MODULAR FORMS: EXAMPLE SHEET 4

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Exercises

Question 1. *Let $m, n \in \mathbb{Z}$. Show that the modular symbol $\{m, n\}$ is 0 in $\mathbb{M}_2(\Gamma_0(N))$ for every N.*

Answer 1. We know that $\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0$ in $\mathbb{M}_2$, so $\{m, n\} = \{\infty, n\} - \{\infty, m\}$. But $\{\infty, n\} - \{\infty, m\}$ is of the form $gx - x$ where $g = \begin{pmatrix} 1 & n-m \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, so it is zero in the quotient $\mathbb{M}_2(\Gamma_0(N))$.

Question 2. *Find a basis for the space $\mathbb{M}_2(\Gamma_0(11), \mathbb{Q})$, and calculate the matrix of the Hecke operator T_3 acting on your space. Verify that its characteristic polynomial is $(x-4)(x+1)^2$.*

(NB: There was a mistake in the circulated example sheet: the characteristic polynomial was given as $(x-3)(x+2)^2$, which the sharp-eyed will have realised is actually the characteristic polynomial of T_2 rather than T_3. The double root comes from the cuspidal subspace; recall that this is isomorphic to a direct sum of two copies of the dual of $S_2(\Gamma, \mathbb{Q})$. The single root corresponds to a lifting of the boundary space $\mathbb{B}_2(\Gamma, \mathbb{Q})$ to a subspace of $\mathbb{M}_2(\Gamma, \mathbb{Q})$, which is isomorphic as a Hecke module to the space of weight 2 Eisenstein series; the latter space is 1-dimensional spanned by a form with T_n-eigenvalue $\sigma_1(n)$ for all n coprime to the level. Hence the single root for T_3 has to be 4.)

Answer 2. We note that a set of coset representatives for $\Gamma_0(11) \backslash \mathrm{SL}_2(\mathbb{Z})$ is given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \right\}_{i \in 0 \dots 10} \cup \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Let these be $r_0, \dots, r_{10}$ and r_{11}. It is helpful at this point to write a computer program which, given an index i and an element $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, will output the index j such that $\Gamma r_i g = \Gamma r_j$. Using such a program, or by hand calculation, one can easily check that the 2-term relations are

$$\left\{ \begin{matrix} r_0 + r_{11} = 0, & r_1 + r_{10} = 0, & r_2 + r_5 = 0, \\ r_3 + r_7 = 0, & r_4 + r_8 = 0, & r_6 + r_9 = 0 \end{matrix} \right\}.$$

Similarly the 3-term relations are

$$\left\{ \begin{matrix} r_0 + r_{10} + r_{11} = 0, & r_1 + r_5 + r_9 = 0, \\ r_2 + r_4 + r_7 = 0, & r_3 + r_6 + r_8 = 0 \end{matrix} \right\}.$$

Thus the space of modular symbols is isomorphic to $\mathrm{Man}(\Gamma) = \mathbb{Z}^{12}$, modulo the submodule generated by the 10 vectors above. One can easily check that the subspace $I \subseteq \mathbb{Q}^{12}$ obtained by tensoring the relation submodule with $\mathbb{Q}$ – which is just the $\mathbb{Q}$-span of the vectors corresponding to the relations – has dimension 9, and if J denotes the subspace generated by the basis vectors $\{r_0, r_2, r_3\}$, then $\mathrm{Man}(\Gamma, \mathbb{Q}) = I \oplus J$. So the images of r_0, r_2 and r_3 are a basis for $\mathbb{M}_2(\Gamma, \mathbb{Q})$.

