MODULAR FORMS: EXAMPLE SHEET 2 – HINTS AND OUTLINE SOLUTIONS

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Exercises

Question 1. *Let Γ be an arbitrary finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Show that Γ contains a finite index normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Hence (or otherwise) show that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ for some $h \neq 0$.*

Solution. Let $h_1, \dots, h_j$ be coset representatives for $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$. Then any conjugate $\alpha^{-1}\Gamma\alpha$ with $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ is equal to $h_i^{-1}\Gamma h_i$ for some i; so the intersection of these finitely many subgroups is a normal subgroup. Since the intersection of two finite index subgroups is a finite index subgroup (isomorphism theorems), this intersection is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

In particular, if P denotes the stabiliser of ∞ in $\mathrm{SL}_2(\mathbb{Z})$, then $P \cap \Gamma$ is normal in P and the quotient is finite; so the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has finite order in the quotient. (Of course this is true without using the first part, since P is abelian and any subgroup of an abelian group is normal.) □

Question 2. *Find a non-constant holomorphic function f on $\mathcal{H}$ which is weakly modular of weight 0 and level $\Gamma_0(2)$ and holomorphic at ∞. Deduce that the space of functions on $\mathcal{H}$ which are weakly modular of weight k for $\Gamma_0(2)$ and holomorphic on $\mathcal{H} \cup \{\infty\}$ is infinite-dimensional for any k for which it is nonzero.*

Solution. Clearly $j(2z)$ has an order 2 pole at ∞ (since we are just substituting $q \mapsto q^2$ into the q-expansion). Thus there are constants a, b such that $j(2z) + aj(z)^2 + bj(z)$ is holomorphic at infinity, and since $j(2z)$ is not a level 1 modular function (or just by calculating the q-expansion) it is not constant.

Let ϕ be this function, and let k be such that the space $M_k^{(\infty)} = \{f \in \mathcal{A}_k(\Gamma) | f \text{ is holomorphic at } \infty\}$ is nonzero. Let $t \in M_k^{(\infty)}$, and consider $t, \phi t, \phi^2 t, \dots$. These are all in $M_k^{(\infty)}$, and they are linearly independent, since ϕ is not constant and thus does not satisfy any polynomial over $\mathbb{C}$. So $M_k^{(\infty)}$ must be infinite-dimensional. □

Question 3. *Is $\mathcal{H}^*$ compact?*

Solution. Of course not, since $i, i+1, i+2, \dots$ is a sequence with no convergent subsequence. (In fact the quotient of $\mathcal{H}$ by the map $z \mapsto z+1$ is not compact either, but this is harder to see.) □

Question 4. *(a) Show that $\Gamma_1(5)$ has 4 cusps: the orbits of $0, \frac{2}{5}, \frac{1}{2}, \infty$. For each cusp x, find the width $h_{\Gamma_1(5)}(x)$ of the cusp and a generator for its stabiliser.*

(b) Show that $\Gamma_0(5)$ has no elliptic points of order 3, and deduce that the same is true of $\Gamma_1(5)$. Show that $\Gamma_0(5)$ has precisely 2 elliptic points of order 2, and for each one, show that none of its four preimages in $\Gamma_1(5)$ is elliptic.

(c) What is the genus of $X_1(5) = X(\Gamma_1(5))$?

(Hint: the calculations can be shortened somewhat by using the fact that $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$.)

Solution. (a) Let x be a cusp of $X_1(5)$. Then x must map to either 0 or ∞ on $X_0(5)$; and the group $\Gamma_0(5)/\Gamma_1(5)$ acts transitively on the preimages of either 0 or ∞. We check that a set of coset representatives for Γ_0/Γ_1 is given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -3 & -1 \\ -5 & -2 \end{pmatrix} \right\}.$$

Clearly $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially, so if $\alpha = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$, then every cusp of $X_1(5)$ is equivalent to one of the cusps $\{\infty, \alpha(\infty), 0, \alpha(0)\} = \{\infty, \frac{3}{5}, 0, \frac{1}{2}\}$. Since $\begin{pmatrix} 1 & -1 \\ 5 & -4 \end{pmatrix} \left(\frac{3}{5}\right) = \frac{2}{5}$, the list of 4 cusps given above is complete.

