split metabolic space of the form $M(\mathcal{F}, \varphi)$ is isomorphic to $\mathbb{H}_\epsilon(\mathcal{F})$.*

The following is then immediate

Corollary 6.1.6. *Let be X any quasi-compact scheme such that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. Then for every rank n ϵ-inner product space $V = (\mathcal{F}, \varphi)$ we have a morphism $f : V \hookrightarrow M(V) \cong \mathbb{H}_\epsilon(\mathcal{F}) =: \mathbb{H}(V)$ given by the inclusion $(\mathcal{F}, \varphi) \hookrightarrow M(\mathcal{F}, \varphi) : x \mapsto \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix} \right)$ comes as a restriction via f.*

6.2 Towards a Riou theorem for orthogonal K-theory

Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. A symmetric bilinear form is an 1-symmetric bilinear form on $\mathcal{F}$ as defined in the previous section. Same terminology for (symmetric) inner product spaces. We say that a scheme X is even if $2 \in \Gamma(X, \mathcal{O}_X)^*$. From now on we fix a quasi-compact and quasi-separated even base scheme S.

Definition 6.2.1. Let $F = (\mathcal{F}, \varphi)$ be a quasi-coherent module over S with a symmetric bilinear form $\varphi : \mathcal{F} \otimes_S \mathcal{F} \to \mathcal{O}_S$ (that can be degenerate). We define the *Orthogonal Grassmannian of non degenerate subspaces of* $\mathcal{F}$ as the presheaf $\mathrm{GrO}_S(\mathcal{F}) := \mathrm{GrB}_S(F)$. We have for every $d \in \mathbb{N}$ a subpresheaf $\mathrm{GrO}_d(F) \subset \mathrm{GrO}(F)$ given by considering only the locally free sheaves of rank d as in Definition 6.1.1.

We have the following proposition, which has been proved in [ST15] Lemma 2.2

Proposition 6.2.2. *Let (V, φ) a symmetric inner product space of rank n over S and d an integer $0 \le d \le n$. Then $\mathrm{GrO}_d(V)$ is represented by a scheme smooth and affine over S.*

Proof. It follows from 6.1.2. $\qquad \square$

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For a scheme $X \in \mathrm{Sch}_S$, we denote as H_X the so called *hyperbolic plane* over X, which is the inner product space $(\mathcal{O}_X^2, \varphi_1)$ where the form is defined by the inner product $(x, y) \cdot (x', y') = xx' - yy'$. Notice this is isomorphic to $\mathbb{H}_1(\mathcal{O}_X)$ of Definition 6.1.4. We let $H_X^n := \perp_n H_X$ to be its n-fold orthogonal sum. Explicitly note that we have maps $H_X^n \to H_X^{n+1}$ induced by the standard inclusion $i_n : \mathcal{O}_X^{2n} \hookrightarrow \mathcal{O}_X^{2n+2}$ noticing that $\varphi_{n+1|H_X^n} = \varphi_n$. These maps define a functor $F_X : (\mathbb{N}, \leq) \to \mathrm{QCoh}(X)$ $n \mapsto \mathcal{O}_X^{2n}$; we can use it to define the infinite hyperbolic space $H_X^\infty := \operatorname*{colim}_{n \in \mathbb{N}} H_X^n$. This is a quasi-coherent $\mathcal{O}_X$-module with a symmetric bilinear form. We now define the filtering category $\mathcal{V}$ having as objects the non degenerate subspaces of H_X^∞ and as maps the inclusions of subspaces. Giving an inclusion of two such non degenerate inner product subspaces $V \subset V'$, we will denote as $V' - V$ the orthogonal complement of V in V'.

Definition 6.2.3 ([ST15] Definition 2.3). We define the *infinite orthogonal grass-mannian* over S as the ind-scheme

$$\mathrm{GrO}_\bullet := \operatorname*{colim}_{V \subset H_X^\infty} \mathrm{GrO}_{|V|}(V \perp H_S^\infty)$$

where $|V|$ denotes the rank of V and the colimit is taken over the full subcategory $\mathcal{V}^c \subset \mathcal{V}$ of non degenerate subbundles of H_X^∞ of constant rank. The transitions maps are given, for every inclusion $V \subset V' \in \mathcal{V}^c$ by the map

$$\mathrm{GrO}_{|V|}(V \perp H_X^\infty) \to \mathrm{GrO}_{|V'|}(V' \perp H_X^\infty) \quad E \mapsto E \perp (V' - V)$$

Remark 6.2.4. Note that every $V \in \mathcal{V}^c$ is going to be a non degenerate subform of some H_S^n for some $n \in \mathbb{N}$ by corollary 6.1.6. Hence the family $\{H_S^n \mid n \in \mathbb{N}\}$ is a cofinal subset of $\mathrm{Ob}(\mathcal{V}^c)$ and so in order to define $\mathrm{GrO}_\bullet$ is sufficient to take the colimit over this full subcategory.

As a consequence we have the following simple lemma

Lemma 6.2.5. *As an ind-scheme,* $\mathrm{GrO}_\bullet \cong \operatorname*{colim}_{n \in \mathbb{N}} \mathrm{GrO}_{2n}(H_S^n \perp H_S^n) =: \operatorname*{colim}_{n \in \mathbb{N}} \mathrm{GrO}_{n,n}$, *where we denote the transition maps involved in the colimit as* $f_n : \mathrm{GrO}_{n,n} \hookrightarrow \mathrm{GrO}_{n+1,n+1}$. *In particular* $\mathrm{GrO}_\bullet \in \mathbf{Pre}(\mathrm{Sm}/S)$.

Proof. By the preceding remark,

$$\mathrm{GrO}_\bullet \cong \operatorname*{colim}_{n \in \mathbb{N}} \mathrm{GrO}_{2n}(H_S^n \perp H_S^\infty) \cong \operatorname*{colim}_{(a,b) \in \mathbb{N}^2} \mathrm{GrO}_{2a}(H_S^a \perp H_S^b)$$

since $\{H_S^n \perp H_S^n \mid n \in \mathbb{N}\}$ is cofinal in the last index category. $\qquad\square$

From now on we let Sm/S be the category of smooth schemes over S having an ample family of line bundles and such that 2 is invertible in $\Gamma(S, \mathcal{O}_S)$. S is also assumed to be regular. We have then the following fundamental theorem which has to be regarded as the hermitian analogue of [MV99] Theorem 3.13

Theorem 6.2.6 ([ST15] Theorem 1.1)**.** *The hermitian K theory* GW *(say* $\mathrm{GW}^{[0]}$ *of [Sch17] Definition 9.1, see Appendix B.5) as an element of $\mathcal{H}(S)$ is representable by $\mathbb{Z} \times \mathrm{GrO}_\bullet$, as a consequence, for every $X \in \mathrm{Sm}/S$ we have*

$$\mathrm{Hom}_{\mathcal{H}(S)}(X, \mathbb{Z} \times \mathrm{GrO}_\bullet) \cong \mathrm{GW}_0(X)$$

so that $\pi_0(\mathbb{Z} \times \mathrm{GrO}_\bullet) \cong \mathrm{GW}_0(-)$ as elements of $\mathbf{Pre}(\mathrm{Sm}/S)$.

We now define the analogue of the system $\mathcal{K}_\bullet$ used to study K-theory. We denote $\mathcal{KO}_n := \sqcup_{i=-n}^{n} \mathrm{GrO}_{n,n}$ and we define maps $\rho_n : \mathcal{KO}_n \to \mathcal{KO}_{n+1}$ as

$$\sqcup_{i=-n}^{n} \mathrm{GrO}_{n,n} \xrightarrow{\sqcup_{2n+1} f_n} \sqcup_{i=-n}^{n} \mathrm{GrO}_{n+1,n+1} \hookrightarrow \sqcup_{i=-n-1}^{n+1} \mathrm{GrO}_{n+1,n+1}$$

So we have an inductive system $\mathcal{KO}_\bullet$ whose colimit is indeed $\mathbb{Z} \times \mathrm{GrO}_\bullet$. The first step into the achievement of a Riou theorem for hermitian K-theory is then the following

Proposition 6.2.7. *The presheaf $\mathbb{Z} \times \mathrm{GrO}_\bullet$ as an element of $\mathbf{Pre}(\mathrm{Sm}/S)$ satisfies the property (ii) (see Definition A.3.4).*

Proof. In the lingo introduced in the appendix, it suffices to show that for every $U \in \mathrm{Sm}/S$ which is affine over $\mathbb{Z}$, the canonical map

$$\tau_{\mathbb{Z} \times \mathbb{Z} \mathrm{GrO}_\bullet} : \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(U, \mathbb{Z} \times \mathrm{GrO}_\bullet) \to \mathrm{Hom}_{\mathcal{H}(S)}(U, \mathbb{Z} \times \mathrm{GrO}_\bullet) \cong \mathrm{GW}_0(U)$$

is surjective. To do this we can follow Riou. In particular we start with an analogue of Assertion III.4 of [Rio06] which is spelt out in [Zib11b] page 38 or in [Zib11a] page 477 and follows from the work [ST15]. It says that under the isomorphism of Theorem 6.2.6, if $r - d$ is even, the inclusion

$$i_{d,n} : \mathrm{GrO}_{d,n} \hookrightarrow \{r\} \times \mathrm{GrO}_{d,n} \hookrightarrow \mathbb{Z} \times \mathrm{GrO}_\bullet$$

where $\mathrm{GrO}_{d,n} := \mathrm{GrO}_d(H_S^d \perp H_S^n)$ corresponds to the element

$$([\mathcal{T}_{d,n}] + \frac{r - d}{2}[H_{\mathrm{GrO}_{d,n}}]) \in \mathrm{GW}_0(\mathrm{GrO}_{d,n})$$

where $\mathcal{T}_{d,n}$ is the universal rank d symmetric bundle over $\mathrm{GrO}_{d,n}$, whose underlying vector bundle is the pullback of the rank d vector bundle over the appropriate grassmannian. Now, since U is affine, every element $\gamma \in \mathrm{GW}_0(U)$ can be written as $[P] + \frac{r-d}{2}[H_U]$ so that using the universal property of the orthogonal grassmannian, we find an arrow $f : U \to \mathrm{GrO}_{d,n}$ such that the composite $i_{d,n} \circ f$ is mapped via $\tau_{\mathbb{Z} \times \mathbb{Z} \mathrm{GrO}_\bullet}$ to γ in $\mathrm{GW}_0(U)$, completing the proof. $\qquad\square$

Corollary 6.2.8. *The map*

$$\pi_0 : [\mathrm{GW}, \mathrm{GW}]_{\mathcal{H}(S)} \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{GW}_0, \mathrm{GW}_0)$$

is surjective.

Proof. This follows from Theorem A.3.6 in virtue of the previous Proposition. $\qquad\square$

Now we would like to have that the tower

$$... \to \mathrm{GW}_1(\mathrm{GrO}_{n+1,n+1}) \xrightarrow{\boldsymbol{f}_n^*} \mathrm{GW}_1(\mathrm{GrO}_{n,n}) \to ...$$

satisfies the Mittag-Leffler property. If we did have that, we could use the system $\mathcal{KO}_\bullet$ to run Riou's argument and finally prove the following

Conjecture 6.2.9. *Let S be an even regular noetherian base schemes. Then*

$$\pi_0 : [\mathrm{GW}^n, \mathrm{GW}]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{GW}_0^n, \mathrm{GW}_0)$$

and the pointed version of this bijection is true as well. Also, $\mathrm{GW} \in \mathcal{H}(S)$ has a structure of λ-ring and this structure gives a lambda ring structure on $\oplus_{n \in \mathbb{N}} \mathrm{GW}_n(\mathcal{X})$ for every $\mathcal{X} \in \mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)$.

Remark 6.2.10. Using corollary 6.2.8 one can lift the operations defined by Zibrowius in [Zib18] (note that in *op. cit.* all the schemes are assumed to be over a field of characteristic not two), but then one could not verify the axioms of λ-ring in $\mathcal{H}(S)$ for GW. This means that we can define maps $\lambda^r : \mathrm{GW} \to \mathrm{GW}$ in $\mathcal{H}(S)_\bullet$ getting maps $\lambda_n^r : \mathrm{GW}_n(\mathcal{X}) \to \mathrm{GW}_n(\mathcal{X})$ for any siplicial presheaf $\mathcal{X} \in \mathrm{s}\mathbf{Pre}(\mathrm{Sm}/S)$. However we cannot use Proposition 5.1.21 to put a lambda ring structure on $\oplus_{n \in \mathbb{N}} \mathrm{GW}_n(\mathcal{X})$. It seems that Marcus Zibrowius (private communication) has a way to overcome this problem for the $\mathrm{GW}_n(X)$ of every $X \in \mathrm{Sm}/S$, but it is still unknown if we can go further. In addition, the author has ongoing work that will use the results of [Zib15] and [Elh00] in order to build the lambda operations on hermitian K-theory for any reasonable (noetherian and of finite type over some nice base) following the

construction of [GS99]. After work of Alexander Efimov [Efi17], the author in joint work with Dylan Madden, Heng Xie and Marco Schlichting has computed some of the Grothendieck-Witt groups of the standard Grassmannians. One could hope to do the same for the orthogonal ones to prove the conjecture above, although this seems much more complicated and will be subject of further work.

