MODULAR FORMS: EXAMPLE SHEET 2

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Exercises

Question 1. Let Γ be an arbitrary finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Show that Γ contains a finite index *normal* subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Hence (or otherwise) show that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ for some $h \neq 0$.

Question 2. Find a non-constant holomorphic function f on $\mathcal{H}$ which is weakly modular of weight 0 and level $\Gamma_0(2)$ and holomorphic at ∞. Deduce that the space of functions on $\mathcal{H}$ which are weakly modular of weight k for $\Gamma_0(2)$ and holomorphic on $\mathcal{H} \cup \{\infty\}$ is infinite-dimensional for any k for which it is nonzero.

Question 3. Is $\mathcal{H}^*$ compact?

Question 4. (a) Show that $\Gamma_1(5)$ has 4 cusps: the orbits of $0, \frac{2}{5}, \frac{1}{2}, \infty$. For each cusp x, find the width $h_{\Gamma_1(5)}(x)$ of the cusp and a generator for its stabiliser.

(b) Show that $\Gamma_0(5)$ has no elliptic points of order 3, and deduce that the same is true of $\Gamma_0(5)$. Show that $\Gamma_0(5)$ has precisely 2 elliptic points of order 2, and for each one, show that none of its four preimages in $\Gamma_1(5)$ is elliptic.

(c) What is the genus of $X_1(5) = X(\Gamma_1(5))$?

(Hint: the calculations can be shortened somewhat by using the fact that $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$.)

Question 5. Fill in the details of the proof of Corollary 2.3.5, by checking the formulae given for the sums $\sum_{x \in \pi^{-1} y} (e_x(\pi) - 1)$ for the three non-ordinary points $y = \infty, i, \rho$ of $X(\mathrm{SL}_2(\mathbb{Z}))$.

Question 6. Let G be any group. Recall that we say subgroups H_1, H_2 of G are **commensurable** if $H_1 \cap H_2$ has finite index in either H_1 or H_2. Let us write this relation by $H_1 \sim H_2$.

(a) Show that commensurability is an equivalence relation.

(b) Let H be any subgroup. Define the **commensurator** of H in G, $\mathrm{Comm}_G(H)$, to be the set of $g \in G$ such that $g^{-1} H g$ is commensurable with H. Show that this is a group.

(c) Show that if $H_1 \sim H_2$, then $\mathrm{Comm}_G(H_1) = \mathrm{Comm}_G(H_2)$.

(d) What is the commensurator of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{GL}_2(\mathbb{R})$?

Question 7. Let U_1, U_2 be open subsets of $\mathbb{C}$ and $\alpha : U_1 \to U_2$ a holomorphic map. Let $\mu \in \Omega^1(U_2)$ be a differential that is holomorphic on U_2, and let γ be a path in U_1 (a continuous map $[0, 1] \to U_1$ that is differentiable except at finitely many points). Show that

$$\int_\gamma \alpha^* \mu = \int_{\alpha(\gamma)} \mu,$$

where the integrals are defined in the obvious way (so for $\omega = f(z) dz \in \Omega^1(U)$ and λ a path in U, $\int_\lambda \omega = \int_\lambda f(z) \mathrm{d}z$). Show that if X is a Riemann surface and γ is a path in X, γ can be partitioned into finitely many pieces each of which is contained in a single coordinate chart. Deduce that for any $\omega \in \Omega^1(X)$, there is a well-defined integral

$$\int_\gamma \omega$$

whenever γ does not intersect any pole of ω.

Question 8. Let X be a compact Riemann surface.

(a) Let $f : X \to \mathbb{C}$ be meromorphic and nonzero. Show that f has finitely many zeros and poles, so $\mathrm{div}(f)$ is a finite sum.

(b) Let D be a divisor on X. Show (without using Riemann-Roch) that $L(D)$ is finite-dimensional, and $\ell(D) = \dim L(D)$ satisfies $\ell(D) \leq 1 + \deg D$.
(c) Let D_1, D_2 be divisors with $D_1 - D_2 = \mathrm{div}(f)$ for some meromorphic f. Show that multiplication by f gives an isomorphism from $L(D_1)$ to $L(D_2)$, so $\ell(D_1) = \ell(D_2)$.

