MODULAR FORMS: EXAMPLE SHEET 3

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Exercises

Question 1. Let w_N be the matrix $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$.

(a) Show that $w_N \Gamma_1(N) w_N^{-1} = \Gamma_1(N)$. Deduce that if $f \in S_k(\Gamma_1(N))$, then $w_N(f) = f|_k w_N \in S_k(\Gamma_1(N))$.
(b) Check that $w_N(i_{1,p} f) = i_{2,p}(w_{N/p} f)$ and $w_N(i_{2,p} f) = p^{k-1} i_{1,p}(w_{N/p} f)$.
(c) Show that w_N defines a normal operator with respect to the Petersson product. Deduce that it preserves $S_k(\Gamma_1(N))^{\mathrm{new}}$.
(d) If $f \in S_k(\Gamma_1(N), \chi)$, show that $w_N(f) \in S_k(\Gamma_1(N), \overline{\chi})$.

Question 2. Let $\sigma_{k-1}^{\chi,\psi}$ be the *generalised divisor sum* function defined by

$$\sigma_{k-1}^{\chi,\psi}(n) = \sum_{m|n} \psi(m)\chi(n/m) m^{k-1}.$$

Show that if m, n are coprime, then

$$\sigma_{k-1}^{\chi,\psi}(mn) = \sigma_{k-1}^{\chi,\psi}(m)\sigma_{k-1}^{\chi,\psi}(n).$$

Hence show that the functions $G_k^{\chi,\psi}(tz)$ occuring in Theorem 2.9.2 are eigenforms for the Hecke operators T_ℓ with $\ell \nmid N$, assuming that their q-expansions are as stated in the theorem.

Question 3. Let N, t, d be positive integers with $d < N$ and $td \mid N$. Show that if $f \in S_2(\Gamma_1(d))$, then $f(tz) \in S_2(\Gamma_1(N))^{\mathrm{old}}$.

Question 4. (a) Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ with p prime. Suppose $p \mid N$. Show that $\Gamma_0(Np)\alpha\Gamma_0(Np) = \Gamma_0(Np)\alpha\Gamma_0(N)$. Deduce that if $f \in S_k(\Gamma_0(Np))$, then $f|_k U_p \in S_k(\Gamma_0(N))$.
(b) Hence show that U_p is the zero map on $S_k(\Gamma_0(Np))^{\mathrm{new}}$.
(c) Show similarly that if $f \in S_k(\Gamma_0(Np))^{\mathrm{new}}$ and $p \nmid N$, and f is a U_p-eigenform, then the U_p-eigenvalue of f is $\pm p^{(k-1)/2}$.

Question 5. Let χ be the Dirichlet character $(\mathbb{Z}/5\mathbb{Z})^\times \to \{\pm 1\}$ with $\chi(d) = 1$ when d is a square mod 5 and -1 otherwise. Let $N \in \mathbb{N}$ be divisible by 5, and let

$$\Gamma_\chi(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; d \text{ is a square mod } 5 \right\}.$$

Show that for any even k,

$$S_k(\Gamma_\chi(N)) = S_k(\Gamma_0(N)) \oplus S_k(\Gamma_1(N), \chi).$$

Hence verify that the dimension of $S_4(\Gamma_1(25), \chi)$ is 4, as stated in lectures.

(You may assume that $X(\Gamma_\chi(25))$ has genus 0, 12 cusps and no elliptic points, while $X_0(25)$ has genus 0, 6 cusps, 2 elliptic points of order 2 and no elliptic points of order 3.)

Question 6. Suppose χ is the unique character of $(\mathbb{Z}/7\mathbb{Z})^\times$ with $\chi(3) = e^{2\pi i/3}$. Use question 1(d) above to show that $S_k(\Gamma_0(7)) + 2\dim S_k(\Gamma_1(7), \chi) = \dim S_k(\Gamma_1(7))$ for all even k. Hence find the dimension of $S_{12}(\Gamma_1(7), \chi)$.