6.3 A Riou theorem for symplectic K-theory

In the context of Symplectic K-theory, thanks to the computations made by Panin and Walter, we can go further than in the previous section. The first step is to repeat all the arguments in the previous section to obtain the analogue results. To this aim, we will simply introduce the objects involved stating the results we need providing references for them. We fix a regular noetherian base scheme S such that 2 is invertible in $\Gamma(S, \mathcal{O}_S)$. We denote by $KSp \in \mathcal{H}(S)$ the object in the unstable motivic homotopy category representing symplectic K-theory, aka $\mathrm{GW}^{[2]}$ of [Sch17] Section 9 (see Appendix B.5). Such object is also known as GW^-. We now introduce the quaternionic or symplectic grassmannians. Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. A symplectic bilinear form is a -1-symmetric bilinear form as defined in the previous Section 6.1. Same terminology for (symplectic) inner product spaces.

Definition 6.3.1. If we consider $(F_n, \varphi_n) = (\mathcal{O}_S^{2n}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right))^{\perp^n} =: \mathbb{H}^n \cong (\mathbb{H}_{-1}(\mathcal{O}_X))^{\perp n}$ we define as $\mathrm{GrH}_{d,n}$ the presheaf of $\mathrm{GrB}_{2d}(\mathbb{H}^{n+d})$ (see Definition 6.1.1) parametrizing non-degenerate subbundles of rank $2d$. These will be called *symplectic or quaternionic grassmannians*.

Lemma 6.3.2. *For all the integers d, n, the symplectic grassmannians $\mathrm{GrH}_{d,n}$ are representable by a smooth and affine scheme over S. This scheme is an open subscheme of $\mathrm{Gr}_{2d}(\mathcal{O}_S^{2(n+d)})$.*

The proof is the same than the one of Lemma 2.2 in [ST15] and it follows by Theorem 6.1.2. We notice ([PW10a] page 22) that there are closed immersions $\mathrm{GrH}_{d,n} \hookrightarrow \mathrm{GrH}_{d,n+1}$ and $\mathrm{GrH}_{d,n} \hookrightarrow \mathrm{GrH}_{d+1,n}$ classified by the inclusions $\mathcal{T}_{d,n} \oplus 0 \subset \mathbb{H}^{n+d} \oplus \mathbb{H}$ and $\mathbb{H} \oplus \mathcal{T}_{d,n} \subset \mathbb{H} \oplus \mathbb{H}^{n+d}$ where $\mathcal{T}_{d,n}$ is the restriction of the universal symplectic bundle on $\mathrm{GrH}_{d,n}$ induced by $\mathbb{H}_S^{n+d}$ on the restriction of the universal rank $2d$ bundle on $\mathrm{Gr}_{2d}(\mathcal{O}_S^{2(n+d)})$. As before we can form a system $\mathcal{K}Sp_\bullet$ indexed by $\mathbb{N}$ having $\mathcal{K}Sp_n := \sqcup_{2n+1}\mathrm{GrH}_{n,n}$ having colimit $\mathbb{Z} \times \mathrm{GrH}$ where we have denoted $\mathrm{GrH} := \underset{d,n}{\mathrm{colim}}\mathrm{GrH}_{d,n}$. Hence we have the following representability result (remind $\mathbb{Z} \times \mathrm{GrH}$ is an H-group pointed by $(0, \mathrm{GrH}_{0,0})$)

Theorem 6.3.3 ([PW10a] or Theorem 8.2. [ST15]). *We have $\mathbb{Z} \times \mathrm{GrH} \simeq KSp$ in $\mathcal{H}(S)$. In particular for any $X \in \mathrm{Sm}/S$ we have*

$$\mathrm{Hom}_{\mathcal{H}(S)}(X, \mathbb{Z} \times \mathrm{GrH}) \cong KSp_0(X)$$

so that $\pi_0(\mathbb{Z} \times \mathrm{GrH}) \cong KSp_0(-)$ as elements of $\mathbf{Pre}(\mathrm{Sm}/S)$.

To go on with the theorem of Riou, we need as in the proof of 6.2.7 to know the analogue of Assertion III.4 of [Rio06] which is the following fact contained in [PW10a]. If $r - d$ is even, the inclusion

$$i_{d,n} : \mathrm{GrH}_{d,n} \hookrightarrow \{r\} \times \mathrm{GrH}_{d,n} \hookrightarrow \mathbb{Z} \times \mathrm{GrH}$$

is mapped to the element $([\mathcal{T}_{d,n}] + \frac{r-d}{2}[\mathbb{H}_{\mathrm{GrH}_{d,n}}]) \in KSp_0(\mathrm{GrH}_{d,n})$ by the isomorphism in Theorem 6.3.3 (see [Ana15] Theorem 6.3). Using this we obtain as in the case of orthogonal grassmannian the following result, whose proof goes as the one of Proposition 6.2.7 mutatis mutandis.

Proposition 6.3.4. *The presheaf $\mathbb{Z} \times \mathrm{GrH}$ as element of $\mathbf{Pre}(\mathrm{Sm}/S)$ satisfies the property (ii).*

Now to end up with a Riou like theorem for symplectic K-theory we shall need to study further the system $\mathcal{K}Sp_\bullet$, in particular we have to show that

$$R^1 \varprojlim_{n \in \mathbb{N}} KSp_1(\mathcal{K}Sp_n) = 0$$

which follows as in the case of K-theory using the explicit calculations of [PW10b] Theorem 11.4 (see indeed [PW10a] Theorems 9.4, 9.5) to show that the involved tower satisfies the Mittag-Leffler property. Indeed from the computations contained in *op.cit.* we have that for any i, the object $\Omega_f^i KSp$ satisfies the property (K) (see Definition A.3.7) with respect to the system $\mathcal{K}Sp_\bullet$, the maps $KSp_{i+1}(\mathcal{K}Sp_{n+1}) \to KSp_{i+1}(\mathcal{K}Sp_n)$ being surjective. As a consequence we have the following result

Theorem 6.3.5. *If S is a regular scheme such that 2 is invertible in $\Gamma(S, \mathcal{O}_S)$, then for every natural number n one has the following isomorphisms*

$[KSp, \Omega_f^i KSp]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathcal{H}(S)}(\mathbb{Z} \times \mathrm{GrH}, \Omega_f^i(\mathbb{Z} \times \mathrm{GrH})) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(KSp_0(-), KSp_i(-)) \cong KSp_i(S)[[b_1, b_2, ...]]$

$[KSp^n, \Omega_f^i KSp]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathcal{H}(S)}(\mathbb{Z} \times \mathrm{GrH}^n, \Omega_f^i(\mathbb{Z} \times \mathrm{GrH})) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(KSp_0^n(-), KSp_i(-))$

$[KSp^n, \Omega_f^i KSp]_{\mathcal{H}_\bullet(S)} \cong \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbb{Z} \times \mathrm{GrH}^n, \Omega_f^i(\mathbb{Z} \times \mathrm{GrH})) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)_\bullet}(KSp_0^n(-), KSp_i(-))$

the b_i being the Borel classes described in [PW10b].

Proof. We only notice that the case with n factors follows by considering the system $\mathcal{K}Sp_\bullet^n$. Indeed the computations of [PW10a] Theorems 9.4, 9.5 allow us to conclude

the argument even in this case since they allow us to handle the products of the symplectic grassmannians involved. $\qquad\square$

We can extend further the previous result to smooth affine schemes

Theorem 6.3.6. *Let S be a noetherian regular base scheme such that 2 is invertible in $\Gamma(S, \mathcal{O}_S)$, Sm/S the category of divisorial smooth schemes over S and SmAff/S its full subcategory of affine smooth schemes. Then all the arrows in the following diagram are isomorphisms*

$$
\begin{array}{ccc}
[KSp^n, KSp]_{\mathcal{H}(S)} & \longrightarrow & [KSp^n, KSp]_{\mathcal{H}^{aff}(S)} \\
\downarrow{\scriptstyle \pi_0} & & \downarrow \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(KSp_0^n, KSp_0) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(KSp_0^n, KSp_0)
\end{array}
$$

Proof. This is obtained as Theorem 3.2.7 so one can look at its proof. Indeed the top map can be seen to be an isomorphism using Theorem 3.3.2 in [AHW17], for example, and the fact that the lower horizontal map is an isomorphism too can be deduced from Corollary 3.2.6 taking into account that KSp_0 is $\mathbb{A}^1$-invariant on regular schemes. $\qquad\square$

Remark 6.3.7. Notice that in general we have $\mathcal{H}(S) \simeq \mathcal{H}^{aff}(S)$ for all the possible choices of the category Sm/S, i.e. we can take our smooth schemes to be separated, divisorial or only smooth, as long as our base scheme is noetherian. Indeed this can be seen using Theorem 3.3.2 in [AHW17] together with Lemma 5.1.2 [AHW17].

6.4 Going to divisorial schemes

In the case where we have obtained a Riou like theorem, we can wonder whether we can obtain an extension to divisorial schemes, which can be singular in principle. We remind that we can argue as for K-theory to prove the following (this is Theorem 1.8.11)

Theorem 6.4.1. $[KSp^n, KSp]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \cong [(\mathbb{Z}\times\mathrm{BSp})^n, KSp]_{\mathcal{I}_{Nis}^l\mathrm{Sm}/S} \cong [KSp^n, KSp]_{\mathcal{H}(S)}$ *for every $n \in \mathbb{N}$.*

We are then able to draw the following diagram as in Section 3.1

$$[KSp^n, KSp]_{\mathcal{H}_{Zar}^{\mathrm{Sch}\,S}} \xrightarrow{\ \cong\ } [KSp^n, KSp]_{\mathcal{H}(S)}$$

$$\pi_0 \downarrow \qquad\qquad\qquad\qquad \cong \downarrow \pi_0$$

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(KSp_0^n(-), K_0(-)) \xrightarrow{\ \mathrm{res}\ } \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(KSp_0^n(-), K_0(-))$$

To show that all the maps in the previous diagram are isomorphisms we are then left to show that

Theorem 6.4.2. *For every $n \in \mathbb{N}$, the restriction map*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(KSp_0^n(-), KSp_0(-)) \twoheadrightarrow \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(KSp_0^n(-), KSp_0(-))$$

is injective.

We are going to do this in such a way to have a similar result also for orthogonal K-theory.

6.4.1 Pulling back forms via bilinear grassmannnian

In this section all rings and schemes we will consider are supposed to be even, i.e. 2 is invertible in them. We are then in the position to prove the following

Theorem 6.4.3. *Assume X is a divisorial scheme of finite type over a scheme S which is quasi-projective over a noetherian affine scheme R where 2 is invertible. Let be $V = (\mathcal{F}, \varphi)$ an ϵ-inner product space over X where $\mathcal{F}$ is a vector bundle. Then there exists a divisorial smooth scheme Y_V over S, an ϵ-inner product space $E_V = (\mathcal{E}_V, \varphi_V)$ over Y_V and a map $f : X \to Y_V$ over S such that $f^*(E_V) \cong V$. If X and S are affine schemes, then we can take Y_V to be affine.*

Proof. We first assume that X is connected so that $\mathcal{F}$ is a vector bundle of rank n. We can use 3.1.11 and 3.2.14 to find a scheme W which is divisorial and smooth over S together with a vector bundle $\mathcal{E}$ on it and a map $g : X \to W$ such that $g^*(\mathcal{E}) \cong \mathcal{F}$. If X and S are affine, we remark that we may choose W to be affine. Now, we can consider the bilinear Grassmannian $\mathrm{GrB}_{n,W}(\mathbb{H}_\epsilon(\mathcal{E}))$. This is a divisorial smooth scheme affine over W. In particular, if W is affine, then it is affine in the absolute sense. Now the universal property of the bilinear grassmannians 6.1.2 together with Corollary 6.1.6 give us a map $f : X \to \mathrm{GrB}_{n,W}(\mathbb{H}_\epsilon(\mathcal{E})) =: Y_V$ over W and then over S and an ϵ-inner product space E_V over Y_V such that $f^*(E_V) \cong V$, as wanted. Now if X is not connected we can reason componentwise and then glue together the resulting schemes to get the assert, as in the proof of Proposition 2.5.3. $\qquad\square$

As a corollary we have the following "many variables" version of the previous Theorem which extends it to a finite family of ϵ-inner product spaces, and whose proof is identical *mutatis mutandis* to the one of Proposition 3.1.13

Proposition 6.4.4. *Let X be a divisorial scheme of finite type over S quasi-projective over a noetherian affine scheme R where 2 is invertible. Then given a finite number of ϵ-inner product spaces over X, $V_1 = (\mathcal{E}_1, \varphi_1), ..., V_n = (\mathcal{E}_n, \varphi_n)$, there is a smooth scheme Y_V over S and ϵ-inner product spaces $V_{1,Y_V}, ..., V_{n,Y_V}$ over it together with a morphism $\psi_V : X \to Y_V$ such that $\psi_V^*(V_{i,Y_V}) \cong V_i$ for every $i = 1, ..., n$. If X and S are affine schemes, then we can take Y_V to be affine.*

Proof. We just observe that we do not require all our inner product spaces to have the same value of ϵ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

With this result is now easy to prove the following using Proposition 3.1.10

Theorem 6.4.5. *The natural restriction maps*

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(\mathrm{GW}_0(-)^n, \mathrm{GW}_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{GW}_0(-)^n, \mathrm{GW}_0(-))$$
$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(KSp_0(-)^n, KSp_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(KSp_0(-)^n, KSp_0(-))$$

are injective if S is a quasi-projective (affine if we consider the categories of affine schemes) noetherian finite type over a noetherian affine scheme R where 2 is invertible. The following maps

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}/S)}(\mathrm{GW}_0(-)^n, \mathrm{GW}_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(\mathrm{GW}_0(-)^n, \mathrm{GW}_0(-))$$
$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Aff}/S)}(KSp_0(-)^n, KSp_0(-)) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{SmAff}/S)}(KSp_0(-)^n, KSp_0(-))$$

are injective as well if in addition to the previous hypothesis, S is affine.