Question 9. (a) Let $\mathcal{H}_3$ be the space $\{(z, r) \mid z \in \mathbb{C}, r \in \mathbb{R}, r > 0\}$, sometimes called *hyperbolic 3-space*. Let $\mathrm{SL}_2(\mathbb{C})$ act on $\mathcal{H}_3$ by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z, r) = \left(\frac{(az+b)\overline{(cz+d)} + a\bar{c}r^2}{|cz+d|^2 + |c|^2 r^2}, \frac{r}{|cz+d|^2 + |c|^2 r^2} \right).$$

(You may assume that this is a left action, although you can check it if you like!). Show that any discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ acts properly discontinuously on $\mathcal{H}_3$, using the same approach we used in lectures for discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathcal{H}$.
(b) Show that if K is an imaginary quadratic field, the *Bianchi group* $\mathrm{SL}_2(\mathcal{O}_K)$ is discrete in $\mathrm{SL}_2(\mathbb{C})$. Show that $\mathrm{SL}_2(\mathbb{C})$ acts on the "boundary" of $\mathcal{H}_3$ – that is, $\mathbb{P}^1(\mathbb{C})$, identified with $\{(z, 0) \mid z \in \mathbb{C}\} \cup \{\infty\}$ – and $\mathrm{SL}_2(\mathcal{O}_K)$ preserves $\mathbb{P}^1(K)$. Does it act transitively?

Question 10. (a) Let $p \neq 2$ be prime. Show that any cusp for $\Gamma_1(p)$ is equivalent to one and only one of the set

$$\left\{ \frac{d}{p} \;\middle|\; 1 \leq d \leq \frac{p-1}{2} \right\} \cup \left\{ \frac{1}{d} \;\middle|\; 1 \leq d \leq \frac{p-1}{2} \right\}.$$

(b) Show that the genus of $X_1(31) = X(\Gamma_1(31))$ is 26. *(You may assume that $X_1(31)$ has no elliptic points.)*
(c) Show that $\dim M_k(\Gamma_1(31)) = 40k - 25$ for all even $k \geq 2$.
(d) Assume that there exists a nonzero weight 1 modular function $f \in \mathcal{A}_1(\Gamma_1(31))$. By considering the differential attached to f^2, show that the space of meromorphic functions g on $X_1(31)$ such that gf is a holomorphic modular form of weight 1 is $\mathcal{L}(E)$ for some divisor E of degree 40. Deduce that the dimension of $M_k(\Gamma_1(31))$ is $40k - 25$ for all $k > 1$.
(e) Why doesn't this work for $k = 1$? Show that $d = \dim M_k(\Gamma_1(31))$ satisfies $15 \leq d \leq 41$. *(In fact $d = 16$, but this is quite hard to show.)*

Question 11. Let $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$, and $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Write $\Gamma' = \Gamma \cap \alpha^{-1}\Gamma\alpha$. Show that

$$\Gamma = \begin{cases} \Gamma' \cdot \tau_0 \cup \cdots \cup \Gamma' \cdot \tau_{p-1} & \text{if } p \mid N \\ \Gamma' \cdot \tau_0 \cup \cdots \cup \Gamma' \cdot \tau_{p-1} \cup \Gamma' \cdot \tau_\infty, & \text{if } p \nmid N \end{cases}$$

where $\tau_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ and τ_∞ is any element of the form $\begin{pmatrix} rp & s \\ N & 1 \end{pmatrix}$ with $rp - sN = 1$. Deduce the formulae for the double coset operators $[\Gamma\alpha\Gamma]$ from Proposition 2.6.5.

Harder Problems

Question 12. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup. Show that if $\Gamma \supseteq \mathrm{SL}_2(\mathbb{Z})$, then $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, so $\mathrm{SL}_2(\mathbb{Z})$ is a maximal discrete subgroup.

Question 13. Let $G = \mathrm{SL}_2(\mathbb{Z})$.
(a) Let N be an integer and $t \in \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$. Show that there is some $\tilde{t} \in G$ such that $\tilde{t} \bmod m = t$.
(b) Let J be the directed set of normal congruence subgroups of G. Use part (a) to show that the completion $\overline{G} = \varprojlim_{\Gamma \in J} G/\Gamma$ is equal to $\mathrm{SL}_2(\hat{\mathbb{Z}})$, where the topological ring $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.
(c) Let I be the directed set of all normal finite index subgroups of G. Let $\hat{G} = \varprojlim_{\Gamma \in I} G/\Gamma$. Show that there is a canonical continuous surjective homomorphism $\hat{G} \to \overline{G}$.
(d) The kernel of the above map is called the *congruence kernel*. Show that the congruence kernel is trivial if and only if every finite index subgroup is a congruence subgroup.