Now let's find out how T_3 acts on these vectors. Recall that the image of r_i is the modular symbol $r_i \cdot \{0, \infty\}$.

$$\begin{aligned} T_3(r_0 \cdot \{0, \infty\}) = T_3(\{0, \infty\}) &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \right] \{0, \infty\} \\ &= \{0, \infty\} + \{\tfrac{1}{3}, \infty\} + \{\tfrac{2}{3}, \infty\} + \{0, \infty\} \end{aligned}$$

Using Manin's continued fraction algorithm, we see that

$$\begin{aligned} \{\tfrac{1}{3},\infty\} &= \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}\right]\cdot\{0,\infty\} &= r_0 - r_3 \\ \{\tfrac{2}{3},\infty\} &= \left[-\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}2 & -1\\ 3 & -1\end{pmatrix}\right]\cdot\{0,\infty\} &= -r_8 - r_{11} \end{aligned}$$

We note that modulo the relations, $-r_8 - r_{11} = r_0 - r_2 + r_3$. Thus $T_3 r_0 = 2r_0 + (r_0 - r_3) + (r_0 - r_2 + r_3) = 4r_0 - r_2$. Similar calculations show that $T_3 r_2 = -r_2$ and $T_3 r_3 = -r_3$.

So the matrix of T_3 is $\begin{pmatrix}4 & 0 & 0\\ -1 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$, which has the characteristic polynomial above.

(Note that one can verify this in Sage as follows:

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sage: ModularSymbols(Gamma0(11), 2).hecke_matrix(3)
[ 4  0 -1]
[ 0 -1  0]
[ 0  0 -1]
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This looks slightly different from the result above, since Sage writes matrices on the right rather than on the left, and also Sage chooses a different subset of Manin symbols – r_{11}, r_8 and r_9 – which are in fact equivalent to the ones we use up to sign, but ordered differently.)

Question 3. *Use Manin symbols to show that the quotient* $\mathbb{M}_2(\mathbb{Z})/K_2(\Gamma_0(13),\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}\times\mathbb{Z}/2\times\mathbb{Z}/3$.

Answer 3. As in the previous question, we know that $\mathbb{M}_2(\mathbb{Z})/K_2(\Gamma,\mathbb{Z})$ is isomorphic to a space of Manin symbols modulo 2-term and 3-term relations. In this case the space of Manin symbols is 14-dimensional and there are 28 relations.

A standard theorem in linear algebra (sometimes known as "Smith normal form") states that if A is a matrix over a Noetherian PID R, then there exist invertible matrices U, V and a diagonal matrix D, all over R, such that $A = UDV$, with D a diagonal matrix whose diagonal entries d_i (the elementary divisors of A) satisfy $d_i | d_{i+1}$ for all i. It is clear that if A denotes the matrix with 14 rows and 28 columns whose ith column is the ith relation, then we want to calculate $\mathbb{Z}^{14}/(A\cdot\mathbb{Z}^{28})$. If we factorise A in the above form, then $A\cdot\mathbb{Z}^{28} = UD\cdot\mathbb{Z}^{28}$, and since U is invertible, $\mathbb{Z}^{14}/(UD\cdot\mathbb{Z}^{28}) \cong U^{-1}\cdot\mathbb{Z}^{14}/D\cdot\mathbb{Z}^{28} = \mathbb{Z}^{14}/D\cdot\mathbb{Z}^{28}$.

The diagonal entries of the Smith normal form of A are twelve 1's, a 6 and the remaining entries zero. So $\mathbb{M}_2(\mathbb{Z})/K_2(\Gamma,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^{14}$ modulo the subspace spanned by the basis vectors $e_1,\dots,e_{12}$ and $6e_{13}$, so it is isomorphic as an abelian group to $\mathbb{Z}\oplus\mathbb{Z}/6$.

(Smith normal form over $\mathbb{Z}$ is essentially a more explicit variant of the structure theorem for finite abelian groups; see any basic algebra textbook.)

Question 4. *Let x be the element of* $H_1(\Gamma_0(11),\mathbb{R})$ *corresponding to the modular symbol* $\{0,\infty\}$. *Using the methods we used to prove the Manin-Drinfeld theorem, show that $5x$ is in* $H_1(\Gamma_0(11),\mathbb{Z})$. *Hence show that the unique cusp form* $f = \sum a_n q^n$ *of weight 2 and level* $\Gamma_0(11)$ *satisfies* $a_n = \sigma_1(n) \bmod 5$.