6.4.2 The end of the story

Going back to symplectic hermitian K-theory, using Theorem 6.4.5 we have the following

Theorem 6.4.6. *Fix S which is a regular quasi-projective noetherian scheme of finite type over a noetherian affine scheme R where 2 is invertible. Then all the arrows in the following diagram are isomorphisms.*

$$
\begin{array}{ccc}
[KSp^n, KSp]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} & \xrightarrow{\quad\cong\quad} & [KSp^n, KSp]_{\mathcal{H}(S)} \\
{\scriptstyle \pi_0}\Big\uparrow & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle \pi_0} \\
\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sch}_S)}(KSp_0^n(-), KSp_0(-)) & \xrightarrow{\ \mathrm{res}\ } & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(KSp_0^n(-), KSp_0(-))
\end{array}
$$

Proof. It follows from Theorems 6.4.5 and and what we already knew. $\square$

Finally, we have

Theorem 6.4.7. *Fix S an affine regular noetherian base scheme with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Then all the arrows in the following commutative cube are isomorphisms for every $n \in \mathbb{N}$*

$$
\begin{array}{ccc}
 & [KSp^n, KSp]_{\mathcal{H}_{Zar}^{\mathrm{Aff}/S}} \longrightarrow [KSp^n, KSp]_{\mathcal{H}^{aff}(S)} & \\
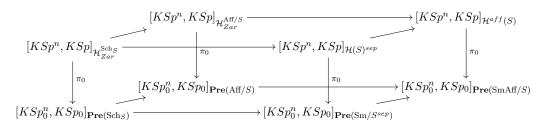[KSp^n, KSp]_{\mathcal{H}_{Zar}^{\mathrm{Sch}_S}} \longrightarrow [KSp^n, KSp]_{\mathcal{H}(S)} & & \\
 & [KSp_0^n, KSp_0]_{\mathbf{Pre}(\mathrm{Aff}/S)} \longrightarrow [KSp_0^n, KSp_0]_{\mathbf{Pre}(\mathrm{SmAff}/S)} & \\
[KSp_0^n, KSp_0]_{\mathbf{Pre}(\mathrm{Sch}_S)} \longrightarrow [KSp_0^n, KSp_0]_{\mathbf{Pre}(\mathrm{Sm}/S)} & &
\end{array}
$$

The pointed version of this theorem also holds.

Proof. This puts together Theorems 6.4.6, 6.3.6 and 6.4.5 and it is shown as Theorem 3.2.16. $\square$

6.5 Separated schemes

We can repeat also for symplectic hermitian K-theory the same considerations that we made in Section 3.3. All the arguments go mutatis mutandis so we content ourselves to state the resulting theorem

Theorem 6.5.1. *Let S be an affine regular noetherian base scheme with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$ and consider $\mathrm{Sm}/S^{sep} \subseteq \mathrm{Sch}_S$ the category of separated (in the absolute sense) smooth schemes seen as a full subcategory of the category of divisorial schemes over S and KSp the Schliching's $\mathrm{GW}^{[2]}$. Then all the arrows in the following commutative cube are isomorphisms for every $n \in \mathbb{N}$*
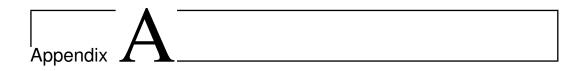
$$[KSp^n, KSp]_{\mathcal{H}^{\mathrm{Aff}/S}_{Zar}} \longrightarrow [KSp^n, KSp]_{\mathcal{H}^{aff}(S)}$$

$$[KSp^n, KSp]_{\mathcal{H}^{\mathrm{Sch}_S}_{Zar}} \xrightarrow{\quad \pi_0 \quad} [KSp^n, KSp]_{\mathcal{H}(S)^{sep}}$$

$$\downarrow \pi_0 \qquad [KSp_0^n, KSp_0]_{\mathbf{Pre}(\mathrm{Aff}/S)} \xrightarrow{\quad \pi_0 \quad} [KSp_0^n, KSp_0]_{\mathbf{Pre}(\mathrm{SmAff}/S)} \quad \downarrow \pi_0$$

$$[KSp_0^n, KSp_0]_{\mathbf{Pre}(\mathrm{Sch}_S)} \longrightarrow [KSp_0^n, KSp_0]_{\mathbf{Pre}(\mathrm{Sm}/S^{sep})}$$

The pointed version of this theorem also holds.

Proof. The proof as the one of Theorem 3.2.16 mutatis mutandis. $\qquad\square$

6.6 Non-divisorial schemes

We remark that if we assume our even base scheme to be regular (and possibly non divisorial) as in Section 3.4 and we allow all the schemes in Sch_S and Sm/S to be possibly non divisorial, then defining hermitian K-theory using the construction of Schlichting recalled in the appendix ad using perfect complexes instead of vector bundles, we still have a cohomology theory satisfying Zariski and Nisnevich descent and which is homotopy invariant on Sm/S. So we can extend the results on the cube drawn in Theorem 6.4.7 obtaining a result exactly analogue to the one obtained in Section 3.4 and we can repeat the same considerations. If one wants to have homotopy invariance for non regular schemes, then one could define and employ an homotopy invariant hermitian K-theory, but so far such theory has not been discussed in literature and we refrain from any further comment on this matter.

Some results of Riou

In this appendix we present a sketch of the proof of one of the main results contained in the thesis of Joel Riou [Rio06] together with a recollection of some facts concerning algebraic geometry and homotopy theory that we need in the thesis. We provide here some details that are somewhat abridged or available only in French in the works of Riou ([Rio10],[Rio06]) while we will usually omit or sketch the proofs whose details can be found elsewhere. In places, we deviate from the exposition and the arguments found in literature, but all the material here which is not explicitly contained in [Rio10] or [Rio06] should be considered known to the experts or deriving from Riou's work.

Remark A.0.1. We deviate a bit from the exposition of Riou in the sense that for us Sm/S will be from now on (unless otherwise stated) the category of divisorial smooth schemes over a regular base scheme S as assumed at the beginning of this thesis (see our Assumption 0.1), while Riou uses separated schemes instead of the divisorial ones. This will make little difference since the Riou's argument run through even in this setting as the reader can check reading the details of what follows. The reader interested only in separated schemes can consider all the schemes to be separated (see also 3.3 to see that this is not an issue for the main theorems contained in the thesis).

A.1   Towers and Milnor's exact sequence in motivic geometry

Given $\mathcal{A}$ abelian category with enough injectives and I a small category, then $\mathcal{A}^I$ has enough injectives too and the projective limit functor $\varprojlim_{i \in I} \mathcal{A}^I \to \mathcal{A}$ is left exact so

that we can define its right derived functors. If $\mathcal{A}$ satisfies in addition the condition (AB4*) as in [Wei94] 3.5 (i.e. $\mathcal{A}$ is complete and the left exact functor $\prod : \mathcal{A}^I \to \mathcal{A}$ is exact for every discrete I) we know that when I is $\mathbb{N}^{op}$ (in this case the elements $X_\bullet$ of $\mathcal{A}^{\mathbb{N}^{op}}$ are called *towers*, and the arrows between the X_n *transition morphisms*), these derived functors have a special form $(\mathrm{R}^0 \varprojlim_{\mathbb{N}^{op}} X_\bullet = \varprojlim_{\mathbb{N}^{op}} X_\bullet$, $\mathrm{R}^n \varprojlim_{\mathbb{N}^{op}} X_\bullet = 0$ for $n \geq 2$ and $\mathrm{R}^1 \varprojlim_{\mathbb{N}^{op}} X_\bullet = \varprojlim_{\mathbb{N}^{op}}^1 X_\bullet)$ as detailed in [Wei94] 3.5 for example. $\mathrm{R}^1 \varprojlim$ will be particularly important to our aims.

Definition A.1.1. A tower $X_\bullet$ satisfies the Mittag-Leffler condition if for every k there exists $j \geq k$ such that $im(X_j \to X_k) = im(X_i \to X_k)\ \forall i \geq j$.

We remark that in the case of $\mathcal{A} = \mathrm{Ab}$, if all the transition maps of a tower $X_\bullet$ are onto, then $X_\bullet$ satisfies the Mittag-Leffler condition. The following is standard (see [Wei94])

Proposition A.1.2. *Consider $\mathcal{A}$ an abelian category with enough injectives and satisfying (AB4*). Then if a tower $X_\bullet$ satisfies the Mittag-Leffler condition, it holds $\mathrm{R}^1 \varprojlim X_\bullet = 0$.*

Since we will not be interested only in the cases where $\mathbb{N}^{op}$ is our index category, we will need the notion of cofinal functor.

Definition A.1.3. Let I be a directed ordered set and let $(x_n)_{n \in \mathbb{N}}$ be an increasing sequence of elements in I. $(x_n)_{n \in \mathbb{N}}$ is called cofinal if for every $y \in I$, there exists $n \in \mathbb{N}$ such that $x_n \geq y$. This is equivalent to give a cofinal functor $x : \mathbb{N} \to I$ (i.e. for every functor $\varphi : I \to \mathcal{C}$, the natural transformation $\varinjlim_{\mathbb{N}} \varphi \circ x \to \varinjlim_{I} \varphi$ is an isomorphism). For a discussion of the equivalence between this two notions, see [SGA72] I 8.1.1-8.1.3 (see also [Mac71] page 217).

The following is standard, see [Rio06] Proposition II.5

Proposition A.1.4. *Let I be a directed ordered set, $x : \mathbb{N} \to I$ a cofinal functor and $\mathcal{A}$ as above. Denote $x^* : \mathcal{A}^{I^{op}} \to \mathcal{A}^{\mathbb{N}^{op}}$ the functor obtained by composition from x. Hence $\forall X_\bullet \in \mathcal{A}^{I^{op}}$ and for any integer n we have $R^n \varprojlim_{I^{op}} X_\bullet \cong R^n \varprojlim_{\mathbb{N}^{op}} x^* X_\bullet$*

With this proposition, we will be in the position to use the Mittag-Leffler condition to calculate the first derived functor of the inverse limit also in the case of directed ordered sets admitting a cofinal sequence. We turn now to some model category theory. We denote by $\mathrm{Tow}(\mathcal{S})_\bullet = \mathcal{S}_\bullet^{\mathbb{N}^{op}}$ the category of pointed towers of simplicial sets. This category can be endowed with an injective model category

structure (weak equivalences and cofibrations sectionwise, see [GJ09] for the details). In this context one can then build the total right derived functor of the functor $\varprojlim$, $R\varprojlim : \mathrm{Ho}(\mathrm{Tow}(\mathcal{S})_\bullet) \to \mathrm{Ho}(\mathcal{S}_\bullet)$ and use it to define the homotopy limit of a tower of simplicial set (given such a tower $X_\bullet$, $\mathrm{holim}X_\bullet = \varprojlim(X_{\bullet f})$ where $X_{\bullet f}$ is a fibrant replacement of our tower, and a tower is fibrant if X_0 is fibrant and all the transition morphisms are fibrations). We now remind what an H-group is in general

Definition A.1.5. If $\mathcal{C}$ is a model category with a zero object, then an object X is called H-group if it is a group object in $\mathrm{Ho}(\mathcal{C})$. If $\mathcal{C}$ has no zero object, then X will be called H-group if it is an H-group in $\mathcal{C}_\bullet$.

We are now in the position to quote the Milnor's exact sequence

Theorem A.1.6. *Let $X_\bullet$ be an H-group in $\mathrm{Tow}(\mathcal{S})_\bullet$ and denote $X = \mathrm{holim}X_\bullet$. Then for every $i \in \mathbb{N}$ there is an exact sequence (of groups)*

$$* \to R^1 \varprojlim_{n \in \mathbb{N}^{op}} \pi_{i+1}(X_n) \to \pi_i X \to \varprojlim_{n \in \mathbb{N}^{op}} \pi_i(X_n) \to *$$

The proof of this theorem is given in [GJ09]VI Prop.2.15 in the case $i \neq 0$ for groups and $i = 0$ for sets. The version given here results from the structure of H-group specified in the assumptions. We apply the previous theorem to a case relevant to us. Let S be a noetherian scheme and consider the standard unstable motivic homotopy categories $\mathcal{H}_\bullet(S)$ and $\mathcal{H}(S)$ (here the schemes can be allowed to be also non divisorial). Consider E fibrant H-group and $Y_\bullet = (Y_i)_{i \in \mathbb{N}}$ directed system of schemes in Sm/S (hence in $\mathrm{sShv}(\mathrm{Sm}/S)$ and in $\mathcal{H}(S)$). Let Y_{i+} be their pointed egos. We then obtain, using the simplicial structure on simplicial presheaves, a projective system of pointed simplicial sets $X_\bullet = \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{\bullet+}, E)$, $\mathfrak{Map}_{\mathcal{H}_\bullet(S)}(-,-)$ being the simplicial mapping space of $\mathcal{H}_\bullet(S)$. Since for any i, $\mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{i+}, E)$ is an H-group, then $X_\bullet$ is an H-group in $\mathrm{Tow}(\mathcal{S})_\bullet$ and in particular it is pointed. Now we can can compute, denoting $Y = \varinjlim_{\mathbb{N}} Y_\bullet$

$$R\varprojlim_{\mathbb{N}^{op}}\mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{\bullet+}, E) = \mathrm{holim}_{\mathbb{N}^{op}}\mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{\bullet+}, E)$$

$$\simeq \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(\mathrm{hocolim}_{\mathbb{N}}Y_{\bullet+}, E)$$

$$\simeq \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(\varinjlim_{\mathbb{N}}Y_{\bullet+}, E)$$

$$= \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_+, E)$$

Moreover we have that $\pi_0 \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_+, E) = [Y_+, E]_{\mathcal{H}_\bullet(S)} = \mathrm{Hom}_{\mathcal{H}(S)}(Y, E)$ and, since the Y_i are representable, $\mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{i+}, E) \cong E(Y_i)$. This equalities, together with Milnor's exact sequences, yield in the case $i = 0$ the following exact sequence

$$* \to R^1 \varprojlim_{i \in \mathbb{N}^{op}} \pi_1 E(Y_i) \to \mathrm{Hom}_{\mathcal{H}(S)}(Y, E) \to \varprojlim_{i \in \mathbb{N}^{op}} \pi_0 E(Y_i) \to *$$

