MODULAR FORMS: EXAMPLE SHEET 3 – HINTS AND OUTLINE SOLUTIONS

DAVID LOEFFLER

Question 1. *Let w_N be the matrix $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$.*

(a) Show that $w_N \Gamma_1(N) w_N^{-1} = \Gamma_1(N)$. Deduce that if $f \in S_k(\Gamma_1(N))$, then $w_N(f) = f|_k w_N \in S_k(\Gamma_1(N))$.
(b) Check that $w_N(i_{1,p} f) = i_{2,p}(w_{N/p} f)$ and $w_N(i_{2,p} f) = p^{k-1} i_{1,p}(w_{N/p} f)$.
(c) Show that w_N defines a normal operator with respect to the Petersson product. Deduce that it preserves $S_k(\Gamma_1(N))^{\mathrm{new}}$.
(d) If $f \in S_k(\Gamma_1(N), \chi)$, show that $w_N(f) \in S_k(\Gamma_1(N), \overline{\chi})$.

Solution. (a) If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, then $w_N \gamma w_N^{-1} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix} \in \Gamma_1(N)$, from which the first statement is immediate. Hence $\Gamma_1(N) w_N \Gamma_1(N) = \Gamma_1(N) w_N = w_N \Gamma_1(N)$, and the result follows.

(b) We note that $(f|_k w_N)(z) = N^{k-1}(Nz)^{-k} f(-1/Nz)$. Meanwhile $i_{1,p}(f)(z) = f(z)$ and $i_{2,p}(f)(z) = p^{k-1} f(pz)$.
$$(i_{1,p} f)|_k w_N = N^{-1} z^{-k} f(-1/Nz)$$
and
$$i_{2,p}(f|_k w_{N/p}) = i_{2,p}((N/p)^{-1} z^{-k} f(-p/Nz)) = p^{k-1}(p/N)(pz)^{-k} f(-1/Nz)$$
so the first equality holds.

The second is actually not true: one easily checks that the constant should be p^{k-2}, not p^{k-1}. Sorry. (This is wrong in the lecture notes as well)

(c) The Petersson adjoint of $\Gamma\alpha\Gamma$ is $\Gamma\alpha'\Gamma$ where $\alpha' = (\det \alpha)\alpha^{-1}$. For $\alpha = w_N$ we have $\alpha' = -\alpha$; thus w_N certainly commutes with its adjoint, so it is normal. We saw in lectures that w_N stabilises $S_k(\Gamma_1(N))^{\mathrm{old}}$, and hence (as it is normal) it also preserves $S_k(\Gamma_1(N))^{\mathrm{new}}$.

(d) Since $w_N \langle d \rangle w_N^{-1} = \langle d \rangle^{-1}$, w_N sends the χ-eigenspace to the $\overline{\chi}$-eigenspace. □

Question 2. *Let $\sigma_{k-1}^{\chi,\psi}$ be the generalised divisor sum function defined by*
$$\sigma_{k-1}^{\chi,\psi}(n) = \sum_{m|n} \psi(m)\chi(n/m) m^{k-1}.$$
Show that if m, n are coprime, then
$$\sigma_{k-1}^{\chi,\psi}(mn) = \sigma_{k-1}^{\chi,\psi}(m)\sigma_{k-1}^{\chi,\psi}(n).$$
Hence show that the functions $G_k^{\chi,\psi}(tz)$ occuring in Theorem 2.9.2 are eigenforms for the Hecke operators T_ℓ with $\ell \nmid N$, assuming that their q-expansions are as stated in the theorem.

Solution. If m, n are coprime, then any divisor of mn may be uniquely written in the form de with $d|m$ and $e|n$. Thus if $g, h : \mathbb{Z} \to \mathbb{C}$ are any weakly multiplicative functions, $f(n) = \sum_{d|n} g(d)h(n/d)$ is also weakly multiplicative. The sum defining $\sigma_{k-1}^{\chi,\psi}$ is clearly of this type. Similarly, one can check that $a_n = \sigma_{k-1}^{\chi,\psi}(n)$ satisfies $a_{p^e} = a_p a_{p^{e-1}} - \chi(p)\psi(p)p^{k-1} a_{p^{e-2}}$, with the second term interpreted as 0 if p divides the conductor of either character; and this is sufficient to prove that the functions $G_k^{\chi,\psi}(z)$ are eigenforms (for *all* Hecke operators, not just those away from the level). The weaker statement for $G_k^{\chi,\psi}(tz)$ is now immediate. □