We must now check that $\frac{2}{5}$ is not equivalent to ∞, and $\frac{1}{2}$ is not equivalent to 0; but this is immediate, since the orbit of ∞ is all rationals $\frac{r}{s}$ in lowest terms with $5 \mid s$ and $r = 1 \bmod 5$, and the orbit of 0 is all rationals $\frac{r}{s}$ in lowest terms with $s = 1 \bmod 5$.

Finally, since each cusp of $X_0(5)$ has two preimages on $X_1(5)$, and these are interchanged by the quotient group, the two preimages have equal widths. Since $X_0(5)$ contains -1, but $X_1(5)$ does not, it follows that the index $\mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{Z})}(x) : \mathrm{Stab}_{\Gamma/\pm 1}(x)]$ does not change as we pass from $\Gamma = \Gamma_0(5)$ to $\Gamma_1(5)$. Hence ∞ and $\frac{2}{5}$ have width 1, and 0 and $\frac{1}{2}$ have width 5.

It's clear that we can calculate the stabiliser of a cusp x by finding $\delta \in \mathrm{SL}_2(\mathbb{Z})$ with $\delta(\infty) = x$ and checking small values of h to see if $\pm\delta \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \delta^{-1} \in \Gamma_1(5)$ (this will also give the widths, checking the above argument.) We find generators

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad \infty$$
$$\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \quad \text{for} \quad 0$$
$$\begin{pmatrix} -9 & 4 \\ -25 & 11 \end{pmatrix} \quad \text{for} \quad \frac{2}{5}$$
$$\begin{pmatrix} 11 & -5 \\ 20 & -9 \end{pmatrix} \quad \text{for} \quad \frac{1}{2}$$

(Note that the inverses of these also work, which are respectively $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$, $\begin{pmatrix} 11 & -4 \\ 25 & -9 \end{pmatrix}$ and $\begin{pmatrix} -9 & 5 \\ 20 & 11 \end{pmatrix}$.)

(b) We use a similar method based on cosets, but more care is needed since $\Gamma_0(5)$ is not normal in $\mathrm{SL}_2(\mathbb{Z})$. Any elliptic point of order 3 is in the orbit $\mathrm{SL}_2(\mathbb{Z})\rho$, so if $\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_j \Gamma_0(5)\alpha_j$, then every elliptic point is equivalent under $\Gamma_0(5)$ to one of the points $\alpha_j\rho$. The stabiliser of $\alpha_j\rho$ in $\Gamma_0(5)$ is $\alpha_j(\mathrm{Stab}_{SL_2(\mathbb{Z})}\,\rho)\alpha_j^{-1} \cap \Gamma_0(5)$.

We choose the coset representatives $\alpha_i = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$, $i = 0..4$, and $\alpha_5 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For each of these, we check that if $\mu = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ is the order 3 element of $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}\,\rho$, then $\alpha_j w \alpha_j^{-1}$ is not in $\Gamma_0(5)$. So the stabiliser of $\alpha_j\rho$ is ± 1.

A similar check using $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ shows that the stabiliser of $\alpha_j(i)$ is ± 1 unless $\alpha_j w \alpha_j^{-1} \in \Gamma_0(5)$, which occurs if $j = 2$ or 3. These points are not equivalent, since $\alpha_2 \mu \alpha_3^{-1}$ is not in $\Gamma_0(5)$ for any $\mu \in \mathrm{Stab}_{\mathrm{SL}_2}(i)$. Hence there are exactly 2 elliptic points of order 3 on $X_0(5)$.

Since $\Gamma_1(5)$ is normal in $\Gamma_0(5)$, if any one preimage of a point of $X_0(5)$ is elliptic on $X_1(5)$ then all of them are. But it is clear that the points $\alpha_2(i)$ and $\alpha_3(i)$ are not elliptic for $X_1(5)$, since the nontrivial stabiliser elements we found were not in $\Gamma_1(5)$.