Finally, if we change the index set, i.e. if $Y_\bullet = (Y_i)_{i \in I}$ with I direct filtered set admitting a cofinal sequence $x : \mathbb{N} \to I$, using the previous notation and the previous proposition, we obtain immediately

Proposition A.1.7. *If S is a noetherian scheme and $Y_\bullet$ and E are as above, we get an exact sequence*

$$* \to R^1 \varprojlim_{i \in I^{op}} \pi_1 E(Y_i) \to \mathrm{Hom}_{\mathcal{H}(S)}(Y, E) \to \varprojlim_{i \in I^{op}} \pi_0 E(Y_i) \to *$$

which coincides with the one written above

Under the same assumptions, we easily get the following chain of isomorphisms (use also [GJ09] II Lemma 2.3)

$$\pi_1 E(Y_i) \cong \pi_1 \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{i+}, E) \cong \pi_0 \Omega \, \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{i+}, E) \cong \pi_0 \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{i+}, E)^{(\mathbb{S}^1, *)}$$
$$\cong \pi_0 \mathfrak{Map}_{\mathcal{S}_\bullet}((\mathbb{S}^1, *), \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(Y_{i+}, E)) \cong \pi_0 \mathfrak{Map}_{\mathcal{H}_\bullet(S)}(\mathbb{S}^1 \wedge Y_{i+}, E)$$
$$\cong \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbb{S}^1 \wedge Y_{i+}, E)$$

where by $\mathbb{S}^1$ we denote the standard simplicial 1-sphere.

A.2 Grassmannians, K-theory and Jouanolou's trick

Definition A.2.1. If X is a scheme, we denote as $K_0(X)$ the Grothendieck group of the exact category of vector bundles over X. Fix now a base scheme S and let $(d, r) \in \mathbb{N}^2$ (from now on $\mathbb{N}^2$ will be always considered with its natural order, i.e. $(d, r) \leq (d', r')$ if $d \leq d'$ and $r \leq r'$). Denote as $\mathrm{Gr}_{d,r,S} : \mathrm{Sch}_S \to \mathrm{Sets}$ the representable Grassmannian functor (usually we omit the subscript S) defined by $\mathrm{Gr}_{d,r}(X) = \{F \subseteq \mathcal{O}_X^{d+r} \mid \mathcal{O}_X^{d+r}/F \text{ locally free of rank } r\}$ on the objects and which is defined by pullback on the arrows. If $(d, r) \leq (d', r')$ we get a closed immersion $f_{(d,r)(d',r')} : \mathrm{Gr}_{d,r} \hookrightarrow \mathrm{Gr}_{d',r'}$ defined, for any $X \in \mathrm{Sch}_S$ as $F \subseteq \mathcal{O}_X^{d+r} \in \mathrm{Gr}_{d,r}(X) \mapsto F \oplus \mathcal{O}_X^{d'-d} \oplus \{0\}^{r'-r} \subseteq \mathcal{O}_X^{d'+r'} \in \mathrm{Gr}_{d',r'}(X)$. Notice this convention is dual to the one of Grothendieck in [GD71] and Morel in [Mor06].

We remark that $(n,n)_{n\in\mathbb{N}}$ is a cofinal sequence in $\mathbb{N}^2$. Denote $\mathrm{Gr} := \mathrm{colim}_{\mathbb{N}^2}\mathrm{Gr}_{d,r}$, $\mathrm{Gr}_{d,\infty} := \mathrm{colim}_{r\in\mathbb{N}}\mathrm{Gr}_{d,r}$, where the colimits are calculated in $\mathbf{Pre}(\mathrm{Sm}/S)$ (so that infinite grassmannians are what is sometimes called ind-objects). We have that $\mathrm{Gr} \cong \mathrm{colim}_{d\in\mathbb{N}}\mathrm{Gr}_{d,\infty}$. Riou shows that we can build a directed system having a cofinal sequence whose colimit is $\mathbb{Z} \times \mathrm{Gr}$. We let this system to be $\mathcal{K}_\bullet = (\mathcal{K}_{d,r})_{(d,r)\in\mathbb{N}^2}$, with $\mathcal{K}_{(d,r)} = \bigsqcup_{2d+1}\mathrm{Gr}_{d,r}$. Notice that this system can be seen to be equivalent to the system described in [Rio06] page 84 by considering $2d + 1$ as the subset $\{-d,\cdots,0,1,\cdots,d\}$ of $\mathbb{Z}$. Hence $\mathrm{colim}_{\mathbb{N}^2}\mathcal{K}_\bullet \cong \mathbb{Z} \times \mathrm{Gr}$. Since $\mathbb{N}^2$ has a cofinal sequence $i : \mathbb{N} \to \mathbb{N}^2$, $n \mapsto (n,n)$, we can denote as $\mathcal{P}_\bullet := \mathcal{K}_\bullet \circ i$. Notice this cofinal sequence is the same than the one of [Rio06] Remarque III.2. We have $\mathrm{colim}\mathcal{P}_\bullet \cong \mathbb{Z} \times \mathrm{Gr}$ as well. We point $\mathbb{Z} \times \mathrm{Gr}$ by the inclusion $\{0\} \times \mathrm{Gr}_{0,0} \hookrightarrow \mathbb{Z} \times \mathrm{Gr}$. Morel and Voevodsky proved in [MV99] the following theorem (here one can consider as Sm/S the category of possibly non divisorial smooth schemes over S)

Theorem A.2.2. *Let S be a regular scheme and allow Sm/S to be the category of possibly non divisorial smooth schemes over S. Then we have an H-group structure on $\mathbb{Z} \times \mathrm{Gr}$ such that we have canonical functorial isomorphisms*

$$\mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbb{S}^n \wedge X_+, \mathbb{Z} \times \mathrm{Gr}) \cong K_n(X)$$

where n is any integer and K_n denotes the Quillen's higher K-theory if $X \in \mathrm{Sm}/S$ is divisorial and the higher thomason's K-theory of perfect complexes otherwise. In particular, for $n = 0$ one has

$$\mathrm{Hom}_{\mathcal{H}(S)}(X, \mathbb{Z} \times \mathrm{Gr}) \cong K_0(X)$$

where, again, $K_0(X)$ has to be interpreted in the sense of perfect complexes or in the one of vector bundles if X is divisorial or not. Moreover $K \cong \mathbb{Z}\times\mathrm{Gr} \cong \mathbb{Z}\times\mathrm{BGL}^+ \cong \mathbb{Z} \times \mathrm{BGL}$ in $\mathcal{H}(S)$.

The previous theorem provide an explicit geometric model of algebraic K-theory in the motivic setting. We also explicitly remind that smooth schemes over a regular separated base can be non-divisorial (see Remark 2.1.7). Indeed being smooth is a local property, so that they have to be regular, but being separated or having affine diagonal (i.e. being semi-separated over some base in the lingo of [TT90]) is not a local property so that a priori Lemma 2.1.5 does not apply and therefore such schemes might not have an ample family of line bundles or might not satisfy the so called resolution property. We will also need the following technical fact, known in literature as Jouanolou's trick ([Wei89], [Jou73]).

A.2.1 Jouanolou's trick

We start with a scheme S together with a locally free sheaf $\mathcal{E} \in \mathrm{Vect}(S)$. To this vector bundle we can associate a scheme $\mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$ over S as customary. We want to give to this scheme the structure of algebraic S-group scheme. In order to do this we need to give a functorial group structure to all the sets $\mathrm{Hom}_S(Y, \mathrm{Spec}(\mathrm{Sym}(\mathcal{E})))$ of S-scheme morphisms between a scheme $h : Y \to S$ over S and $\mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$. We remind ([GW10] 11.1.5 and Proposition 11.1) that we have functorial bijections

$$\mathrm{Hom}_S(Y, \mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))) \cong \mathrm{Hom}_{\mathcal{O}_S - \mathrm{algebras}}(\mathrm{Sym}(\mathcal{E}), h_* Y) \cong \mathrm{Hom}_{\mathcal{O}_S - \mathrm{modules}}(\mathcal{E}, h_* Y)$$

and the latter has a canonical group structure. We notice that on trivializing open subsets of S for $\mathcal{E}$, $\mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$ as a group is isomorphic to $\mathbb{G}_{a,S}^n$ (i.e. to $\mathbb{A}_S^n$, see [DG11] I 4.3.1). This suffices to give an S-group structure on $\mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$. We now remind the definition of torsor.

Definition A.2.3. ([Sta18, Tag 0497] 38.11) Let S be some base scheme, $\pi : X \to S$ a scheme over S, G an algebraic group S-scheme which acts on X via $\sigma : G \times_S X \to X$. X is called a *principal homogeneous space* or G-torsor if

1) The induced morphism of schemes $(\sigma \times id) : G \times_S X \to X \times_S X$ is an isomorphism. Equivalently for every S-scheme Y, the group $G(Y)$ acts simply transitively on $X(Y)$ or $X(Y)$ is empty.

2) There exists a Zariski covering $\{\gamma_i : U_i \to S\}_{i \in I}$ of S such that $X(U_i) \neq \emptyset$ for all $i \in I$.

A G-torsor X is called *trivial* if $X(S) \neq \emptyset$

Remark A.2.4. One can prove that a G-torsor is trivial if and only if it is isomorphic to G. This can be shown to be equivalent to having a section of the structure map $\pi : X \to S$. Hence condition 2) of the definition is equivalent to ask that there exists a Zariski covering $\{\gamma_i : U_i \to S\}_{i \in I}$ of S such that the G_{U_i}-torsors $\pi_i : \gamma_i^{-1}(X) \to U_i$ are trivial, which means that all the maps $\gamma_i^{-1}(X) \to S_i$ have a section, i.e. $\gamma_i^{-1}(X) \cong \gamma_i^{-1}(G)$ for all $i \in I$. Notice that π is always surjective and it is smooth, étale, flat, affine, etc. if and only if G is such (see [Mil80] Proposition III 4.2).

Definition A.2.5. [Wei89] An *affine vector bundle torsor* over a scheme X is an affine (over $\mathrm{Spec}(\mathbb{Z})$) scheme W together with an affine map $\pi : W \to X$ making W a G-torsor where $G = \mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$ for some vector bundle $\mathcal{E}$ over X. This

means that $\mathrm{Spec}(\mathrm{Sym}(\mathcal{E}))$ acts on W and that W is locally isomorphic to it, but the patching maps need not to be linear.

We can then state the Jouanaolou's trick.

Proposition A.2.6 (Jouanolou's Trick, [Wei89] Proposition 4.4)**.** *Let X be a scheme admitting an ample family of line bundles. Then there exists an affine (over $\mathbb{Z}$) torsor $T \to X$ under a vector bundle E.*

A.3 Functor φ and properties (ii) and (K). The theorem of Riou

All schemes in this section are supposed to be divisorial. The material presented here can be found in [Rio06] III.2.

Definition A.3.1. Let S be noetherian scheme and $X \in \mathcal{H}(S)$. Define a presheaf $\varphi X : (\mathrm{Sm}/S)^{op} \to \mathrm{Sets}$ as

$$\varphi X(U) := \mathrm{Hom}_{\mathcal{H}(S)}(U, X)$$

Remark A.3.2. From A.2.2 we have that if S is a regular scheme, we have an isomorphism $\varphi(\mathbb{Z} \times \mathrm{Gr}) \cong K_0(-)$ in $\mathbf{Pre}(\mathrm{Sm}/S)$. In some parts of the text we used the notation $\pi_0(X)$ instead of φX as in [Rio10]. However here we prefer to stick to the notation introduced in [Rio06].

Definition A.3.3. Every element $X \in \mathbf{Pre}(\mathrm{Sm}/S)$ can be seen as element in $\mathcal{H}(S)$ and one can apply to it φ to obtain a presheaf of sets φX. We define a morphism $\tau_X : X \to \varphi X$ in $\mathbf{Pre}(\mathrm{Sm}/S)$ in such a way that for every $U \in \mathrm{Sm}/S$,

$$\tau_{X,U} : X(U) \to \varphi(X)(U)$$

associate to $x \in X(U)$ the element in $\varphi(X)(U)$ obtained first by looking at x as an arrow $U \to X$ via the Yoneda lemma and then making this arrow simplicial (trivially) to obtain an arrow in $\mathcal{H}(S)$.

Definition A.3.4. (Property (ii)) Let S be a regular scheme and $X \in \mathbf{Pre}(\mathrm{Sm}/S)$. We say that X satisfies the *property (ii)* if for every $U \in \mathrm{Sm}/S$ affine (over $\mathrm{Spec}(\mathbb{Z})$), the arrow $\tau_{X,U}$ defined above is surjective.

This means that in this case every morphism $U \to X$ in $\mathcal{H}(S)$ with U affine comes from a genuine morphism of presheaves. Moreover, we explicitly note that if two presheaves satisfy the property (ii), then also their product does.

