# MODULAR FORMS: EXAMPLE SHEET 4

DAVID LOEFFLER

## Exercises

*Question* 1. Let $m, n \in \mathbb{Z}$. Show that the modular symbol $\{m, n\}$ is 0 in $\mathbb{M}_2(\Gamma_0(N))$ for every $N$.

*Question* 2. Find a basis for the space $\mathbb{M}_2(\Gamma_0(11), \mathbb{Q})$, and calculate the matrix of the Hecke operator $T_3$ acting on your space. Verify that its characteristic polynomial is $(x-3)(x+2)^2$.

*Question* 3. Use Manin symbols to show that the quotient $\mathbb{M}_2(\Gamma_0(13), \mathbb{Z})/K_2(\Gamma_0(13), \mathbb{Z})$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/3$.

*Question* 4. Let $x$ be the element of $H_1(\Gamma_0(11), \mathbb{R})$ corresponding to the modular symbol $\{0, \infty\}$. Using the methods we used to prove the Manin-Drinfeld theorem, show that $5x$ is in $H_1(\Gamma_0(11), \mathbb{Z})$. Hence show that the unique cusp form $f = \sum a_n q^n$ of weight 2 and level $\Gamma_0(11)$ satisfies $a_n = \sigma_1(n) \bmod 5$.

*Question* 5. Let $K = \mathbb{Q}(\sqrt{-1})$, and let $\mathrm{SL}_2(\mathcal{O}_K)$ act on $\mathbb{P}^1(K)$ in the obvious way (as in Question 9 of sheet 2). Let $\mathbb{M}_2^K(\mathbb{Z})$ be the space of formal $\mathbb{Z}$-linear combinations of pairs $\{\alpha, \beta\}$, with $\alpha, \beta \in \mathbb{P}^1(K)$, modulo the usual relation $\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0$. Show that $\mathbb{M}_2^K(\mathbb{Z})$ is cyclic as a module over $\mathbb{Z}[\mathrm{SL}_2(\mathcal{O}_K)]$.

*Question* 6 (*For amusement value only).* We saw on sheet 3 that the function $f = q\prod_{n\geq 1}(1-q^n)(1-q^{23n})$ was an element of $S_2(\Gamma_1(23), \chi)$ where $\chi$ is the unique quadratic character of conductor 23. The Galois group of the splitting field of $X^3 - X - 1$ is isomorphic to $S_3$ and thus has a unique 2-dimensional irreducible representation $\rho$. Calculate $\mathrm{Tr}\,\rho(\mathrm{Frob}_p)$ for some primes $p \neq 23$ and verify that these are equal to the coefficients $a_p$ of $f$.

Assuming Artin's conjecture, show that $S_1(\Gamma_1(47))$ has a 2-dimensional subspace spanned by two eigenforms with coefficients in $\mathbb{Q}(\zeta_5)$, and calculate the $q$-expansions of these forms up to the $q^4$ term. (Hint: the class group of $\mathbb{Q}(\sqrt{-47})$ is cyclic of order 5.)

## Revision Questions

Past papers are generally an excellent way of revising for exams, but for Part III this is rendered more difficult by the fact that syllabuses vary heavily from year to year. Past papers from 2001 onwards are available from the Maths Faculty website; there has been a Modular Forms course in each of these years with the exception of 2005. The following past examination questions are accessible using only the material we have studied in the course:

- 2001: 1,2,3,5,6
- 2002: 1,4,5 and possibly 2
- 2003: 1,2,5 and possibly 3
- 2004: 1 and 3(i)
- 2006: 2,3,5
- 2007: 1,2,3,4,5,6
- 2008: 2,3

(Here "possibly" is intended to signify that the question is possible using only what we covered this year, but that additional material covered in that year's course would render the question somewhat easier.)

A selection of additional revision questions, in a style vaguely resembling that of a Part III examination, are given below.

*Question* 7. If $\Gamma$ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, define the Riemann surface structure on $X(\Gamma)$. By considering the map $X(\Gamma) \to X(\mathrm{SL}_2(\mathbb{Z}))$, or otherwise, show that the genus of $X(\Gamma)$ is

$$1 + \frac{d}{12} - \frac{\epsilon_\infty}{2} - \frac{\epsilon_3}{3} - \frac{\epsilon_2}{4}$$

where $d$ is the index of $\Gamma/\{\pm 1\}$ in $\mathrm{PSL}_2(\mathbb{Z})$, and $\epsilon_\infty$, $\epsilon_2$, $\epsilon_3$ denote the number of cusps and elliptic points of order 2 and 3 respectively.

Hence (or otherwise) show that if $p$ is a prime congruent to 1 mod 12, the genus of $X_0(p)$ is $\frac{p-1}{12} - 1$.