(c) We note that

$$g(X(\Gamma)) = 1 + \frac{d}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{3} - \frac{\epsilon_\infty}{2}$$

(Corollary 2.3.5). Here d is the degree of the morphism from $X(\Gamma)$ to $X(\mathrm{SL}_2(\mathbb{Z}))$, which is $[\mathrm{PSL}_2(\mathbb{Z}) : \pm\Gamma]$. In this case, since $\Gamma_1(5)$ has index 24 but does not contain -1, the degree of the map is 12. Plugging in the appropriate constants from the above argument,

$$g(X_1(5)) = 1 + \frac{12}{12} - \frac{0}{4} - \frac{0}{3} - \frac{4}{2} = 0.$$

□

Question 5. *Fill in the details of the proof of Corollary 2.3.5, by checking the formulae given for the sums* $\sum_{x\in\pi^{-1}y}(e_x(\pi)-1)$ *for the three non-ordinary points* $y=\infty, i, \rho$ *of* $X(\mathrm{SL}_2(\mathbb{Z}))$.

Solution. Firstly, we note that for any point $y \in X(\mathrm{SL}_2(\mathbb{Z}))$, $\sum_{x\in\pi^{-1}y} e_x(\pi) = d$, since d is the degree of the map; so the sum is just $d - |\pi^{-1}y|$. For $y=\infty$, the points of $\pi^{-1}y$ are exactly the cusps of Γ, so the sum is $d-\varepsilon_\infty$.

For $y=i$, the preimage of i on $X(\Gamma)$ consists of ε_2 elliptic points of order 2 on $X(\Gamma)$, at which the π is unramified, and some other points x for which $e_x(\pi)=2$. So there must be $\frac{d-\varepsilon_2}{2}$ of the latter, and thus the sum is equal to $\frac{d-\varepsilon_2}{2}$.

For $y=\rho$, we see similarly that if $x\in\pi^{-1}(\rho)$, $e_x(\pi)$ is 1 if x is elliptic of order 3 and 3 otherwise; there are ε_3 of the former, hence $\frac{d-\varepsilon_3}{3}$ of the latter, so the sum is $\frac{2(d-\varepsilon_3)}{3}$. □

Question 6. *Let* G *be any group. Recall that we say subgroups* H_1, H_2 *of* G *are* **commensurable** *if* $H_1\cap H_2$ *has finite index in either* H_1 *or* H_2. *Let us write this relation by* $H_1 \sim H_2$.

(a) Show that commensurability is an equivalence relation.

(b) Let H *be any subgroup. Define the* **commensurator** *of* H *in* G, $\mathrm{Comm}_G(H)$, *to be the set of* $g\in G$ *such that* $g^{-1}Hg$ *is commensurable with* H. *Show that this is a group.*

(c) Show that if $H_1\sim H_2$, *then* $\mathrm{Comm}_G(H_1) = \mathrm{Comm}_G(H_2)$.

(d) What is the commensurator of $\mathrm{SL}_2(\mathbb{Z})$ *in* $\mathrm{GL}_2(\mathbb{R})$?

Solution. We note that the intersection of any two finite index subgroups of G is a finite index subgroup of G.

(a) The only nontrivial check is that commensurability is transitive; but if $H_1, H_2, H_3 \subseteq G$ with $H_1\sim H_2$ and $H_2 \sim H_3$, then clearly $H_1\cap H_2$ and $H_2\cap H_3$ are finite index subgroups of H_2, hence so is $H_1\cap H_2\cap H_3$. But

$$\begin{aligned}[H_1 : H_1\cap H_2\cap H_3] &= [H_1 : H_1\cap H_2][H_1\cap H_2 : H_1\cap H_2\cap H_3]\\ &\le [H_1 : H_1\cap H_2][H_2 : H_1\cap H_2\cap H_3]\\ &< \infty.\end{aligned}$$

So certainly $[H_1 : H_1\cap H_3] < \infty$, and similarly $[H_3 : H_1\cap H_3]<\infty$. Thus $H_1\sim H_3$.

(b) The commensurator is closed under multiplication because $\sim$ is transitive, and closed under inversion because $\sim$ is symmetric.

(c) Again this is trivial from transitivity of $\sim$.