Proposition A.3.5 ([Rio06] Proposition III.10). *Let S be a regular scheme, and $X \in \mathbf{Pre}(\mathrm{Sm}/S)$ satisfying the property (ii). Let E be an object in $\mathcal{H}(S)$. Then the application*

$$\tau_X^* : \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\varphi X, \varphi E) \to \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(X, \varphi E)$$

is injective.

Proof. Suppose we have two morphisms $f, f' : \varphi X \rightrightarrows \varphi E$ such that $f \circ \tau_X = f' \circ \tau_X$. We want to show that for every $U \in \mathrm{Sm}/S$, the maps $f_U, f'_U : \varphi X(U) \rightrightarrows \varphi E(U)$ are equal. Thanks to the fact that X satisfies the property (ii), if $U \in \mathrm{Sm}/S$ is affine over $\mathbb{Z}$, the arrow $\tau_{X,U}$ is an epi and then $f_U = f'_U$. Let $U \in \mathrm{Sm}/S$, then using the Jouanolou's trick one gets an affine vector bundle torsor $\pi : T \to U$ so that the following diagram commutes

$$
\begin{array}{ccc}
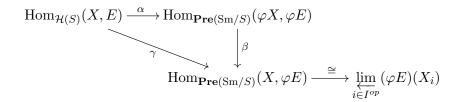\varphi X(T) & \overset{f_T}{\underset{f'_T}{\rightrightarrows}} & \varphi E(T) \\
\pi^* \big\uparrow & & \big\uparrow \pi^* \\
\varphi X(U) & \overset{f_U}{\underset{f'_U}{\rightrightarrows}} & \varphi E(U)
\end{array}
$$

Arguing as above, since T is affine, one has that $f_T = f'_T$. Hence $\pi^* \circ f_U = \pi^* \circ f'_U$, so if π^* is at least an injection (i.e. a mono), it will follow that $f_U = f'_U$. But since $\pi : T \to U$ is an affine vector bundle torsor, by [MV99] 3.2.3 $\pi^* : T \cong^{\mathbb{A}^1} U$ and so π^* is an isomorphism. $\qquad\square$

The main use of the property (ii) is the following

Theorem A.3.6. *[[Rio06] Theorem III.16] Let S be a regular scheme and $X_\bullet = (X_i)_{i \in I}$ an inductive system in Sm/S indexed by a directed ordered set I having a cofinal sequence. Set $X = \mathrm{colim} X_\bullet$ and suppose that X satisfy the (ii) property. Then for every H-group E we can form the diagram*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{H}(S)}(X, E) & \overset{\alpha}{\longrightarrow} & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\varphi X, \varphi E) \\
& \overset{\gamma}{\searrow} & \big\downarrow \beta \\
& & \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(X, \varphi E) \overset{\cong}{\longrightarrow} \varprojlim_{i \in I^{op}} (\varphi E)(X_i)
\end{array}
$$

where the maps α and γ are surjective and β is bijective. Moreover $\mathrm{Ker}\,\gamma = \mathrm{Ker}\,\alpha \cong R^1 \varprojlim_{i \in I^{op}} \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbb{S}^1 \wedge X_{i+}, E)$

Proof. α is induced by φ and β is the arrow τ_X^* considered before. γ is their composition. Since X satisfies the property (ii), then β is injective by the previous proposition. Moreover the Yoneda lemma gives us

$$\mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(\mathrm{colim}_{i \in I} X_i, \varphi E) \cong \varprojlim_{i \in I^{op}} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(X_i, \varphi E) \cong \varprojlim_{i \in I^{op}} (\varphi E)(X_i)$$

In addition, after taking a fibrant replacement $E \to E_f$

$$\pi_0 E(X_i) = \pi_0 \mathfrak{Map}(X_i, E_f) = [X_i, E]_{\mathcal{H}(S)} = (\varphi E)(X_i)$$

and so the arrow $\Lambda : \mathrm{Hom}_{\mathcal{H}(S)}(X, E) \to \varprojlim_{i \in I^{op}} (\varphi E)(X_i)$ is the one invoked by the Milnor's theorem A.1.6. Hence we can conclude that it is surjective, so β is also surjective and γ has the same property too. Invoking again A.1.7 and the calculation made at the end of Subsection A.1, one gets that $\mathrm{Ker}\,\gamma = \mathrm{Ker}\,\alpha$ is exactly of the desired form. $\qquad\square$

The pointed variant of the previous theorem also holds, see [Rio06] Theorem III.18.

Definition A.3.7. [Rio06] III.25] Let S be a regular scheme and $X_\bullet = (X_i)_{i \in I}$ an inductive system in Sm/S indexed by a directed ordered set I having a cofinal sequence such that $X = \mathrm{colim} X_\bullet$ satisfies the property (ii). Let E be an H-group in $\mathcal{H}_\bullet(S)$. We say that E satisfies the property (K) with respect to the system $X_\bullet$ if the arrow α as in the previous theorem is bijective, i.e. $R^1 \varprojlim_{i \in I^{op}} \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbb{S}^1 \wedge X_{i+}, E) = 0$.

Remark A.3.8. It is possible to show, using explicit calculations detailed in [Rio06] Proposition III.14, that $\mathbb{Z} \times \mathrm{Gr}$ colimit of the system $\mathcal{K}$ satisfies the property (ii).

Proposition A.3.9. *Let S be a regular scheme. If E is an H-group in $\mathcal{H}_\bullet(S)$ and for every $(d, r) \leq (d', r')$ the arrow*

$$f_{(d,r)(d',r')}^* : \mathrm{Hom}_{\mathcal{H}(S)}(\mathrm{Gr}_{d',r'}, R\Omega E) \to \mathrm{Hom}_{\mathcal{H}(S)}(\mathrm{Gr}_{d,r}, R\Omega E)$$

is surjective, then E satisfies the property (K) with respect to the system $\mathcal{K}$.

Proof. We said in the remark before that $\mathbb{Z} \times \mathrm{Gr}$ satisfies the property (ii) and so by the previous theorem, we just need to prove that $R^1 \varprojlim_{i \in (\mathbb{N}^2)^{op}} \mathrm{Hom}_{\mathcal{H}_\bullet(S)}(\mathbb{S}^1 \wedge \mathcal{K}_{i+}, E) =$

0. Since $\mathbb{N}^2$ has a cofinal sequence, we can simply calculate $R^1 \varprojlim_{n \in \mathbb{N}^{op}} \mathrm{Hom}_{\mathcal{H}_{\bullet}(S)}(\mathbb{S}^1 \wedge \mathcal{K}_{n+}, E) = 0$ where $\mathcal{K}_n = \bigsqcup_{2n+1} \mathrm{Gr}_{n,n}$. We just need to show that the tower $(\mathrm{Hom}_{\mathcal{H}_{\bullet}(S)}(\mathbb{S}^1 \wedge \mathcal{K}_{n+}, E))$ satisfies Mittag-Leffler (this is a tower in Ab since the fundamental group of an H-space is abelian). Now, in Ab

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{H}_{\bullet}(S)}(\mathbb{S}^1 \wedge \mathcal{K}_{n+}, E) &\cong \mathrm{Hom}_{\mathcal{H}(S)}(\mathcal{K}_n, R\Omega E) \\
&= \mathrm{Hom}_{\mathcal{H}(S)}(\bigsqcup_{2n+1} \mathrm{Gr}_{n,n}, R\Omega E) \\
&= \prod_{2n+1} \mathrm{Hom}_{\mathcal{H}(S)}(\mathrm{Gr}_{n,n}, R\Omega E)
\end{aligned}
$$

The "dual" transition morphisms $f_n : \bigsqcup_{2n+1} \mathrm{Gr}_{n,n} \to \bigsqcup_{2(n+1)+1} \mathrm{Gr}_{n+1,n+1}$ are given by

$$
i \circ (\sqcup f_{(n,n)(n+1,n+1)}) : \bigsqcup_{2n+1} \mathrm{Gr}_{n,n} \to \bigsqcup_{2n+1} \mathrm{Gr}_{n+1,n+1} \hookrightarrow \bigsqcup_{2(n+1)+1} \mathrm{Gr}_{n+1,n+1}
$$

and so the transition morphisms

$$
f_n^* : \prod_{2(n+1)+1} \mathrm{Hom}_{\mathcal{H}(S)}(\mathrm{Gr}_{n+1,n+1}, R\Omega E) \to \prod_{2n+1} \mathrm{Hom}_{\mathcal{H}(S)}(\mathrm{Gr}_{n,n}, R\Omega E)
$$

are given by

$$
(\prod f_{(n,n)(n+1,n+1)}^*) \circ i^* : \prod_{2(n+1)+1} \mathrm{Hom}_{\mathcal{H}(S)}(\mathrm{Gr}_{n+1,n+1}, R\Omega E) \twoheadrightarrow
$$

$$
\twoheadrightarrow \prod_{2n+1} \mathrm{Hom}_{\mathcal{H}(S)}(\mathrm{Gr}_{n+1,n+1}, R\Omega E) \to \prod_{2n+1} \mathrm{Hom}_{\mathcal{H}(S)}(\mathrm{Gr}_{n,n}, R\Omega E)
$$

But in Ab products of surjective maps are surjective, so $\prod f_{(n,n)(n+1,n+1)}^*$ is surjective and hence the f_n^* are all surjective. It follows that our tower satisfies the Mittag-Leffler condition. $\square$

Lemma A.3.10. *Let be E an H-group in $\mathcal{H}_{\bullet}(S)$ or in $\mathcal{I}_{Zar}^l \mathrm{Sch}_{S\,\bullet}$. Then for every object X of $\mathcal{H}_{\bullet}(S)$ (or of $\mathrm{Ho}(\mathcal{I}_{Zar}^l \mathrm{Sch}_{S\,\bullet})$) the evident morphism*

$$
\mathrm{Hom}_{\mathcal{H}_{\bullet}(S)}(X, E) \to \{f \in \mathrm{Hom}_{\mathcal{H}(S)}(X, E), f^{\star}(\bullet) = \bullet \in \mathrm{Hom}_{\mathcal{H}(S)}(S, E)\}
$$

is a bijection (same for $\mathrm{Ho}(\mathcal{I}_{Zar}^l \mathrm{Sch}_{S\,\bullet})$), where with $f^{\star}$ we have denoted the composition with f.

Proof. This is Lemme III.19 of [Rio06], the case of $\mathcal{I}^l_{Zar}\mathrm{Sch}_{S\,\bullet}$ being the same. We translate here the argument of Riou from the French for the reader's convenience. First one notices that we can assume E to be fibrant. Now denoting as $\mathcal{C}$ the model categories $\mathcal{I}^l_{Zar}\mathrm{Sch}_S$ or $\mathcal{I}^l_{Nis}\mathrm{Sm}/S$ localised at the A^1-weak equivalences, we have a cofiber sequence

$$S_+ \to X_+ \to X$$

in both $\mathcal{C}$ and $\mathcal{C}_\bullet$, obtained as the pushout of the two maps $S_+ \to X_+$ and $S_+ \to *$ (remark that all the objects involved are cofibrant and that $S_+ \to X_+$ is a cofibration so this pushout is actually an homotopy pushout). We can then apply to this cofiber sequence the pointed mapping space $\mathfrak{Map}_{\mathcal{C}_\bullet}(-, E)$ to get a fibration sequence

$$\mathfrak{Map}_{\mathcal{C}_\bullet}(X, E) \to \mathfrak{Map}_{\mathcal{C}_\bullet}(X_+, E) \to \mathfrak{Map}_{\mathcal{C}_\bullet}(S_+, E)$$

which induces a long exact sequence on the homotopy groups. Now, since E is an H-group, we have that the π_0 terms of this sequence are groups, so that, using the fact that the map $S_+ \to X_+$ has a retract induced by the terminal map $X \to S$ we can split the long exact sequence of the homotopy groups in short exact sequences, obtaining for the π_0 terms the following exact sequence

$$1 \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C}_\bullet)}(X, E) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, E) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(S, E) \to 1$$

that allows us to conclude the proof. $\qquad\square$

Remark A.3.11. The proof of the above lemma shows that the same result holds replacing $\mathcal{H}(S)$ with any model category coming from any simplicial model category $\mathcal{C}$ where every object is cofibrant, X with an element of $\mathcal{C}_\bullet$ so that its distinguished point is given by a cofibration and E with a fibrant H-group having the same property of X.

Remark A.3.12. Notice that if $\mathcal{C} = \mathbf{Pre}(\mathrm{Sch}_S)(\mathbf{Pre}(\mathrm{Sm}/S))$ then, S being final, if we take $F, G \in \mathcal{C}$ pointed by $a \in F(S)$ and $b \in G(S)$ then $f \in \mathrm{Hom}_{\mathcal{C}}(G, F)$ is pointed if and only if $f(S)(b) = a$ in $F(S)$.

From A.3.9 and the previous lemma, we can draw the following

Theorem A.3.13. *Let be S a regular scheme and $E \in \mathcal{H}(S)$ satisfying the property (K) with respect to the system $\mathcal{K}_\bullet$. Then one has the following bijections*

$$\mathrm{Hom}_{\mathcal{H}(S)}(\mathbb{Z} \times \mathrm{Gr}, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-), \varphi E)$$

$$\mathrm{Hom}_{\mathcal{H}_{\bullet}(S)}(\mathbb{Z} \times \mathrm{Gr}, E) \tilde{\rightarrow} \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)_{\bullet}}(K_0(-), \varphi E)$$

We can state and sketch the proof of the main theorem contained in [Rio06] and [Rio10].