Question 14 (*For those also doing Elliptic Curves)*. Let Λ be a lattice in $\mathbb{C}$, with basis ω_1, ω_2. Then you will have seen in the Elliptic Curves course that the Weierstrass $\wp$-function $\wp(z) = \wp(z, \omega_1, \omega_2)$ satisfies

$$\wp'(z) = \wp(z)^3 - g_4\wp(z) - g_6$$

for complex numbers g_4, g_6 independent of z, and the map $z \mapsto (x, y) = (\wp(z), \wp'(z))$ gives an isomorphism between the quotient $\mathbb{C}/\Lambda$ and the elliptic curve in $\mathbb{P}^1$ defined by this equation.

Show that if ω is a holomorphic 1-form on an elliptic curve E over $\mathbb{C}$, then integrating with respect to ω maps the integral homology $H^1(E(\mathbb{C}), \mathbb{Z})$ to a lattice Λ in $\mathbb{C}$; and these constructions are mutually inverse, with ω corresponding to the differential dz on $\mathbb{C}/\Lambda$.

Show that g_4 and g_6 are (up to normalisation[1]) exactly the functions on lattices associated to the Eisenstein series G_4 and G_6. Deduce that the definition in this course of the j-invariant (as a weight 0 modular function) and the definition in the Elliptic Curves course (as an invariant attached to the Weierstrass equation of an elliptic curve) coincide.

Let f be a modular form of weight k and level 1, and F the corresponding function on the space $\mathcal{L}$ of lattices. Let (E, ω) be a pair of an elliptic curve and a holomorphic 1-form. Set $F(E, \omega)$ to be F evaluated at the lattice in $\mathbb{C}$ corresponding to (E, ω). Deduce that $f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$ for $\lambda \in \mathbb{C}^\times$. (This is the starting point for the algebraic theory of modular forms.)

Question 15. Let $\mathcal{O}$ denote the ring $\mathbb{Z} + \mathbb{Z}\sqrt{7}$. For $x = a + b\sqrt{7} \in \mathcal{O}$, define $\overline{x} = a - b\sqrt{7}$. Show that

$$\Gamma = \left\{ \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} \;\middle|\; x, y \in \mathcal{O},\ x\overline{x} + y\overline{y} = 1 \right\}$$

is a discrete subgroup of $SL_2(\mathbb{R})$. Is it maximal?

Question 16. If $z \in \mathrm{SL}_2(\mathbb{R})$, we say z is *parabolic* if z is conjugate to $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that any parabolic element z generates an infinite discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ consisting solely of parabolic elements; and z fixes no points of $\mathcal{H}$ and a unique point of $\mathbb{P}^1(\mathbb{R})$.

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ (a *Fuchsian group*). If $v \in \mathbb{P}^1(\mathbb{R})$ is a fixed point of a parabolic element of Γ, then we say v is a *parabolic point* for Γ.

(a) Show that if Γ' is commensurable with Γ, then the sets of parabolic points for Γ and Γ' coincide.
(b) Show that the parabolic points for $\mathrm{SL}_2(\mathbb{Z})$ are $\mathbb{P}^1(\mathbb{Q})$.
(c) For the example of question 15, show that there are no parabolic points. Deduce that Γ is not commensurable with any conjugate of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{R})$. What if $\sqrt{7}$ is replaced by $\sqrt{5}$? What about $\sqrt{N}$ for a general integer N?
(d) Show that if $C(\Gamma)$ is the set of Γ-orbits of parabolic points for Γ, and $X(\Gamma) = Y(\Gamma) \cup C(\Gamma)$, then $X(\Gamma)$ has a natural topology and a complex structure extending that of $Y(\Gamma)$.
(e) Show that $X(\Gamma)$ is compact if and only if $C(\Gamma)$ is finite. Describe $X(\Gamma)$ when Γ is the subgroup generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Question 17. Let p be prime. Show that $X_0(p) = X(\Gamma_0(p))$ has genus 0 if and only if $p \in \{2, 3, 5, 7, 13\}$. If N is a general integer and $X_0(N)$ has genus 0, show that all prime factors of N must be from the above set. Are there infinitely many such N?

[1] I don't know what normalisation Dr Fisher is using.