(d) If $x\in \mathrm{Comm}_{GL_2(\mathbb{R})}\mathrm{SL}_2(\mathbb{Z})$, then so is λx for all $\lambda\in\mathbb{R}^\times$. So we may as well assume that $x\in\mathrm{SL}_2(\mathbb{R})$. Write $x = \begin{pmatrix} r & s\\ t & u\end{pmatrix}$, with $ru - st = 1$. Since $x^{-1}\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}x$ must be in $\mathrm{SL}_2(\mathbb{Z})$ for all h in some finite index subset of $\mathbb{Z}$. But this conjugate is equal to

$$\begin{pmatrix}1+hut & hu^2\\ -ht^2 & 1-hut\end{pmatrix}$$

Hence $u^2, t^2, ut\in\mathbb{Q}$. Similarly,

$$x^{-1}\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}x = \begin{pmatrix}1-hrs & -hs^2\\ hr^2 & 1+hrs\end{pmatrix}$$

so $r^2, s^2, rs\in\mathbb{Q}$. Manipulating these relations, one deduces that $x = \begin{pmatrix}a\sqrt{\lambda} & b\sqrt{\lambda}\\ c\sqrt{\mu} & d\sqrt{\mu}\end{pmatrix}$ with $a,b,c,d,\lambda,\mu\in\mathbb{Q}$. But since the determinant is 1, $(ad-bc)\sqrt{\lambda\mu} = 1$, and consequently we may write $x=\sqrt{\lambda}y$ for some $y\in\mathrm{GL}_2(\mathbb{Q})$. So $\mathrm{Comm}_{\mathrm{GL}_2\mathbb{R}}\mathrm{SL}_2\mathbb{Z}\subseteq\mathbb{R}^\times\cdot\mathrm{GL}_2(\mathbb{Q})$. Conversely any element of $\mathbb{R}^\times\cdot\mathrm{GL}_2(\mathbb{Q})$ does work.

□

Question 7. *Let U_1, U_2 be open subsets of $\mathbb{C}$ and $\alpha : U_1 \to U_2$ a holomorphic map. Let $\mu \in \Omega^1(U_2)$ be a differential that is holomorphic on U_2, and let γ be a path in U_1 (a continuous map $[0,1] \to U_1$ that is differentiable except at finitely many points). Show that*

$$\int_\gamma \alpha^* \mu = \int_{\alpha(\gamma)} \mu,$$

where the integrals are defined in the obvious way (so for $\omega = f(z)dz \in \Omega^1(U)$ and λ a path in U, $\int_\lambda \omega = \int_\lambda f(z)\mathrm{d}z$). Show that if X is a Riemann surface and γ is a path in X, γ can be partitioned into finitely many pieces each of which is contained in a single coordinate chart. Deduce that for any $\omega \in \Omega^1(X)$, there is a well-defined integral

$$\int_\gamma \omega$$

whenever γ does not intersect any pole of ω.

Solution. Routine check. The first statement is a restatement of the chain rule. The second follows because $[0,1]$ is compact, hence so is the image of γ; and the image of γ is covered by its intersections with the coordinate charts, so we can pick a finite set of charts that also cover it. □

Question 8. *Let X be a compact Riemann surface.*

(a) Let $f : X \to \mathbb{C}$ be meromorphic and nonzero. Show that f has finitely many zeros and poles, so $\operatorname{div}(f)$ is a finite sum.

(b) Let D be a divisor on X. Show (without using Riemann-Roch) that $L(D)$ is finite-dimensional, and $\ell(D) = \dim L(D)$ satisfies $\ell(D) \le 1 + \deg D$.

(c) Let D_1, D_2 be divisors with $D_1 - D_2 = \operatorname{div}(f)$ for some meromorphic f. Show that multiplication by f gives an isomorphism from $L(D_1)$ to $L(D_2)$, so $\ell(D_1) = \ell(D_2)$.

Solution. (a) Clearly every pole or zero has a neighbourhood containing no other pole or zero, since this is a local property and it is true for meromorphic functions on open subsets of $\mathbb{C}$. Thus by compactness there are finitely many of these.

(b) Without loss of generality suppose D is effective (all coefficients are ≥ 0). Fix a choice of coordinate chart containing each point in the support of D, and define a holomorphic map from $L(D)$ to $\mathbb{C}^{\deg D}$ by sending f to the negative coefficients in its Laurent expansion at each pole. The kernel of this map consists of functions holomorphic everywhere on X, which must be constant.