Theorem A.3.14 (Riou [Rio06] III.31)**.** *If S is a regular scheme, then one has the following isomorphisms*

$$[K, K]_{\mathcal{H}(S)} \cong \mathrm{End}_{\mathcal{H}(S)}(\mathbb{Z} \times \mathrm{Gr}) \cong \mathrm{End}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-))$$

$$[K, K]_{\mathcal{H}_{\bullet}(S)} \cong \mathrm{End}_{\mathcal{H}_{\bullet}(S)}(\mathbb{Z} \times \mathrm{Gr}) \cong \mathrm{End}_{\mathbf{Pre}(\mathrm{Sm}/S)_{\bullet}}(K_0(-))$$

Proof. We sketch the proof of the unpointed case only, the pointed case being the same because of A.3.10. All we need to show is that $\mathbb{Z} \times \mathrm{Gr}$ satisfies the property (K) with respect to the system $\mathcal{K}_{\bullet}$. We want to use A.3.9. Hence all we need to show is that for every $(d, r) \leq (d', r')$, the arrow

$$f_{(r,d)(r',d')}^* : K_1(\mathrm{Gr}_{d',r'}) \cong \mathrm{Hom}_{\mathcal{H}(S)}(\mathbb{S}^1 \wedge \mathrm{Gr}_{d',r'+}, \mathbb{Z} \times \mathrm{Gr}) \rightarrow$$
$$\rightarrow \mathrm{Hom}_{\mathcal{H}(S)}(\mathbb{S}^1 \wedge \mathrm{Gr}_{d,r+}, \mathbb{Z} \times \mathrm{Gr}) \cong K_1(\mathrm{Gr}_{d,r})$$

is surjective. From the explicit calculations given in [SGA71], it is possible to show that for every $(d, r) \leq (d', r')$, the arrow $f_{(r,d)(r',d')}^* : K_0(\mathrm{Gr}_{d',r'}) \rightarrow K_0(\mathrm{Gr}_{d,r})$ is surjective. But now one can note that the Grassmannians admit a cellular decomposition (i.e. there is a sequence of closed subschemes $\emptyset = Z_0 \subset Z_1 \subset ... \subset Z_n = X$ such that for every $1 \leq i \leq n$, $Z_i - Z_{i-1} \cong$ an affine space $\mathbb{A}^d$ over S) and for such schemes the natural maps $K_0(X) \otimes_{K_0(S)} K_1(S) \rightarrow K_1(X)$ are bijections. Hence the surjectivity at the level of K_1 follows from the surjectivity at the level of K_0 and we conclude. Alternatively, one might use the semi-orthogonal decomposition of the Grasmannians to show that indeed $f_{(r,d)(r',d')}^* : K_n(\mathrm{Gr}_{d',r'}) \rightarrow K_n(\mathrm{Gr}_{d,r})$ is surjective for any natural number n. $\square$

To define operations on higher K-theory, however, we need to strengthen the former theorem a little, in order to consider morphisms from K^n to K. The result is the following, which is obtained with minor modifications from the above reasoning.

Theorem A.3.15 (Riou)**.** *If S is a regular scheme, then one has the following isomorphisms*

$$[K^n, K]_{\mathcal{H}(S)} \cong \mathrm{Hom}_{\mathcal{H}(S)}((\mathbb{Z} \times \mathrm{Gr})^n, \mathbb{Z} \times \mathrm{Gr}) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)}(K_0(-)^n, K_0(-))$$

$$[K^n, K]_{\mathcal{H}_{\bullet}(S)} \cong \mathrm{Hom}_{\mathcal{H}_{\bullet}(S)}((\mathbb{Z} \times \mathrm{Gr})^n, \mathbb{Z} \times \mathrm{Gr}) \cong \mathrm{Hom}_{\mathbf{Pre}(\mathrm{Sm}/S)_{\bullet}}(K_0(-)^n, K_0(-))$$

A.4 Recollections on generalizing the (ii) and the (K) properties

In this section we recall that the (ii) property and the (K) property defined by Riou in [Rio06] make sense in a wider context, which we need. Concerning the property (ii) this has been studied in Appendix B of the Riou's thesis [Rio06]. To ease the terminology and proving some simple lemmas we will need we prefer however to recollect here some easy to prove facts, deviating from *op. cit.*, for future use. Even if this discussion does not appear explicitly in the works of Riou, it is certainly contained there in a different form. We fix some base scheme S and we let Sch_S to be the category of schemes over S. We let $\mathcal{C}$ to be some Grothendieck site having underlying category some full subcategory of Sch_S. We assume that $s\mathbf{Pre}(\mathcal{C})$ comes endowed with a simplicial model category structure which comes as a left Bousfield localization of the Jardine injective local model structure on $s\mathbf{Pre}(\mathcal{C})$. We denote as $\mathcal{H}$ the homotopy category of $s\mathbf{Pre}(\mathcal{C})$ with this model structure. We can then give the following definition

Definition A.4.1. Assume we have $X \in \mathcal{H}$. Define a presheaf $\varphi X : (\mathcal{C})^{op} \to \mathrm{Sets}$ as
$$\varphi X(U) := \mathrm{Hom}_{\mathcal{H}}(U, X)$$

If $X \in \mathbf{Pre}(\mathcal{C})$, we define a morphism $\tau_X : X \to \varphi X$ in $\mathbf{Pre}(\mathcal{C})$ using the Yoneda lemma as usual.

We now come to the generalised version of the property (ii).

Definition A.4.2. Let $X \in \mathbf{Pre}(\mathcal{C})$. Assume we have a full subcategory $\mathcal{A}$ of $\mathcal{C}$ such that for every $B \in \mathcal{C}$ we have an element $A \in \mathcal{A}$ and an arrow $A \to B$ in $\mathcal{C}$ which induces an isomorphism in $\mathcal{H}$. Then we say that X satisfies the property (ii) relative to $\mathcal{A}$ if for every $U \in \mathcal{A}$, the arrow $\tau_{X,U}$ defined above is surjective.

Remark A.4.3. If $\mathcal{C} = \mathrm{Sm}/S$ is the category of smooth divisorial schemes, $\mathcal{A} = \mathrm{SmAff}/S$, S is regular noetherian and we consider the $\mathbb{A}^1$-localised Nisnevich injective local model structures over $s\mathbf{Pre}(\mathcal{C})$, then we are in the situation described in Appendix A.3, i.e. we have the usual property (ii). Note also that if two presheaves satisfy the property (ii) then also their product does.

For the rest of this section we will assume that we are in the situation described in the previous definition. It follows that the following Proposition can be proved exactly as its counterpart in Appendix A.3.
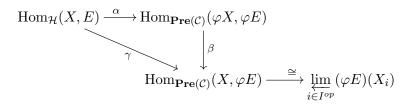
Proposition A.4.4. *Let $X \in \mathbf{Pre}(\mathcal{C})$ and E be an object in $\mathcal{H}$. Assume that X satisfies the property (ii) relative to $\mathcal{A}$. Then the application*

$$\tau_X^* : \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(\varphi X, \varphi E) \to \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(X, \varphi E)$$

is injective.

As a corollary we have the following theorem, whose proof is again the same of its counterpart in Appendix A.3.

Theorem A.4.5. *Let $X_\bullet = (X_i)_{i \in I}$ an inductive system in $\mathcal{C}$ indexed by a directed ordered set I having a cofinal sequence. Set $X = \mathrm{colim} X_\bullet$ and suppose that X satisfy the (ii) property relative to $\mathcal{A}$. Then for every H-group E we can form the diagram*

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{H}}(X, E) & \xrightarrow{\ \alpha\ } & \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(\varphi X, \varphi E) \\
& {\scriptstyle \gamma} \searrow & \downarrow {\scriptstyle \beta} \\
& & \mathrm{Hom}_{\mathbf{Pre}(\mathcal{C})}(X, \varphi E) \xrightarrow{\ \cong\ } \varprojlim_{i \in I^{op}} (\varphi E)(X_i)
\end{array}
$$

where the maps α and γ are surjective and β is bijective. Moreover $\mathrm{Ker}\,\gamma = \mathrm{Ker}\,\alpha \cong R^1 \varprojlim_{i \in I^{op}} \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^1 \wedge X_{i+}, E)$

We now conclude with the generalized version of the property (K).

Definition A.4.6. Let $X_\bullet = (X_i)_{i \in I}$ an inductive system in $\mathcal{C}$ indexed by a directed ordered set I having a cofinal sequence such that $X = \mathrm{colim} X_\bullet$ satisfies the property (ii) relative to $\mathcal{A}$. Let E be an H-group in $\mathcal{H}$. We say that E satisfies the property (K) with respect to the system $X_\bullet$ if the arrow α as in the previous theorem is bijective, i.e. $R^1 \varprojlim_{i \in I^{op}} \mathrm{Hom}_{\mathcal{H}_\bullet}(\mathbb{S}^1 \wedge X_{i+}, E) = 0$.

A.5 Recollections on algebraic structures

In this section we recollect some material from [Rio10] that we need in order to define operations on higher K-theory. What follows is contained in 2.1 and 2.2 of *op. cit.* although we deviate a little from the discussion found there (compare also with the discussion in [Bor94] Section 3.2 page 125).

Definition A.5.1. A language $\mathfrak{L}$ is the datum of a family of operators $(l_i, n_i)_{i \in I}$ of arity $n_i \in \mathbb{N}$, called *abstract operators*

Note that the previous definition is not the usual definition one can find in logic, for example in the context of first order logics, where a language consists of more than this datum, see for example [Men15] or [EFT94].

Definition A.5.2. Given a finite set of variables $(x_j)_{j \in J}$ and a language $\mathfrak{L} = (l_i, n_i)_{i \in I}$ we call *formula* an expression built inductively from the following passages:

- every variable x_j is a formula, called atomic formula;

- given any $i \in I$ and $F_1, ..., F_{n_i}$ formulas, then $l_i(F_1, ..., F_{n_i})$ is a formula.

Moreover we define an *algebraic structure* as the datum of a language $\mathfrak{L} = (l_i, n_i)_{i \in I}$, a finite set of variables $(x_j)_{j \in J}$ and a family of pairs of non atomic formulas in the above variables $(A_r, B_r)_{r \in R}$, called *relations* and denoted as $(A_r = B_r)$.

Note that this is non standard too since if we try to formalize the above in the context of first order logics, we have to be careful because symbols as $=$ are usually contained in the alphabet of a first order logic while $=$ is treated as an abstract operator in our sense, so that in the definitions usually found in literature, a relation is still a formula. Here we want to stick to Riou's notation since it is specific for the algebraic structure we want to consider, but the more scrupulous reader should take these definitions cum grano salis.

Definition A.5.3. Given a language $\mathfrak{L} = (l_i, n_i)_{i \in I}$ and a category $\mathcal{C}$ we say that an $\mathfrak{L}$-object is an object X of the category $\mathcal{C}$ such that all finite products of it exist and for every $i \in I$ we have a family of morphisms $X^{n_i} \to X$, denoted l_i^X.

Zero-arity operators (consider them as constants) are considered as maps $T \to X$ where $T := X^0$ is the empty product (if there is any, i.e. the terminal object) or a distinguished object we choose. $\mathfrak{L}$-objects form a category, arrows $X \to Y$ being maps $F : X \to Y$ in $\mathcal{C}$ such that $\forall i \in I$, the following diagram commutes

$$\begin{array}{ccc} X^{n_i} & \xrightarrow{\ l_i^X\ } & X \\ {\scriptstyle F \times \cdots \times F}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ Y^{n_i} & \xrightarrow[\ l_i^Y\ ]{} & Y \end{array}$$

Notice that if X is an $\mathfrak{L}$-object, one can define an arrow $F^X : X^{\alpha_F} \to X$ for any non atomic formula F, α_F being the sum of the atomic formulas involved in the inductive definition of the formula F.

Definition A.5.4. Let $\mathfrak{S} = ((l_i, n_i)_{i \in I}, (A_r = B_r)_{r \in R}, (x_j)_{j \in J})$ be an abstract algebraic structure. We say that an $\mathfrak{L}$-object in the category $\mathcal{C}$ is equipped with an $\mathfrak{S}$-structure if for every $r \in R$ the morphisms A_r^X and B_r^X in $\mathcal{C}$ are equal. In this case we say that X is an $\mathfrak{S}$-object. We define the category of $\mathfrak{S}$-objects as the full subcategory of $\mathfrak{L}$-objects consisting of the $\mathfrak{S}$-objects.
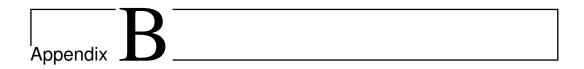
Remark A.5.5. When we interpret the language usually see the variables as elements of $\mathrm{Hom}_{\mathcal{C}}(A, X)$ with $A \in \mathcal{C}$ and in this case we interpret the other formulas via the Yoneda embedding.

Proposition A.5.6. *[[Rio10] Proposition 2.2.3] Fix an abstract algebraic structure $\mathfrak{S}$ and let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories such that finite products exist in $\mathcal{C}$ and F commutes with these products. Then if X is a $\mathfrak{S}$-object, $F(X)$ inherits a natural structure of $\mathfrak{S}$-object from F.*
On the other side, if for any natural n the natural map $\mathrm{Hom}_{\mathcal{C}}(X^n, X) \to \mathrm{Hom}_{\mathcal{D}}(F(X^n), F(X))$ is a bijection for some $X \in \mathrm{Ob}(\mathcal{C})$, then any $\mathfrak{S}$-structure on X uniquely arises from a $\mathfrak{S}$-structure on $F(X)$.
Now, fixed X, Y $\mathfrak{S}$-objects in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X^n, Y) \cong \mathrm{Hom}_{\mathcal{D}}(F(X^n), F(Y))$, we have that $f : X \to Y$ is a morphism of $\mathfrak{S}$-objects if and only if $F(f)$ is such.