Answer 4. We saw above that r_0, r_2 and r_3 are a basis of $\mathbb{M}_2(\mathbb{Q})/K_2(\Gamma,\mathbb{Q})$ in this case. In fact it is easy to see that these three symbols are a basis over $\mathbb{Z}$, and $\mathbb{M}_2(\mathbb{Z})/K_2(\Gamma,\mathbb{Z})$ is torsion-free, so they are a basis for the free $\mathbb{Z}$-module $\mathbb{M}_2(\Gamma) = H_1(X(\Gamma),\mathbb{Z},\mathit{cusps})$.

Now, it's clear that the cusps $\frac{1}{3}$ and $\frac{2}{3}$ are both equivalent to ∞, so r_2 and r_3 are in the space $\mathbb{S}_2(\Gamma)$ of cuspidal symbols, whereas r_0 clearly is not.

However, we saw above that $T_3 r_0 = 4r_0 - r_2$. Hence $(T_3 - 4)r_0$ is in $\mathbb{S}_2(\Gamma)$. However, the matrix of T_3 on $\mathbb{S}_2(\Gamma)$ is simply $-I$ where I is the identity; so the matrix of $T_3 - 4$ on $\mathbb{S}_2(\Gamma)$, and thus on $\mathbb{S}_2(\Gamma,\mathbb{R})$, is $-5I$. Thus if we consider $\{0,\infty\}$ as an element of $\mathbb{S}_2(\Gamma,\mathbb{R})$, we see that $5\{0,\infty\}\in\mathbb{S}_2(\Gamma,\mathbb{Z})$.

Now suppose n is arbitrary. It is easy to see that T_n acts on $\mathbb{B}_2(\Gamma)$ as the integer $\sigma_1(n)$. Thus $T_n\{0,\infty\} = \sigma_1(n)\cdot\{0,\infty\} + s$ where s is a cuspidal symbol. Thus if the (integral) matrix of $T_n - \sigma_1(n)$ on $\mathbb{S}_2(\Gamma)$ were invertible modulo 5, we would be able to deduce that $\{0,\infty\}\in\mathbb{S}_2$, which is absurd. This implies the stated congruence for the eigenvalues of T_n on $\mathbb{S}_2(\Gamma)$, and hence the coefficients of the corresponding cusp form.

(In general, the denominator of the homology class of $\{0,\infty\}$ is divisible by exactly those primes modulo which there are congruences between cusp forms and Eisenstein series.)

Question 5. *Let $K = \mathbb{Q}(\sqrt{-1})$, and let $\mathrm{SL}_2(\mathcal{O}_K)$ act on $\mathbb{P}^1(K)$ in the obvious way (as in Question 9 of sheet 2). Let $\mathbb{M}_2^K(\mathbb{Z})$ be the space of formal $\mathbb{Z}$-linear combinations of pairs $\{\alpha, \beta\}$, with $\alpha, \beta \in \mathbb{P}^1(K)$, modulo the usual relation $\{\alpha,\beta\} + \{\beta,\gamma\} + \{\gamma,\alpha\} = 0$. Show that $\mathbb{M}_2^K(\mathbb{Z})$ is cyclic as a module over $\mathbb{Z}[\mathrm{SL}_2(\mathcal{O}_K)]$.*

Answer 5. This follows immediately from the fact that $\mathcal{O}_K$ is a Euclidean domain. For instance, suppose we consider the symbol $\left\{0, \frac{21i+19}{10i+7}\right\}$. We find that

$$\begin{array}{rcccl} 21i+19 &=& 2(10i+17) &+& (i+5) \\ 10i+17 &=& (2i+2)(i+5) &+& (-2i-1) \\ i+5 &=& (-2i-2)(2i-1) &+& i \\ -2i-1 &=& (i-2)i. && \end{array}$$