(c) Write $D \ge 0$ if the divisor D is effective. Then

$$\begin{aligned} g \in L(D_1) &\iff \operatorname{div} g + D_1 \ge 0 \\ &\iff \operatorname{div}(fg) + D_1 - \operatorname{div} f \ge 0 \\ &\iff \operatorname{div}(fg) + D_2 \ge 0 \\ &\iff fg \in L(D_2). \end{aligned}$$

□

Question 9. *(a) Let $\mathcal{H}_3$ be the space $\{(z, r) \mid z \in \mathbb{C}, r \in \mathbb{R}, r > 0\}$, sometimes called* **hyperbolic 3-space***. Let $\mathrm{SL}_2(\mathbb{C})$ act on $\mathcal{H}_3$ by the formula*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z, r) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}r^2}{|cz+d|^2 + |c|^2 r^2}, \frac{r}{|cz+d|^2 + |c|^2 r^2} \right).$$

(You may assume that this is a left action, although you can check it if you like!). Show that any discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ acts properly discontinuously on $\mathcal{H}_3$, using the same approach we used in lectures for discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathcal{H}$.

(b) Show that if K is an imaginary quadratic field, the Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$ is discrete in $\mathrm{SL}_2(\mathbb{C})$. Show that $\mathrm{SL}_2(\mathbb{C})$ acts on the "boundary" of $\mathcal{H}_3$ – that is, $\mathbb{P}^1(\mathbb{C})$, identified with $\{(z, 0) \mid z \in \mathbb{C}\} \cup \{\infty\}$ – and $\mathrm{SL}_2(\mathcal{O}_K)$ preserves $\mathbb{P}^1(K)$. Does it act transitively?

Solution. For part (a), we really can follow the steps from lectures precisely: all we need is to pick some point P in $\mathcal{H}_3$ for which

- $\mathrm{Stab}_{\mathrm{SL}_2\mathbb{C}} P$ is compact,
- there exists a continuous map $\lambda_P : \mathcal{H}_3 \to \mathrm{SL}_2(\mathbb{C})$ such that $[\lambda_P(Q)](P) = Q$ for all $Q \in \mathcal{H}^3$.

The natural choice of P is $(0,1)$; the stabiliser of this is exactly $\mathrm{SU}_2(\mathbb{C})$, which is compact. And it is easy to write down a λ which works: one example is

$$\lambda(z,r) = \begin{pmatrix} \sqrt{r} & z/\sqrt{r} \\ 0 & 1/\sqrt{r} \end{pmatrix}.$$

For part (b), the Bianchi group is clearly discrete in $\mathrm{SL}_2(\mathbb{C})$ because $\mathcal{O}_K$ is discrete in $\mathbb{C}$. If we formally set $r = 0$ in the formula defining the action, it reduces to $z \mapsto \frac{az+b}{cz+d}$, with the special case ∞ dealt with in the obvious way; and it is clear that this action preserves $\mathbb{P}^1(K)$. It is *not* transitive in general: the orbit of ∞ is the points x/y, $x, y \in \mathcal{O}_K$, such that the ideal generated by (x, y) is principal. The orbits are in bijection with the ideal class group of K. □

Question 10. *(a) Let $p \neq 2$ be prime. Show that any cusp for $\Gamma_1(p)$ is equivalent to one and only one of the set*

$$\left\{ \frac{d}{p} \;\middle|\; 1 \le d \le \frac{p-1}{2} \right\} \cup \left\{ \frac{1}{d} \;\middle|\; 1 \le d \le \frac{p-1}{2} \right\}.$$

(b) Show that the genus of $X_1(31) = X(\Gamma_1(31))$ is 26. (You may assume that $X_1(31)$ has no elliptic points.)
(c) Show that $\dim M_k(\Gamma_1(31)) = 40k - 25$ for all even $k \ge 2$.
(d) Assume that there exists a nonzero weight 1 modular function $f \in \mathcal{A}_1(\Gamma_1(31))$. By considering the differential attached to f^2, show that the space of meromorphic functions g on $X_1(31)$ such that gf is a holomorphic modular form of weight 1 is $\mathcal{L}(E)$ for some divisor E of degree 40. Deduce that the dimension of $M_k(\Gamma_1(31))$ is $40k - 25$ for all $k > 1$.
(e) Why doesn't this work for $k = 1$? Show that $d = \dim M_k(\Gamma_1(31))$ satisfies $15 \le d \le 41$. (In fact $d = 16$, but this is quite hard to show.)