Question 3. *Let N, t, d be positive integers with $d < N$ and $td \mid N$. Show that if $f \in S_k(\Gamma_1(d))$, then $f(tz) \in S_k(\Gamma_1(N))^{\mathrm{old}}$.*

Solution. Pick a prime p dividing N/d. Clearly at least one of $f(tz)$ and $f(t/pz)$ will be in $S_k(\Gamma_1(N/p))$, depending whether or not t/p is an integer. Applying the map $i_{1,p}$ or $i_{2,p}$ as appropriate, then we see that $f(tz)$ is in $S_k(\Gamma_1(N))^{\text{old}}$. □

Question 4. *(a) Let* $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ *with* p *prime. Suppose* $p \mid N$. *Show that* $\Gamma_0(Np)\alpha\Gamma_0(Np) = \Gamma_0(Np)\alpha\Gamma_0(N)$. *Deduce that if* $f \in S_k(\Gamma_0(Np))$, *then* $f|_k U_p \in S_k(\Gamma_0(N))$.
(b) Hence show that U_p *is the zero map on* $S_k(\Gamma_0(Np))^{\text{new}}$.
(c) Show similarly that if $f \in S_k(\Gamma_0(Np))^{\text{new}}$ *and* $p \nmid N$, *and* f *is a* U_p*-eigenform, then the* U_p*-eigenvalue of* f *is* $\pm p^{(k-1)/2}$.

Solution. (a) From a question on the previous sheet we know that the index of $\Gamma_0(N)$ in $\Gamma_0(Np)$ is p; and it is clear that coset representatives for $\Gamma_0(N)/\Gamma_0(Np)$ are $\tau_j = \begin{pmatrix} 1 & 0 \\ Nj & 1 \end{pmatrix}$ for $j = 0 \dots p-1$. But for all j, $\alpha\tau_j \in \Gamma_0(Np)\alpha$, and the result follows. Consequently for any $f \in \Gamma_0(Np)$,

$$f|_k U_p = f|_k(\Gamma_0(Np)\alpha\Gamma_0(Np)) = f|_k(\Gamma_0(Np)\alpha\Gamma_0(N)) \in S_k(\Gamma_0(N)).$$

(b) We know that U_p preserves $S_k(\Gamma_0(Np))^{\text{new}}$; but the above result shows that its image is contained in $S_k(\Gamma_0(Np))^{\text{old}}$. Since the intersection of the old and the new subspaces is 0, U_p must be the zero map.
(c) Let us do the special case $N = 1$ first. A set of coset representatives for $\mathrm{SL}_2(\mathbb{Z})/\Gamma_0(p)$ is the τ_j above together with $\tau_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence

$$\begin{aligned} \Gamma_0(p)\alpha\,\mathrm{SL}_2(\mathbb{Z}) &= \Gamma_0(p)\alpha\Gamma_0(p) \sqcup \Gamma_0(p)\alpha\tau_p\Gamma_(p) \\ &= \Gamma_0(p)\alpha\Gamma_0(p) \sqcup \Gamma_0(p)w_p\Gamma_0(p) \end{aligned}$$

where w_p is as defined above. Hence if $f \in S_k(\Gamma_0(p))$, $(f|_k U_p) + (f|_k w_p) \in S_k(\mathrm{SL}_2(\mathbb{Z}))$; and if f is new, we deduce as before that $f|_k U_p = -f|_k w_p$. Since w_p^2 acts as scalar multiplication by p^{k-1}, the eigenvalues of w_p must be $\pm p^{k-1}/2$, and consequently if f is a U_p-eigenform its U_p-eigenvalue is equal to $\mp p^{(k-1)/2}$. In the general case we can argue in exactly the same way, using the Chinese Remainder Theorem to find elements of $\Gamma_0(N)$ which are congruent to the τ_j modulo p.