*Question* 8. What is meant by a *congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$? Let $\Gamma$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level $N$. Show that every cusp of $\Gamma$ has width dividing $N$.

If $\Gamma$ is an arbitrary finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, let $\Gamma'$ be the *congruence closure* of $\Gamma$, which is the intersection of all congruence subgroups containing $\Gamma$. Show that $\Gamma'$ is itself a congruence subgroup.

*Question* 9. Let $\Gamma$ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let $C(\Gamma)$ be the set of cusps of $\Gamma$. Show that $C(\Gamma)$ is finite.

Let $k \geq 4$ be an even integer. Define $\mathbb{B}_k(\Gamma, \mathbb{C})$ be the free vector space with basis the set $C(\Gamma)$, with a left action of the Hecke algebra $\mathcal{R}(\Gamma)$ defined by $[\Gamma\alpha\Gamma] \cdot c = (\det \alpha)^{k-1} \sum_i \alpha_i c$. Show that this is well-defined.

Let $c \in C(\Gamma)$ and $f \in M_k(\Gamma)$, with $k$ even. Show that if we define $f(c) = (f|_k\delta)(\infty)$, where $\delta$ is some element of $\mathrm{SL}_2(\mathbb{Z})$ such that $\delta(\infty) = c$, then $f(c)$ is independent of the choice of $\delta$ and the choice of representative for the equivalence class $c$.

Define a bilinear map $\langle , \rangle : \mathbb{B}_k(\Gamma, \mathbb{C}) \times M_k(\Gamma) \to \mathbb{C}$ by $\langle c, f\rangle = f(c)$. Show that

$$\langle Xc, f\rangle = \langle c, Xf\rangle$$

for all $X \in \mathcal{R}(\Gamma)$.

*Question* 10. Let $\Gamma$ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let $c$ be a cusp of $\Gamma$, $\delta \in \mathrm{SL}_2(\mathbb{Z})$ an element such that $\delta(\infty) = c$, and $m \geq 0$, $k \geq 3$ integers. Write $h = h_\Gamma(c)$. Show that the function $g_m(z) = e^{2\pi i m z/h} |_k \delta^{-1}$ is weight $k$ invariant under $\Gamma_c = \mathrm{Stab}_\Gamma(c)$. Show that $g$ is independent of the choice of $\delta$ if and only if $h \mid m$.

Suppose that $h \mid m$, and let $f_m(z) = \sum_{\gamma \in \Gamma_c \backslash \Gamma} (g_m |_k \gamma)$. Show that the sum converges absolutely, uniformly on any compact subset of $\mathcal{H}$; and that $f_m(z)$ is in $M_k(\Gamma)$, and in $S_k(\Gamma)$ if $m > 0$.

Now assume that $c = \infty$ and $h = 1$. Show that if $f = \sum a_n q^n$ is in $S_k(\Gamma)$, then

$$\langle f, f_m\rangle = (2k-2)!(4\pi m)^{1-2k} a_m.$$

(You may use without proof the following convergence estimate: for any real $k > 1$, there exists a constant $C$ such that $\sum_{n\in\mathbb{Z}} |z+n|^{-k} \leq C(|\operatorname{Im} z|^{1-k} + |\operatorname{Im} z|^{-k})$.

*Question* 11. Show that if $f \in M_k(\Gamma)$, for $k$ a positive even integer and $\Gamma$ a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of index $d < \infty$, containing $-1$, then we have

$$\sum_{z \in \Gamma\backslash\mathcal{H}} \frac{\operatorname{ord}_z f}{h_\Gamma(z)} + \sum_{P \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})} \operatorname{ord}_P f = \frac{kd}{12}.$$

Deduce that any element of $M_k(\Gamma)$ with $\operatorname{ord}_\infty f > \frac{kd}{12}$ must be zero. (You may use any result from the course about modular curves and differentials, as long as it is clearly stated.)

Let $E$ be the matrix algebra $\mathrm{End}_{\mathbb{C}} S_k(\Gamma_0(N))$, and let $H$ be the $\mathbb{C}$-vector subspace of $E$ spanned by the Hecke operators $T_n$ and $U_n$ for all integers $n \geq 1$. Show that the pairing $H \times S_k(\Gamma_0(N)) \to \mathbb{C}$ defined by $\langle A, f\rangle = a_1(Af)$ is a perfect pairing. Hence show that $H$ is spanned as a vector space by the Hecke operators $T_m$ and $U_m$ with $m \leq \frac{kd}{12}$.

*(Remark: Finding a bound on m such that $T_1, \ldots, T_m$ generate the analogue of H in $\mathrm{End}_{\mathbb{C}} M_k(\Gamma_0(N))$ appears to be much harder; if anyone can find such a bound I'd be very interested to know!)*