Question 7. (a) Let $f \in S_k(\Gamma_1(N), \chi)$ be a simultaneous eigenform for the operators T_ℓ with $\ell \nmid N$. Let ℓ be any one such prime. Show that the matrix of U_ℓ on the subspace of $S_k(\Gamma_1(N\ell^r), \chi)$ spanned by $f(z), f(\ell z), \dots, f(\ell^r z)$ is given by

$$\begin{pmatrix} a_\ell & 1 & 0 & \dots & 0 \\ -\ell^{k-1}\chi(\ell) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(b) Show that this is never diagonalisable if $r \geq 3$. If $r = 1$ or $r = 2$, and the Hecke polynomial $X^2 - a_\ell X + \ell^{k-1}\chi(\ell)$ has distinct roots $\{\alpha, \beta\}$, show that it is diagonalisable and has eigenvalues $\{\alpha, \beta\}$ and $\{\alpha, \beta, 0\}$ respectively. If this quadratic has a repeated root, what happens?

(c) Let f be the function $f(z) = q\prod_{n\geq 1}(1 - q^n)(1 - q^{23n})$. You may assume that this is in $S_1(\Gamma_1(23), \chi)$ where $\chi(n) = \left(\frac{n}{23}\right)$ is the quadratic residue character modulo 23. Using a computer, find a prime ℓ such that the Hecke polynomial of f at ℓ has repeated roots.

Question 8. Let $(u, v) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $k \geq 4$ is even. Show that the sum

$$\sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1 \\ (c,d)=(u,v) \bmod N}} (cz + d)^{-k}$$

is an Eisenstein series for a cusp of $\Gamma(N)$. Hence show that the sum $G_k^{\psi,\chi}$ defined in lectures is in $N_k(\Gamma(N))$.

Question 9. Let $U \subseteq \mathbb{R}^2$ be an open set and let $\Omega^1_{sm}(U)$ denote the $\mathbb{C}$-vector space of expressions of the form $\lambda = p\mathrm{d}x + q\mathrm{d}y$ with p, q smooth functions $U \to \mathbb{C}$. These are the smooth 1-forms of Theorem 3.1.6.

(a) Suppose $f = (u(x, y), v(x, y)) : U_1 \to U_2$ is a smooth function, with U_1, U_2 open sets in $\mathbb{R}^2$ as above. Define $f^* : \Omega^1_{sm}(U_2) \to \Omega^1_{sm}(U_1)$ by $f^*(\mathrm{d}u) = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$, and similarly for $\mathrm{d}v$, extended in the obvious way to the whole space. Show that if we have a map $U_1 \to^g U_2 \to^f U_3$, then $(f \circ g)^* = g^* \circ f^*$. Deduce that $\Omega^1_{sm}(X)$ is well-defined, for any real 2-manifold X.

(b) Define $\star(\lambda) = -q\mathrm{d}x + p\mathrm{d}y$. This is the *Hodge star* operator. If f and λ are as before, calculate $\star(f^*\lambda)$ and $f^*(\star\lambda)$. Show that these are equal for all λ if and only if the map $f(x + iy) = u(x, y) + iv(x, y)$ satisfies the Cauchy-Riemann conditions, so f is a holomorphic map between subsets of $\mathbb{C}$. Hence show that if X is a Riemann surface, $\star$ gives a well-defined map from $\Omega^1_{sm}(X)$ to itself.

(c) Suppose that f is a smooth function on X. In local coordinates define $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$. Show that this gives a well-defined differential (independent of the choice of coordinate chart).

(d) We say $\lambda = p\mathrm{d}x + q\mathrm{d}y$ is *closed* if $\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$. Show that $\mathrm{d}f$ is always closed. When is $\star(\mathrm{d}f)$ closed?

(e) Show that λ and $\star\lambda$ are both closed if and only if we may write $\lambda = r\mathrm{d}z + s\mathrm{d}\bar{z}$ with r holomorphic and s anti-holomorphic. (This completes the proof of Theorem 3.1.6, on the assumption of Hodge's theorem.)

Question 10. Find an element $x \in \mathbb{Z}[\mathrm{SL}_2\,\mathbb{Z}]$ such that $x\{0, \infty\} = \{\frac{1339}{164}, -\frac{19}{28}\}$.

Question 11. Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Check the assertion made in the proof of Prop 3.2.5 that if g is an elliptic element of order h, so $g^h = \pm 1$, and α is any element of $\mathbb{P}^1(\mathbb{Q})$, then the element $h\{\alpha, g\alpha\}$ is zero in $\mathbb{M}_2(\mathbb{Z})/K_2(\Gamma)$.

Harder Problems

Question 12. In this question, we'll calculate some higher level cusp forms using a method similar to the one we used in the level 1 case.