□

Question 5. *Let* χ *be the Dirichlet character* $(\mathbb{Z}/5\mathbb{Z})^\times \to \{\pm 1\}$ *with* $\chi(d) = 1$ *when* d *is a square mod* 5 *and* -1 *otherwise. Let* $N \in \mathbb{N}$ *be divisible by 5, and let*

$$\Gamma_\chi(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; d \text{ is a square mod } 5 \right\}.$$

Show that for any even k,

$$S_k(\Gamma_\chi(N)) = S_k(\Gamma_0(N)) \oplus S_k(\Gamma_1(N), \chi).$$

Hence verify that the dimension of $S_4(\Gamma_1(25), \chi)$ *is 4, as stated in lectures.*

(You may assume that $X(\Gamma_\chi(25))$ *has genus 0, 12 cusps and no elliptic points, while* $X_0(25)$ *has genus 0, 6 cusps, 2 elliptic points of order 2 and no elliptic points of order 3.)*

Solution. The first statement is clear, since χ is trivial on diagonal entries of $\Gamma_\chi(N)$, and conversely any character of $\Gamma_0(N)/\Gamma_\chi(N)$ must be either 1 or χ. We can easily compute the dimension of $S_k(\Gamma_\chi(25))$ and $S_k(\Gamma(25))$ for any k using the data given; it turns out that we have $\dim S_4(\Gamma_\chi(25)) = 9$ and $\dim S_4(\Gamma_0(25)) = 5$, and the result follows. □

Question 6. *Suppose* χ *is the unique character of* $(\mathbb{Z}/7\mathbb{Z})^\times$ *with* $\chi(3) = e^{2\pi i/3}$. *Use question 1(d) above to show that* $S_k(\Gamma_0(7)) + 2\dim S_k(\Gamma_1(7), \chi) = \dim S_k(\Gamma_1(7))$ *for all even* k. *Hence find the dimension of* $S_{12}(\Gamma_1(7), \chi)$.

Solution. By 1(d) we know that w_7 gives an isomorphism between $S_k(\Gamma_1(7), \chi)$ and $S_k(\Gamma_1(7), \overline{\chi})$, so these spaces have the same dimension. Since the only even characters of $(\mathbb{Z}/7\mathbb{Z})^\times$ are $\{1, \chi, \overline{\chi}\}$, it follows that if k is even we have the dimension formula given above.

Now $\Gamma_0(7)$ has index 8, contains -1, and has 2 cusps and 2 elliptic points both with order 3; hence its genus is 0 and $\dim S_{12}(\Gamma_0(7)) = 7$. Meanwhile $\Gamma_1(7)$ has index 48, does not contain -1, and has 6 cusps and no elliptic points; so its genus is also 0, and $\dim S_{12}(\Gamma_1(7)) = 19$. Thus $\dim S_{12}(\Gamma_1(7), \chi) = 6$. □

Question 7. *(a) Let $f \in S_k(\Gamma_1(N), \chi)$ be a simultaneous eigenform for the operators T_ℓ with $\ell \nmid N$. Let ℓ be any one such prime. Show that the matrix of U_ℓ on the subspace of $S_k(\Gamma_1(N\ell^r), \chi)$ spanned by $f(z), f(\ell z), \dots, f(\ell^r z)$ is given by*

$$\begin{pmatrix} a_\ell & 1 & 0 & \dots & 0 \\ -\ell^{k-1}\chi(\ell) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(b) Show that this is never diagonalisable if $r \geq 3$. If $r = 1$ or $r = 2$, and the Hecke polynomial $X^2 - a_\ell X + \ell^{k-1}\chi(\ell)$ has distinct roots $\{\alpha, \beta\}$, show that it is diagonalisable and has eigenvalues $\{\alpha, \beta\}$ and $\{\alpha, \beta, 0\}$ respectively. If this quadratic has a repeated root, what happens?

(c) Let f be the function $f(z) = q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$. You may assume that this is in $S_1(\Gamma_1(23), \chi)$ where $\chi(n) = \left(\frac{n}{23}\right)$ is the quadratic residue character modulo 23. Using a computer, find a prime ℓ such that the Hecke polynomial of f at ℓ has repeated roots.

Solution. (a) It is clear that $U_\ell(f(\ell^r z)) = f(\ell^{r-1}z)$ if $r > 1$, while the first column follows from Case 5 of Prop 2.10.2.