Solution. (a) This is easy given that we know the cusps of $\Gamma_0(p)$, and a set of coset representatives for $\Gamma_0(p)/\Gamma_1(p)$. The first bracket is the cusps that map to ∞, and the second the cusps that map to 0.

(b) The index of $X_1(31)$ is 960, but it does not contain -1, so the degree of the projection $X_1(31) \to X_0(1)$ is 480. From part (a) there are 30 cusps, hence the genus is

$$1 + \frac{480}{12} - \frac{30}{2} = 26.$$

(c) Recall that the space of modular forms of even weight k is isomorphic to $\mathcal{L}(D_k + \frac{k}{2}\mathcal{K})$ where $\mathcal{K}$ is the canonical divisor on $X(\Gamma)$, and D_k is a certain divisor supported at cusps and elliptic points. Since there are no elliptic points, D_k here is just $\frac{k}{2} \cdot \{\text{cusps}\}$ and hence has degree $15k$. The divisor $\mathcal{K}$ has degree $2g - 2 = 50$, so the degree of $D_k + \frac{k}{2}\mathcal{K}$ is $40k$.

For $k \ge 2$, we see that the degree of $\mathcal{K} - (D_k + \frac{k}{2}\mathcal{K})$ is $50 - 40k < 0$, so $\ell(\mathcal{K} - (D_k + \frac{k}{2}\mathcal{K})) = 0$, and thus the Riemann-Roch formula tells us that

$$\dim M_k(\Gamma_1(31)) = 1 - g + \deg(D_k + \frac{k}{2}\mathcal{K}) = 40k - 25.$$

(d) If $f \in \mathcal{A}_1(\Gamma_1(31))$ is fixed, and g is a meromorphic function on $X_1(31)$, then the order of vanishing of fg at any point of $\mathcal{H}$ is clearly given by a formula that is linear in $\mathrm{ord}_P(g)$; so the requirement that $fg \in S_1(\Gamma_1(31))$ is exactly the requirement that $g \in L(D)$ for some divisor D. Similarly, one sees that for any $k \ge 1$ then $f^k g \in S_k(\Gamma_1(31))$ if and only if $g \in L(kD)$, so comparing with the formulae above for even k, one deduces that D has degree 40. As above, this implies the dimension formula.

(e) For $k = 1$, we have $\dim S_1(\Gamma_1(31)) = 15 + \ell(\mathcal{K} - D)$. The degree of $\mathcal{K} - D$ is 10. Hence $\ell(\mathcal{K} - D)$ is not necessarily 0; but clearly it is at most $1 + \deg(\mathcal{K} - D) = 11$, so $15 \le \dim M_1(\Gamma_1(31)) \le 26$. (I can't remember where I got 41 from.)

□

Question 11. *Let* $\Gamma = \Gamma_0(N)$ *or* $\Gamma_1(N)$*, and* $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$*. Write* $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$*. Show that*

$$\Gamma = \begin{cases} \Gamma' \cdot \tau_0 \cup \cdots \cup \Gamma' \cdot \tau_{p-1} & \text{if } p \mid N \\ \Gamma' \cdot \tau_0 \cup \cdots \cup \Gamma' \cdot \tau_{p-1} \cup \Gamma' \cdot \tau_\infty, & \text{if } p \nmid N \end{cases}$$

where $\tau_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ *and* τ_∞ *is any element of the form* $\begin{pmatrix} rp & s \\ N & 1 \end{pmatrix}$ *with* $rp - sN = 1$*. Deduce the formulae for the double coset operators* $[\Gamma\alpha\Gamma]$ *from Proposition 2.6.5.*

Solution. Let's do the Γ_1 case; the Γ_0 case is similar, but easier. Clearly

$$\Gamma' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; c = 0 \bmod N, a = c = 1 \bmod N, b = 0 \bmod p. \right\}.$$

Evidently any element of Γ with $a \neq 0 \bmod p$ may be put into Γ' by multiplication on the right by τ_i^{-1} for a unique i with $0 \leq i \leq p-1$. If $p \mid N$ then this is everything. If $p \nmid N$ then it can happen that $a = 0 \bmod p$, in which case it is easy to check that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} rp & s \\ N & 1 \end{pmatrix}^{-1} \in \Gamma'$. □

That's all the official solutions for this sheet: I don't have solutions written out for the harder problems, but anyone who is stuck on one of those is welcome to email me.