(a) Let E_4 be the level 1 Eisenstein series from Chapter 1. Recall that $E_4(z)$ and $E_4(7z)$ are a basis for $N_4(\Gamma_0(7))$. Show that the forms $E_4(z)^2$ and $E_4(7z)^2$ are a basis for the Eisenstein subspace $N_8(\Gamma_0(7))$.

(b) Show that $E_4(7z)E_4(z)$ is not a linear combination of $E_4(z)^2$ and $E_4(7z)^2$. Deduce that there are constants a and b such that $E_4(z)E_4(7z) - (aE_4(z)^2 + bE_4(7z)^2)$ is in $S_8(\Gamma_0(7))$. Can you identify these constants?

(c) Let f be this cusp form. Calculate the functions $\{(T_2)^r f \mid 0 \le r \le 4\}$ and show that they are a basis for $S_8(\Gamma_0(7))$ (you may assume this space is 5-dimensional). Hence find the matrix of T_2 on this space, and verify that its characteristic polynomial is $(x+6)(x^2+3x-214)$.

Question 13. (a) Let $f \in S_k(\Gamma)$ for some finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, and $f = \sum_{n \ge 0} a_n e^{2\pi i n/h_\Gamma(\infty)}$, show that $f = \sum_{n \ge 0} \overline{a_n} e^{2\pi i n/h_\Gamma(\infty)}$ is also a modular form for some (possibly different) finite index subgroup $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{Z})$.

(b) Show that we may take $\Gamma' = \Gamma$ if $\Gamma = \Gamma_1(N)$. Deduce that $S_k(\Gamma_1(N))$ has a basis of modular forms whose q-expansions have real coefficients. Does this hold for the character subspaces $S_k(\Gamma_1(N), \chi)$?

Question 14 (Todd-Coxeter enumeration). Suppose Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and we suppose that we know an algorithm which, given an element of $\mathrm{SL}_2(\mathbb{Z})$, determines in finitely many steps whether or not it is in Γ.

Consider the following algorithm:

(1) Let $L = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

(2) For each $x \in L$, perform the following steps:

  (2a) Let $y = xS$ where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

  (2b) For each $z \in L$, test whether or not $yz^{-1} \in \Gamma$.

  (2c) If this does not hold for any z, replace L with $L \cup \{y\}$ and start again at step 2 above.

  (2d) Otherwise, let $y = xT$ where $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and repeat the two previous steps.

(3) Output the list L and terminate.

Show that this algorithm must terminate, and that the output is a set of coset representatives for $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$. Hence show that, given a membership testing algorithm for Γ, the we may compute the following data:

- the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$,
- the sets of elliptic points of order 2 and 3 on $X(\Gamma)$,
- a set of representatives for the cusps of Γ and a generator for the stabiliser of each cusp,
- the genus of $X(\Gamma)$,
- the dimension of $M_k(\Gamma)$ and $S_k(\Gamma)$ for any even $k \ge 0$,
- a set of generators of Γ (harder),
- a presentation for Γ (I don't know how to do this one).

Can you determine algorithmically whether or not Γ is a congruence subgroup?

Question 15. (This is a bit of a long-term project.)

(a) Let $F(z) = \Delta(11z)\Delta(z)^4$, which is a modular form for $\Gamma_0(11)$ of weight 60. Show that there is an analytic function $f(z)$ on $\mathcal{H}$ such that $F(z) = f(z)^5$ (that is, the branch of the fifth root may be chosen in a consistent way).

(b) Show that there is a character $\lambda : \Gamma_0(11) \to \mathbb{C}^\times$, with image contained in the group μ_5 of 5th roots of unity, such that if $\gamma \in \Gamma_0(11)$, we have $f|_{12}\gamma = \lambda(\gamma) f$. Show that λ is not the trivial character. Deduce that $f \in S_{12}(\Gamma)$ for a normal subgroup $\Gamma \subseteq \Gamma_0(11)$ of index 5.

(c) Write a computer program that, given an element of $\Gamma_0(11)$, determines whether or not it is in Γ.

(d) Apply Todd-Coxeter, and hence calculate the widths of the cusps of Γ.

(e) Wohlfart's theorem states that if Γ is a congruence subgroup of level exactly N, then the least common multiple of the widths of the cusps of Γ is N. Use this to show that Γ is not a congruence subgroup.