(b) If $r \geq 3$, then consider the action of this matrix M by right multiplication on the row vector $v = (0, 0, 1, 0, \dots)$. Then we find that that $v \cdot M \neq 0$ but $v \cdot M^{r-1}$ is 0, which is imposible for a diagonalisable matrix. For $r = 1, 2$ the result is a direct calculation. If we have $a_\ell = \pm 2\ell^{k-1}\chi(\ell)$ then it is easy to see that M is not diagonalisable.

(c) One finds that $a_{59} = +2$, so the matrix of U_{59} on $S_1(\Gamma_1(23 \times 59))$ is not diagonalisable.

Remark: It is conjectured that this cannot happen in weights $k \geq 2$. Coleman and Edixhoven have proved this for $k = 2$, and for $k > 2$ they have shown that it follows from an existing conjecture on the geometry of varieties over $\mathbb{F}_p$ (Tate's conjecture). □

Question 8. *Let $(u, v) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $k \geq 4$ is even. Show that the sum*

$$\sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1 \\ (c,d)=(u,v) \bmod N}} (cz+d)^{-k}$$

is an Eisenstein series for a cusp of $\Gamma(N)$. Hence show that the sum $G_k^{\psi,\chi}$ defined in lectures is in $N_k(\Gamma(N))$.

Solution. Since $\Gamma(N)$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, it is easy to see that the cusps of $\Gamma(N)$ biject with the points of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/P$ where P is the subgroup of matrices of the form $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$. Thus there is a unique cusp for each $\begin{pmatrix} u \\ v \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^2$ which appears as the left column of some element of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, and the possible left columns are exactly those vectors in $(\mathbb{Z}/N\mathbb{Z})^2$ whose additive order is exactly N.

It is easy to see that if (u, v) is not allowable in this sense, then the above sum is empty, so assume that this is not the case. Fix some element $\delta \in \mathrm{SL}_2(\mathbb{Z})$ with $\delta = \begin{pmatrix} u & r \\ v & s \end{pmatrix}$ for some r, s. Then we see that

$$\sum_{\delta P \delta^{-1} \backslash \Gamma} 1|_k(\delta^{-1}\gamma) = \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1 \\ (c,d)=(-v,u) \bmod N}} (cz+d)^{-k}.$$

So the above series is the Eisenstein series corresponding to the cusp $\frac{-v}{u}$. □

Question 9. *Let $U \subseteq \mathbb{R}^2$ be an open set and let $\Omega^1_{sm}(U)$ denote the $\mathbb{C}$-vector space of expressions of the form $\lambda = p\mathrm{d}x + q\mathrm{d}y$ with p, q smooth functions $U \to \mathbb{C}$. These are the smooth 1-forms of Theorem 3.1.6.*

(a) *Suppose $f = (u(x,y), v(x,y)) : U_1 \to U_2$ is a smooth function, with U_1, U_2 open sets in $\mathbb{R}^2$ as above. Define $f^* : \Omega^1_{sm}(U_2) \to \Omega^1_{sm}(U_1)$ by $f^*(\mathrm{d}u) = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$, and similarly for $\mathrm{d}v$, extended in the obvious way to the whole space. Show that if we have a map $U_1 \to^g U_2 \to^f U_3$, then $(f \circ g)^* = g^* \circ f^*$. Deduce that $\Omega^1_{sm}(X)$ is well-defined, for any real 2-manifold X.*

(b) *Define $\star(\lambda) = -q\mathrm{d}x + p\mathrm{d}y$. This is the* **Hodge star** *operator. If f and λ are as before, calculate $\star(f^*\lambda)$ and $f^*(\star\lambda)$. Show that these are equal for all λ if and only if the map $f(x+iy) = u(x,y) + iv(x,y)$ satisfies the Cauchy-Riemann conditions, so f is a holomorphic map between subsets of $\mathbb{C}$. Hence show that if X is a Riemann surface, $\star$ gives a well-defined map from $\Omega^1_{sm}(X)$ to itself.*

(c) *Suppose that f is a smooth function on X. In local coordinates define $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$. Show that this gives a well-defined differential (independent of the choice of coordinate chart).*