Remark A.5.7. We apply this in the main part of the thesis with $\mathcal{C} = \mathcal{H}(S)$, $\mathcal{I}_{Zar}^l \mathrm{Sch}_S$, $\mathcal{D} = \mathbf{Pre}(\mathrm{Sch}_S, \mathrm{Sm}/S)$ and $F = \pi_0$ (which commutes with finite products).

Appendix B

On the categories $\mathrm{Sm}/S, \mathrm{Sch}_S, \mathcal{H}(S)$ and descent

We are going to be a little bit more precise about the hypothesis we need to work with. In particular we obtained our results avoiding the separated hypothesis and we obtained our embedding 2.0.1 of a divisorial scheme into a smooth one in such a way that the smooth scheme we obtain as a target of our embedding might be not separated. Many authors such as [MV99],[Mor06],[Mor04],[Rio06] and [Rio10] work in the framework of *separated* smooth schemes over a regular noetherian scheme S of finite dimension. Sometimes it is not explicitly pointed out in literature if the theorems we are using still hold if we remove the separation hypothesis even if that is the case, so some remarks concerning this issue are worthy, although everything that appears in this appendix is well known and can be found in literature, if one reads it correctly. For any scheme S (supposed to be noetherian of finite dimension unless otherwise stated) we denote as S_{Zar} the small Zariski site over S (which objects are open embeddings $U \subseteq S$ and we consider the Zariski topology), as $et_{|S}$ the étale site over S (objects being étale separated maps $Y \to S$ and we give to it the étale topology) and as $(et_{|S})_{Nis}$ the site having same category of $et_{|S}$ but the Nisnevich topology. For any scheme S we denote with Sch_S the category of schemes of finite type over S and with Sm/S its full subcategory of smooth schemes over S. Remark that the definition of étale and smooth morphisms a priori does not come with any separated hypothesis, i.e. we agree with EGA terminology, see Section 2.1.1 for detailed references.

B.1 BG property and descent

A good reference for this subsection is [Jar15] or [AHW17]. Everything contained here is well known, besides the exposition.

Definition B.1.1. Let $\mathcal{C}$ any Grothendieck site and let X be a simplicial presheaf. We say that X *satisfies descent* if for some (and hence for any) $\mathcal{I}^l$-fibrant replacement $i : X \to X_f$, i is a sectionwise weak equivalence.

Suppose to have a pullback square of schemes

$$
\begin{array}{ccc}
T & \longrightarrow & U \\
\downarrow & & \downarrow{\scriptstyle f} \\
V & \underset{j}{\longrightarrow} & W
\end{array}
$$

If j, f are open embeddings and $W = U \cup V$ then we denote such square as $\square_{Zar}$ and we call it a *Zariski square*. If j is an open embedding, f is an étale morphism and the induced morphism of closed subschemes (with the reduced induced structure) $f^{-1}(W-V) \to W-V$ is an isomorphism, we denote such a square as $\square_{Nis}$ and we call it *Nisnevich square*. Squares of this form are often called elementary distinguished squares.

Definition B.1.2. Consider a full subcategory $\mathcal{C} \subseteq \mathrm{Sch}_S$ considered as a site with the choice of the Zariski (or the Nisnevich) topology (which has then to be well defined), which for any object S in it contains also all the elements of S_{Zar} (or $et_{|S}$). We say that $X \in \mathrm{s}\mathbf{Pre}(\mathcal{C})$ has the *BG property* if $X(\emptyset)$ is contractible and $X(\square_{Zar})$ (or $X(\square_{Nis})$) is homotopy cartesian for any elementary distinguished square contained in $\mathcal{C}$.

The main theorem is then the following, due to Brown and Gersten [BG73] in the Zariski case and to Morel and Voevodsky [MV99] for the Nisnevich topology

Theorem B.1.3. *Let S be a noetherian scheme of finite dimension. Then if $F \in \mathrm{s}\mathbf{Pre}(S_{Zar})$ ($F \in \mathrm{s}\mathbf{Pre}((et_{|S})_{Nis})$ has the BG property, then it satisfies Zariski or Nisnevich descent. Moreover any $\mathcal{I}^l_{Zar}$ ($\mathcal{I}^l_{Nis}$)-fibrant simplicial presheaf has the BG property.*

Proof. This is proved in [Jar15], Theorems 5.33 and 5.39. $\qquad\square$

Corollary B.1.4. *Consider a full subcategory $\mathcal{C} \subseteq \mathrm{Sch}_S$ considered as a site with the choice of the Zariski (or the Nisnevich) topology as in B.1.2 and let be $F \in \mathrm{s}\mathbf{Pre}(\mathcal{C})$.*

If F has the BG property then it satisfies descent. Moreover any $\mathcal{I}^l_{Zar}$ $(\mathcal{I}^l_{Nis})$-fibrant simplicial presheaf has the BG property.

Proof. It follows from the previous theorem by restricting the big site to the corresponding small site for any object of $\mathcal{C}$ and using Corollary 5.24 of [Jar15] (that we can use because of [SGA72] Proposition III.1.3 and Proposition I.5.2). One might also use [AHW17] Theorem 3.2.5 (due to Voevodsky), see the refernces in *op. cit.*. $\square$

Remark B.1.5. The proof of all the above theorems really relies on the hypothesis of being noetherian of finite dimension: separatedness or smoothness are never used.

In the main text we also use the projective local model structure on simplicial presheaves. In this case, the BG property gives us more. The details are discussed in [DHI04].

Theorem B.1.6. *Let S be a noetherian scheme of finite dimension and $(\mathcal{C}, \tau)$ be (Sch_S, Zar) or (Sch_S, Nis). Assume F is a simplicial presheaf over one of those sites. If F satisfies descent and it is sectionwise fibrant then F is $\mathcal{P}^l$-fibrant.*

Proof. This is a rewriting of Corollary 6.3[DI04]. $\square$

Remark B.1.7. Suppose to have a $\mathcal{P}$-fibrant presheaf X such that there exist a presheaf X' which is $\mathcal{P}^l$-fibrant, a map between them and such that for every $U \in \mathrm{Ob}(\mathrm{Sch}_S)$, $X(U)$ is homotopy equivalent to $X'(U)$. Then because of the weak homotopy invariance of homotopy pullbacks, the presheaf X has the BG-property too and hence it is $\mathcal{P}^l$-fibrant.

Remark B.1.8. We explicitly note that for the categories of affine schemes, one has to modify a little the definition of Zariski and Nisnevich distinguished squares (in order to be coherent with the affine Zariski and Nisnevich topologies) to get a cd structure generating a topology equivalent to the Zariski or the Nisnevich one and to prove the analogue of Theorem B.1.3. However this can be accomplished as explained in [AHW17] so the above results will hold also for the Zariski and Nisnevich sites Aff/S and SmAff/S, provided one modified the definition of BG property as in *op. cit.* as long as S is noetherian.

B.2 Thomason definition of K-theory

We recall briefly the definition of the Waldhausen $S_\bullet$-construction.

Definition B.2.1. Suppose a natural number n is given. We define the *arrow category* $\mathrm{Ar}[n] := \mathrm{Fun}_{\mathrm{Cat}}([1], [n])$. Explicitly, its objects are couples of natural numbers (a, b) with $0 \leq a \leq b \leq n$ and we have an arrow $(a, b) \to (a', b')$ if and only if $a \leq a'$ and $b \leq b'$. One see that the assignment $[n] \mapsto \mathrm{Ar}[n]$ give rise to a cosimplicial category $\mathrm{Ar} : \Delta \to \mathrm{Cat}$.

If $A : \mathrm{Ar}[n] \to \mathcal{C}$ is any functor, we shall use the notation $A_{p,q}$ for the element $A((b, q)) \in \mathcal{C}$.

Definition B.2.2. Assume that $\mathcal{E}$ is an exact category. We can define, for every $n \in \mathbb{N}$, the category $S_n\mathcal{E} \subset \mathrm{Fun}_{\mathrm{Cat}}(\mathrm{Ar}[n], \mathcal{E})$ as the full subcategory of functors $A : \mathrm{Ar}[n] \to \mathcal{E}$ such that for every $0 \leq a \leq b \leq c \leq n$ the sequence

$$A_{a,b} \rightarrowtail A_{a,c} \twoheadrightarrow A_{b,c}$$

is an admissible exact sequence in $\mathcal{E}$ and $A_{d,d} = 0$ for all d.

$S_n\mathcal{E}$ has an exact structure which is given by declaring a sequence $A \to B \to C$ to be exact if for every $0 \leq a \leq b \leq n$ the sequence $A_{a,b} \rightarrowtail B_{a,b} \twoheadrightarrow C_{a,b}$ is exact in $\mathcal{E}$. The weak equivalences in such a category are defined sectionwise. If we have an exact category with weak equivalences $(\mathcal{E}, \omega)$. We write as $\omega\mathcal{E}$ for the category having as objects the same objects of $\mathcal{E}$ and as arrows the weak equivalences. We then get a simplicial category

$$\omega S_\bullet\mathcal{E} : \Delta^{op} \to \mathrm{Cat} \qquad [n] \mapsto \omega S_n\mathcal{E}$$

Taking the nerve in each degree makes $\omega S_\bullet\mathcal{E}$ into a bisimplicial set that can be realised ([TT90] 1.5.2). Hence we have the following

Definition B.2.3. Let be $(\mathcal{E}, \omega)$ an exact category with weak equivalences. We define
$$K^W(\mathcal{E}, \omega) := \Omega|N_\bullet \omega S_\bullet\mathcal{E}|$$

where we have used the geometric realization of a simplicial category.

This is considered to be one of the most useful definition of K theory, because of its nice properties. It can be shown that it gives the right definition of K-theory

Theorem B.2.4 ([Wal85] Appendix 1.9)**.** *If $(\mathcal{E}, i)$ is an exact category with weak equivalences where the weak equivalences are the subcategory of isomorphisms in $\mathcal{E}$, then we have an homotopy equivalence $|Q\mathcal{E}| \simeq |N_\bullet i S_\bullet\mathcal{E}|$.*

The space $|iS_\bullet\mathcal{E}|$ is the zeroth space of a spectrum because of [Wal85] 1.3.3 and 1.5.3. This spectrum is the one used in the work of Thomason on higher algebraic K-theory [TT90]. Indeed Thomason defined for every scheme quasi-compact and quasi-separated X the K-theory space as $\Omega|N_\bullet\omega S_\bullet Perf(X)|$ ([TT90] 3.1) where $Perf(X)$ denotes the exact category of perfect complexes of globally finite Tor-amplitude ([TT90] 2.2.11) and the weak equivalences are the quasi-isomorphism. He noticed that this presheaf of pointed topological space arises as the zeroth space of a presheaf of spectra on the category of quasi-compact and quasi-separated schemes. Applying the singular functor gives us a simplicial presheaf K^T. Now, if our schemes are divisorial, Thomason is able to prove that there is a sectionwise weak equivalence $K^Q \to K^T$ where K is the Quillen's K-theory presheaf built starting from the presheaf $\mathrm{Vect}(-)$ and then defining $K^Q(-) := \mathrm{Sing}(\Omega|NQ\mathrm{Vect}(-)|) \simeq \Omega\mathrm{Ex}^\infty(NQ\mathrm{Vect}(-))$. This follows by [TT90] Corollary 3.9 and then Proposition 3.10.

B.3 Ties with Thomason descent and K-theory

We formulated the notion of descent and BG property using the modern language of model categories but we have in mind applications to K-theory. For those applications the paper [TT90] is particularly relevant but it doesn't make use of the former methods so that since we want to use the results contained in that seminal paper, a brief discussion is due. In particular, we have to make explicit the link between fibrant replacement and hypercohomology. Given any presheaf of spectra $\mathcal{E}$ (for presheaves of spectra see[Jar15] for example) on any site $\mathcal{C} = (\mathcal{C}, \tau)$ with enough points we can define the hypercohomology presheaf $\mathbb{H}_\mathcal{C}(-, \mathcal{E})$ using the Godement resolution (see [Mit97] for an explanation and the relevant references). This way we always get a map $\mathbb{H}_\mathcal{E} : \mathcal{E} \to \mathbb{H}_\mathcal{C}(-, \mathcal{E})$. Moreover the following holds

Proposition B.3.1. *Let $\mathcal{C} = (\mathcal{C}, \tau)$ be a site with enough points, $\mathcal{E}$ a presheaf of simplicial spectra and $\mathbb{H}_\mathcal{E} : \mathcal{E} \to \mathbb{H}_\mathcal{C}(-, \mathcal{E})$ the natural map mentioned above. Then $\mathbb{H}_\mathcal{C}(-, \mathcal{E})$ is $\mathcal{I}^l$-fibrant and if $cd^\mathcal{C}\mathcal{E} < \infty$ ($cd^\mathcal{C}\mathcal{E}$ being the cohomological dimension) dimension, $\mathbb{H}_\mathcal{E}$ is a local weak equivalence.*

For a proof see [Mit97] 3.20. As a simple corollary we have

Corollary B.3.2. *In the hypothesis of the previous proposition, $\mathcal{E}$ satisfies descent if and only if $\mathbb{H}_\mathcal{E}$ is a sectionwise weak equivalence. If $\mathcal{E}$ satisfies descent, the resulting simplicial presheaf we get after taking the 0th space of $\mathcal{E}$ (i.e after taking $\boldsymbol{R}\Omega^\infty$) satisfies descent.*

We can then use the previous result to translate descent results as in [TT90] in the language of model categories, so that they become available in the framework of (un)stable motivic homotopy theory. We state the descent results we need for our study of K-theory using [TT90]. It is a deep theorem of [TT90] (Theorem 7.6) that for schemes quasi-compact and quasi-separated having an ample family of line bundles, the Quillen's K-theory spectrum is weakly equivalent to the Thomason Trubaugh K-theory spectrum (see [TT90] Proposition 3.10 for a proof and *op.cit.* for the precise definitions). Moreover, by B.1.7, if the Thomason Trubaugh K-theory has the BG property, then also the Quillen's one has that property. Hence under the above hypothesis on our category of schemes we can use the descent results proved in [TT90] for our K-theory presheaf.