One now writes

$$\frac{21i+19}{10i+7} = 2 + \frac{i+5}{10i+7} = 2 + \cfrac{1}{2+2i+\cfrac{1}{2i-1+\cfrac{1}{i-2}}}$$

with convergents

$$\left[2, \frac{4i+5}{2i+2}, \frac{6i-11}{2i-5}, \frac{21i+19}{10i+7}\right].$$

Arguing as before we have

$$\left\{0, \frac{21i+19}{10i+7}\right\} = \left[\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 4i+5 & 2 \\ 2i+2 & 1 \end{pmatrix} + \begin{pmatrix} 6i-11 & -4i-5 \\ 2i-5 & -2i-2 \end{pmatrix} + \begin{pmatrix} 21i+19 & 11i+6 \\ 10i+7 & 5i+2 \end{pmatrix}\right] \cdot \{0, \infty\},$$

and one can check that all of these four matrices are in $\mathrm{SL}_2(\mathcal{O}_K)$. (I do not know if the corresponding statement is true for $\mathbb{Q}(\sqrt{-19})$, which is a PID but not Euclidean.)

Question 6 (*For amusement value only*)**.** *(a) We saw on sheet 3 that the function $f = q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$ was an element of $S_2(\Gamma_1(23), \chi)$ where χ is the unique quadratic character of conductor 23. The Galois group of the splitting field of $X^3 - X - 1$ is isomorphic to S_3 and thus has a unique 2-dimensional irreducible representation ρ. Calculate $\mathrm{Tr}\,\rho(\mathrm{Frob}_p)$ for some primes $p \neq 23$ and verify that these are equal to the coefficients a_p of f.*

(b) Assuming Artin's conjecture, show that $S_1(\Gamma_1(47))$ has a 2-dimensional subspace spanned by two eigenforms with coefficients in $\mathbb{Q}(\zeta_5)$, and calculate the q-expansions of these forms up to the q^4 term. (Hint: the class group of $\mathbb{Q}(\sqrt{-47})$ is cyclic of order 5.)

Answer 6. (a) It is a standard result in Galois theory that if K is the splitting field of a polynomial f over $\mathbb{Q}$, one can read off the cycle type of the Frobenius element Frob_p from the degrees of the irreducible factors of f modulo p. One can check that $X^3 - X - 1$ is irreducible modulo 2 and mod 3, and hence the Frobenius elements at any prime above 2 or 3 correspond to 3-cycles in S_3. Hence their traces in this representation are -1. Modulo 5 the polynomial factors as a product of a linear and quadratic term, so the Frobenius element is a 2-cycle and its trace is 0. In fact, for the first few primes p we have

p	2	3	5	7	11	17	19	23	29	...	59
$\mathrm{Tr}\,\rho(\mathrm{Frob}_p)$	-1	-1	0	0	0	0	0	$-$	-1	...	2

Here 59 is the first prime modulo which $X^3 - X - 1$ splits completely. One can easily check that this coincides with the prime coefficients of f.

(b) (One doesn't actually need Artin's conjecture here.) Let $K = \mathbb{Q}(\sqrt{-47})$. It is easy to check that $\mathrm{Cl}(K)$ is cyclic of order 5, and the primes above 2 are both non-principal. Let $\mathfrak{p}$ be a fixed prime above 2, and let ξ be the character $\mathrm{Cl}(K) \to \mathbb{C}^\times$ sending $\mathfrak{p}$ to $\zeta = e^{2\pi i/5}$.

It's easy to see that the theta series f_χ associated to χ satisfies $a_2 = \zeta + \zeta^{-1} = \frac{\sqrt{5}-1}{2}$, $a_3 = \zeta^2 + \zeta^{-2} = \frac{-\sqrt{5}-1}{2}$, and $a_4 = \xi(2) + \xi(\mathfrak{p}^2) + \xi(\overline{\mathfrak{p}}^2) = 1 + \zeta^2 + \zeta^{-2} = \frac{1-\sqrt{5}}{2}$. This form and its Galois conjugate over $\mathbb{Q}(\sqrt{5})$ (which is the theta-series corresponding to ξ^2) span a 2-dimensional subspace of $S_1(\Gamma_1(47))$.