(d) *We say $\lambda = p\mathrm{d}x + q\mathrm{d}y$ is closed if $\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$. Show that $\mathrm{d}f$ is always closed. When is $\star(\mathrm{d}f)$ closed?*

(e) *Show that λ and $\star\lambda$ are both closed if and only if we may write $\lambda = r\mathrm{d}z + s\mathrm{d}\overline{z}$ with r holomorphic and s anti-holomorphic. (This completes the proof of Theorem 3.1.6, on the assumption of Hodge's theorem.)*

Solution. (a) Let $U_1 \to^g U_2 \to^f U_3$ be two smooth maps as above. Then, in the obvious bases, the matrix of $f^* : \Omega^1_{sm}(U_3) \to \Omega^1_{sm}(U_2)$ acting on $\Omega^1_{sm}(U_3)$ is the *Jacobian* matrix

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Hence one checks that

$$g^* \circ f^* = \begin{pmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial y_2}{\partial y_1} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x_3}{\partial x_2} & \frac{\partial y_3}{\partial x_2} \\ \frac{\partial x_3}{\partial y_2} & \frac{\partial y_3}{\partial y_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_3}{\partial x_1} & \frac{\partial y_3}{\partial x_1} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial y_3}{\partial y_1} \end{pmatrix} = (f \circ g)^*.$$

We can thus define an element of $\Omega^1_{sm}(X)$ to be a collection of local differentials $\omega_i \in \Omega^1_{sm}(U_i)$, where $\{\psi_i : U_i \to V_i\}_{i \in I}$ is a coordinate chart, which are compatible on overlaps; and the above result implies that this makes sense.

(b) We check that for $f : U_1 \to U_2$ a smooth map and $p\mathrm{d}u + q\mathrm{d}v \in \Omega^1_{sm}(U_2)$, the matrix of $(f^*) \circ \star$ is

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the matrix of $\star \circ f^*$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

These are equal if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, which are the Cauchy–Riemann equations. So if the overlap maps in the chart are all holomorphic, and $\omega = \{\omega_i\}_{i \in I}$ is an element of $\Omega^1_{sm}(X)$, then $\{\star\omega_i\}_{i \in I}$ is also an element of $\Omega^1_{sm}(X)$ and we may define this to be $\star\omega$.

(c) One checks that if $\psi : U_1 \to U_2$ is a smooth map of subsets of $\mathbb{R}^2$, and f is a smooth function on U_2, then

$$\psi^*(\mathrm{d}f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{pmatrix} = \mathrm{d}(f \circ \psi).$$

Since by definition a smooth function on X is a family of smooth functions f_i on the coordinate charts U_i satisfying $f_i = f_j \circ \psi_{ij}$ where ψ_{ij} is the overlap map between charts U_i and U_j, this implies that $\{\mathrm{d}f_i\}_{i \in I}$ satisfies the compatibility conditions necessary to define an element of $\Omega^1_{sm}(X)$.

(d) We know that for smooth functions f, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, and hence $\mathrm{d}f$ is always closed. We find that $\star(\mathrm{d}f)$ is closed if and only if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$, so f is a harmonic function on each coordinate chart.

(e) The condition is local, so it is enough to prove it for X an open subset of $\mathbb{C}$. Since $\mathrm{d}z$ and $\mathrm{d}\overline{z}$ are a basis for $\Omega^1(X)$ (as a free module over $\Omega^0(X)$), we can write any λ in the form $r\mathrm{d}z + s\mathrm{d}\overline{z}$ for r, s smooth.

We write $r = a + bi$, $s = c + di$ for smooth real functions a, b, c, d. Noting that $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$, and doing a tedious algebraic check, we find that

$$\lambda \text{ is closed} \Leftrightarrow \frac{\partial a}{\partial y} + i\frac{\partial b}{\partial y} + \frac{\partial c}{\partial y} + i\frac{\partial d}{\partial y} = i\frac{\partial a}{\partial x} - \frac{\partial b}{\partial x} - i\frac{\partial c}{\partial x} + \frac{\partial d}{\partial x}$$

$$\lambda \text{ is co-closed} \Leftrightarrow \frac{\partial a}{\partial x} + i\frac{\partial b}{\partial x} + \frac{\partial c}{\partial x} + i\frac{\partial d}{\partial x} = -i\frac{\partial a}{\partial y} + \frac{\partial b}{\partial y} + i\frac{\partial c}{\partial y} - \frac{\partial d}{\partial y}$$