Remark B.3.3. One might wonder if considering divisorial schemes we actually obtain well behaved Grotendieck topologies. Since all our schemes are noetherian, open embeddings are quasi-compact and then defining Zariski covers as families of open embeddings jointly surjective gives us a well defined Grothendieck topology on the category of divisorial schemes since quasi-compact open subscheme of divisorial schemes are divisorial. We restrict now to the category Sm/S of smooth divisorial schemes over a regular divisorial scheme S as in the main text and we ask if the Nisnevich topolgy is well defined. We have that the pullback of an étale map $f : X \to Y$ in Sm/S along any map in Sm/S is divisorial because the resulting scheme is smooth over S (smoothness is stable under base change), hence regular, and has affine diagonal so that the result follows from Corollary 2.1.5 (remind that all the divisorial schemes have affine diagonal). Indeed if we have morphisms $f : X \to Y$ and $g : Y \to Z$ so that g and $g \circ f$ have affine diagonal, then also f has affine diagonal because we can mimick the proof of [Sta18, Tag 01KV] using the fact that every affine morphism is separated ([Sta18, Tag 01S7]) and Remark 9.11 of [GW10], page 230 (or [Sta18, Tag 01SG]). This implies that any morphism between schemes having affine diagonal has affine diagonal. We remark that instead of the notion of having affine diagonal, one can use the equivalent notion of semi-separated schemes and morphisms detailed in [TT90] Appendix B.7 that we find less explicit, although equivalent. in *op. cit.* one can find observations similar to the ones we just made on the schemes and the morphisms having affine diagonal in terms of semi-separatedness. Since étale maps are smooth maps, the Nisnevich topology is well defined over Sm/S (notice that the proof of [MV99] Proposition 1.4 page 96 goes through and that the hypothesis of Theorem 3.2.5 in [AHW17] are met) and the inclusion of Sm/S into the bigger category of possibly non divisorial smooth schemes over S, call it $\widetilde{\mathrm{Sm}/S}$ induces a functor $\mathrm{s}\mathbf{Pre}(\widetilde{\mathrm{Sm}/S}) \to \mathrm{s}\mathbf{Pre}(\mathrm{Sm}/S)$ which sends

Nisnevich locally fibrant objects to Nisnevich fibrant objects and weak equivalences between them to weak equivalences.

Theorem B.3.4. *Let* Sch_S *be the category of quasi-compact and quasi-separated schemes admitting an ample family of line bundles and of finite type over a Noetherian base scheme S of finite dimension. Then the simplicial presheaf K satisfies Zariski descent and Nisnevich descent when restricted to smooth schemes.*

Proof. This follows by deep work of [TT90], in particular [TT90] Theorem 8.1 shows that we have the Zariski BG property, and so Zariski descent (which is also explicitly stated in [TT90] 10.3), and Theorem 10.8 of [TT90] gives the Nisnevich descent. $\square$

As a consequence, we can use the Thomason's presheaf K^T or the Quillen's indifferently since we are in presence of ample families of line bundles in order to represent K-theory in $\mathrm{Ho}(\mathcal{H}^{\mathrm{Sch}_S}_{Zar})$ or in $\mathrm{Ho}(\mathcal{I}^l_{Nis}\mathrm{Sm}/S)$, i.e. for any scheme $X \in \mathrm{Sch}_S$,

$$[\mathbb{S}^n \wedge X_+, K]_{\mathcal{I}^l_{Zar}\mathrm{Sch}_S, \bullet} \cong K_n(X)$$

and the same holds if X is divisorial smooth and we consider the Nisnevich topology. Considering from now on only divisorial schemes, recall once more that there exists a local Zariski-weak equivalence $K \simeq \mathbb{Z} \times \mathrm{BGL}^+$ in the homotopy category of simplicial presheaves over the Zariski site Sch_S for any S noetherian (see [GS99] or [ST15]). Moreover, we assume S to be a regular noetherian of finite Krull dimension and that we consider the category Sm/S. If we Bousfield localize $\mathcal{I}^l_{Nis}\mathrm{Sm}/S$ at the collection of maps of the form $\mathbb{A}^1 \times_S X \to X$ getting $\mathcal{H}(S)$, we get that the K-theory presheaf is $\mathbb{A}^1$-local because of [TT90] Proposition 6.8, therefore its $\mathcal{I}^l$-fibrant replacement is $\mathbb{A}^1$-local too and hence fibrant in this model structure so that we also have

$$[X_+ \wedge \mathbb{S}^n, K]_{\mathcal{H}_\bullet(S)} \cong K_n(X)$$

meaning that actually K represents the usual K-theory. Moreover, we have a map of simplicial presheaves $\mathbb{Z} \times \mathrm{Gr} \to K$ over the category Sm/S, and its construction does not require any separated hypothesis, see [ST15]. It is proved in *op. cit.* that it is a Nisnevich $\mathbb{A}^1$-weak-equivalence. We have then all the ingredients necessary to prove Theorem A.3.14 without the assumption of separatedness.

B.4 On the functoriality of the K-theory functor

Consider Sch_S, the category of schemes of finite type over a noetherian base scheme S. The assignment $X \mapsto \mathrm{Vect}(X)$ strictly speaking does not define a functor but

only a pseudo functor (since $(f \circ g)^* \neq f^* g^*$ in general, we only have an isomorphism). Then in order to define K-theory as a simplicial presheaf using the Quillen's Q-construction we need to strictify this assignment in order to get a functor. This issue has been solved in several ways in literature by consctucting categories $\mathcal{P}(X)$ equivalent to $\text{Vect}(X)$ for any $X \in \text{Sch}_S$ such that it is possible to define an assignment $X \mapsto \mathcal{P}(X)$ which defines a functor. One can then apply the sectionwise the Quillen's Q-construction (i.e consider the assignment $X \mapsto \Omega \text{Ex}^\infty(NQ\mathcal{P}(X))$) in order to get a simplicial presheaf representing K-theory, for example. Whenever we consider the functor Vect through this work we then assume that it has been rectified with one of these methods so that it is really a functor. We now briefly recall the method used in [FS02] Appendix C.4, i.e we recall briefly what the big vector bundles are and how to use them to build a functor $\mathcal{P} : \text{Sch}_S \to \text{Cat}$ with the properties stated before.

Definition B.4.1. Given the category Sch_S, a *big vector bundle* on $X \in \text{Sch}_S$ is a family $(P_Y \in \text{Vect}(Y))_{Y \in \text{Sch}/X}$ of vector bundles together with a datum of isomorphisms $(f^* P_Z \to P_Y)_{\text{Sch}/X \ni f: Y \to Z}$ such that $P : \text{Sch}/X \to \text{Ab}$ is an $\mathcal{O}$-module. We denote as $\mathcal{P}(\text{Sch}/X)$ the category of big vector bundles over X (seen as a full subcategory of $\mathcal{O}$-modules).
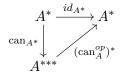
It is possible to check that the functor $\mathcal{P}(\text{Sch}/X) \to \text{Vect}(X)$, $(P_Y)_{Y \in \text{Sch}/X} \mapsto P_X$ is an equivalence of categories, see [FS02] C.4 and [Gra95]. Given any map $f : Y \to X$, the pullback induces a restriction functor $\mathcal{P}(\text{Sch}/X) \to \mathcal{P}(\text{Sch}/Y)$ so that the assignment $X \mapsto \mathcal{P}(\text{Sch}/X)$ defines a functor $\mathcal{P}$ which is equivalent to the pseudo-functor Vect and so has the properties desired.

B.5   Recollections on Hermitian K-theory

We introduce briefly some fundamental tools we need into the thesis

Definition B.5.1. An *exact category with weak equivalences and duality* is the datum $(\mathcal{E}, \omega, *, \text{can})$ of an exact category $\mathcal{E}$, a set of weak equivalences ω (which we can see as a subcategory of $\mathcal{E}$ closed under retracts, push-out along inflations, pullbacks along deflations, containing all the isomorphisms and whose arrows satisfy the 2/3 property), $* : \mathcal{E}^{op} \to \mathcal{E}$ an exact functor with the property that $*(\omega) \subseteq \omega$

and can : $id_{\mathcal{E}} \xrightarrow{\simeq} **^{op}$ a natural transformation such that for every object A of $\mathcal{E}$, $\mathrm{can}_A \in \omega$ and $(\mathrm{can}_A^{op})^* \circ \mathrm{can}_{A^*} = id_{A^*}$ i.e. the following diagram commutes

$$
\begin{array}{ccc}
A^* & \xrightarrow{\;id_{A^*}\;} & A^* \\
\scriptstyle{\mathrm{can}_{A^*}}\downarrow & \nearrow_{\scriptstyle{(\mathrm{can}_A^{op})^*}} & \\
A^{***} & &
\end{array}
$$

Example B.5.2. 1) For any scheme X and any line bundle $\mathcal{L}$, the quadruple $(\mathrm{Vect}(X), iso, \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{L}), \mathrm{can})$ where iso is the subcategory of isomorphisms and can is the canonical natural isomorphism associated with $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{L})$ is an exact category with weak equivalences and duality.

2) For any scheme X, any line bundle $\mathcal{L}$ and any integer n, the quadruple $(\mathrm{Ch}^b\mathrm{Vect}(X), quis, \mathrm{Hom}^\bullet_{\mathcal{O}_X}(-, \mathcal{L}[n]), \mathrm{can})$ where $\mathrm{Ch}^b\mathrm{Vect}(X)$ is the exact category of bounded chain complexes of vector bundles, $quis$ are the quasi-isomorphisms, $\mathcal{L}[n]$ denotes the line bundle $\mathcal{L}$ seen as a chain complex concentrated in degree $-n$, $\mathrm{Hom}^\bullet_{\mathcal{O}_X}(-, \mathcal{L}[n])$ is the internal hom complex and can is the appropriate duality is an exact category with weak equivalences and duality. One can do the same by replacing $\mathrm{Ch}^b\mathrm{Vect}(X)$ with $Perf(X)$, the exact category of perfect complexes on X.

One can have a more general notion of dg category with weak equivalences and duality as in [Sch17] but we do not need it. For any exact category with weak equivalences and duality $(\mathcal{E}, \omega, *, \mathrm{can})$ it is possible to define an H-group

$$\mathrm{GW}(\mathcal{E}, \omega, *, \mathrm{can})$$

whose homotopy groups are the Grothendieck-Witt groups defined classically, see [Sch17] or [Sch10] Definition 3, for example. This construction is functorial in categories with weak equivalences and duality. Moreover, it can be shown that for the category of divisorial schemes over some base S, the assignment

$$X \mapsto (\mathrm{Ch}^b\mathrm{Vect}(X), quis, \mathrm{Hom}^\bullet_{\mathcal{O}_X}(-, \mathcal{L}[n]), \mathrm{can})$$

can be made functorial.

Definition B.5.3. Let be Sch_S the category of divisorial schemes of finite type over some regular base scheme S and $\mathcal{L}$ a line bundle over S. We define the simplicial presheaf $\mathrm{GW}^{[n]}(\mathcal{L}) \in \mathrm{s}\mathbf{Pre}(\mathrm{Sch}_S)$ by the assignment

$$p : X \to S \mapsto \mathrm{GW}(\mathrm{Ch}^b\mathrm{Vect}(X), quis, \mathrm{Hom}^\bullet_{\mathcal{O}_X}(-, p^*\mathcal{L}[n]), \mathrm{can}) =: \mathrm{GW}^{[n]}(X, \mathcal{L})$$

We define the n-shifted Grothendieck-Witt groups of $X \in \mathrm{Sch}_S$ as $\pi_i(\mathrm{GW}^{[n]}(X, \mathcal{L}))$. If $\mathcal{L} = \mathcal{O}_S$ we will denote this simplicial presheaf simply as $\mathrm{GW}^{[n]}$.

Remark B.5.4. The restriction of $\mathrm{GW}^{[n]}$ to smooth schemes gives a pointed simplicial presheaf in $\mathcal{H}(S)$. Moreover, $\mathrm{GW}^{[n]}$ is an H-group in all the homotopy categories we considered in the text. If $n = 0$ we get GW that we considered in Section 6.2, i.e. symmetric hermitian K-theory, while if we take $n = 2$ we get the symplectic hermitian K-theory we considered in Section 6.3.

We also have the same descent results we have for K-theory. In particular

Theorem B.5.5 ([Sch17] Theorems 9.7-9.8)**.** *Consider the category of divisorial schemes Sch_S finite dimensional and of fine type over a regular base scheme S with $\frac{1}{2} \in \Gamma(S, \mathcal{O}_S)$. Then for any $n \in \mathbb{Z}$, $\mathrm{GW}^{[n]}$ satisfies Zariski and Nisnevich descent and it is $\mathbb{A}^1$-invariant over regular schemes.*

Proof. See *op. cit.* and notice that the assumption of separatedness is used in Theorem 9.8 there to make Theorem 3.4 of [Bal01] work. However, we can replace the hypothesis of being separated with the hypothesis of having an affine diagonal in the proof of this last theorem (just start with a semi-separated affine open covering, i.e. with an open covering consisting of affine schemes such that their intersection is still affine) and the proof goes through as well. $\qquad\square$

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