Taking real and imaginary parts (remembering that a and b are real functions) we have

$$\frac{\partial a}{\partial y} + \frac{\partial c}{\partial y} = -\frac{\partial b}{\partial x} + \frac{\partial d}{\partial x}$$
$$\frac{\partial b}{\partial y} + \frac{\partial d}{\partial y} = \frac{\partial a}{\partial x} - \frac{\partial c}{\partial x}$$
$$\frac{\partial a}{\partial x} + \frac{\partial c}{\partial x} = \frac{\partial b}{\partial y} - \frac{\partial d}{\partial y}$$
$$\frac{\partial b}{\partial x} + \frac{\partial d}{\partial x} = -\frac{\partial a}{\partial y} + \frac{\partial c}{\partial y}$$

The second and third equations are equivalent to $\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$ and $\frac{\partial c}{\partial x} = -\frac{\partial d}{\partial y}$; while the first and fourth hold if and only if $\frac{\partial a}{\partial y} = -\frac{\partial b}{\partial x}$ and $\frac{\partial c}{\partial y} = \frac{\partial d}{\partial x}$. Consequently λ is both closed and co-closed if and only if r is holomorphic and s is anti-holomorphic.

□

Question 10. *Find an element $x \in \mathbb{Z}[\mathrm{SL}_2 \mathbb{Z}]$ such that $x\{0, \infty\} = \{\frac{1339}{164}, -\frac{19}{28}\}$.*

Solution. We find that

$$-\frac{19}{28} = -1 + \cfrac{1}{3 + \cfrac{1}{9}}$$

so the convergents are $\infty, 0, -1, -\frac{2}{3}, -\frac{19}{28}$. Hence we have

$$\begin{aligned}\{0, -\tfrac{19}{28}\} &= \{0, \infty\} + \{\infty, -1\} + \{-1, -\tfrac{2}{3}\} + \{-\tfrac{2}{3}, -\tfrac{19}{28}\} \\ &= \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix} + \begin{pmatrix}2 & 1\\ -3 & -1\end{pmatrix} + \begin{pmatrix}-19 & 2\\ 28 & -3\end{pmatrix}\right] \cdot \{0, \infty\}\end{aligned}$$

Similarly

$$\frac{1339}{164} = 8 + \cfrac{1}{6 + \cfrac{1}{13 + \cfrac{1}{2}}}$$

and the convergents are $0, \infty, 8, 49/6, 645/79, 1339/164$, so one finds that

$$\{0, \tfrac{1339}{164}\} = \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}8 & -1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}49 & 8\\ 6 & 1\end{pmatrix} + \begin{pmatrix}645 & -49\\ 79 & -6\end{pmatrix} + \begin{pmatrix}1339 & 645\\ 164 & 79\end{pmatrix}\right] \cdot \{0, \infty\}$$

Taking the difference between these gives the required element.

□

Question 11. *Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Check the assertion made in the proof of Prop 3.2.5 that if g is an elliptic element of order h, so $g^h = \pm 1$, and α is any element of $\mathbb{P}^1(\mathbb{Q})$, then the element $h\{\alpha, g\alpha\}$ is zero in $\mathbb{M}_2(\mathbb{Z})/K_2(\Gamma)$.*

Solution. It is clear that the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{M}_2(\mathbb{Z})$ factors through $\mathrm{PSL}_2(\mathbb{Z})$; hence we can assume $h = 2$ or $h = 3$. We have $\{\alpha, g\alpha\} = \{g^t\alpha, g^{t+1}\alpha\} \bmod K_2$, so

$$h\{\alpha, g\alpha\} = \{\alpha, g\alpha\} + \{g\alpha, g^2\alpha\} + \cdots + \{g^{h-1}\alpha, \alpha\}.$$

Thus $h\{\alpha, g\alpha\}$ is congruent modulo K_2 to the boundary of a (possibly degenerate) triangle, and is thus 0. $\square$