MODULAR FORMS: EXAMPLE SHEET 1

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Remark: A computer is helpful for some of these questions. I suggest you use the Web interface to William Stein's Sage (Software for Algebra and Geometry Experimentation) – follow the links from my web page `http://www.dpmms.cam.ac.uk/~dl267/`.

Exercises

Question 1. Show that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ is transitive, and the stabiliser of i is the group $\mathrm{SO}_2(\mathbb{R})$. (Thus $\mathcal{H}$ is a quotient of the semisimple real Lie group $\mathrm{SL}_2(\mathbb{R})$ by a maximal compact subgroup.)

Question 2. Let $\mathcal{L}$ be the ring of holomorphic functions on $\mathcal{H}$, with the usual (weight 0) action of $\mathrm{SL}_2(\mathbb{R})$. Show that the function $j_k : \mathrm{SL}_2(\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ defined by $j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = (cz+d)^k$ satisfies

$$j_k(\gamma_1\gamma_2, z) = j_k(\gamma_1, \gamma_2 z) \cdot j_k(\gamma_2, z).$$

Deduce that the weight k action on $\mathcal{L}$ defined in lectures really is a right action, i.e.

$$f|_k(\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2.$$

Question 3. Complete the proof of Theorem 1.2.2 by showing that if $z, z' \in \mathcal{D}$ and $\gamma z = z'$ for some $\gamma \in G = \mathrm{PSL}_2(\mathbb{Z})$ with $\gamma \neq 1$, then one of the following holds: $\operatorname{Re} z = \pm\frac{1}{2}$ and $\gamma = \begin{pmatrix} 1 & \mp 1 \\ 0 & 1 \end{pmatrix}$; $|z| = 1$ and $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $z = z' = i$; or z and z' are in the set $\{\rho, 1+\rho\}$ where ρ is a cube root of unity. Find the elements of $\mathrm{Stab}_G(i)$ and $\mathrm{Stab}_G(\rho)$, and verify that they have order 2 and 3 respectively.

Question 4. Express $\begin{pmatrix} 70 & 213 \\ 23 & 70 \end{pmatrix}$ in terms of the generators $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$.

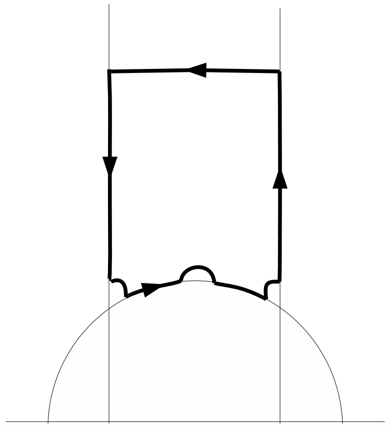
Question 5. Show that for $k \geq 4$ the sum defining the Eisenstein series G_k,

$$G_k(z) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m+nz)^k}$$

is absolutely convergent, locally uniformly in z. Deduce that $G_k(z)$ is holomorphic and independent of the order of summation.

Question 6. Show that E_6 has a single simple zero at i, and E_4 a single simple zero at ρ where ρ is a cube root of unity; and neither has any other zeroes.

Question 7. Let f be a nonzero modular function of weight k. For simplicity, assume f has no zeros on the boundary of the fundamental domain $\mathcal{D}$, except possibly at the points i, ρ, $1+\rho$. Note that $\frac{\mathrm{d}f}{f}$ is a meromorphic differential on $\mathcal{H}$ with a simple pole at each zero or pole of f, with residue at z equal to $\operatorname{ord}_z f$. Consider the contour $\mathcal{C}$ illustrated below:



Deduce the valence formula by finding the limit of the integral $\oint_{\mathcal{C}} \frac{\mathrm{d}f}{f}$ (as the height of the top horizontal line tends to ∞ and the radius of the three small arcs tends to zero) in two ways, first by using the residue theorem, and secondly by splitting up the integral into its component arcs.

Question 8. Give a careful proof of the claim made in the course of the proof of Prop 1.4.5: if L is a lattice in $\mathbb{C}$, and L' is a sublattice of index p^{n+1}, then the number of lattices L'' with $L \supset L'' \supset L'$ and $|L/L''| = p$ is $p+1$ if $L'' \subseteq pL$ and 1 otherwise.

Question 9. Let $k = 24$. Calculate the q-expansions of a basis for the 3-dimensional space M_k, up to and including the q^6 term. Using the formulae for the Hecke operators in terms of q-expansions, compute the first 3 terms of the q-expansons of $T_2 f$ for each f in your basis, and hence find the matrix of T_2. Verify that the characteristic polynomial of T_2 is

$$(x - 8388609)(x^2 - 1080x - 20468736).$$

(You will need a computer for this question. Feel free to do it how you like, but you might want to use the Sage worksheet linked from my web page.)

Question 10. Show that if $k \geq 0$ is even, and $d = \dim M_k$, there are unique elements $f_0, \dots, f_{d-1}$ in M_k such that for $0 \leq i, j \leq d-1$, the coefficient of q^i in f_j is 1 if $i = j$ and 0 otherwise. Show also that the f_j are in $\mathbb{Z}[[q]]$, so they give a $\mathbb{Z}$-basis for the module of weight k modular forms with integral coefficients. (This basis is called the *Victor Miller basis*). Compute the first 5 terms of the q-expansion of each function in the Victor Miller basis for M_{24}.

Question 11. Show that the index of $\Gamma(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is

$$N^3 \prod_{\substack{p \mid N \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right).$$

Find similar formulae for $\Gamma_0(N)$ and $\Gamma_1(N)$.

Question 12. Show that for any prime p, $\Gamma_0(p)$ has exactly 2 cusps. What can you say about $\Gamma_1(p)$?

Question 13. Let f be weakly modular of some weight k and level $\mathrm{SL}_2(\mathbb{Z})$. Show that $z \mapsto f(Nz)$ is weakly modular of level $\Gamma_0(N)$. If f is a modular form, is $f(Nz)$?

Harder Problems

Question 14. Show that j defines an element of the group cohomology $H^1(\mathrm{SL}_2(\mathbb{R}), \mathcal{L}^\times)$, where $\mathcal{L}^\times$ is considered as a right $\mathrm{SL}_2(\mathbb{R})$-module via the usual (weight 0) action. Is it a coboundary? (In other words, is there a nowhere vanishing function f on $\mathcal{H}$ such that

$$j_k(\gamma, z) = f(\gamma z)/f(z)$$

for all $z \in \mathcal{H}, \gamma \in SL_2(\mathbb{R})$?) What if we restrict to $\gamma \in \mathrm{SL}_2(\mathbb{Z})$?

Question 15. Show that any automorphism of $\mathcal{H}$ which is biholomorphic (holomorphic with holomorphic inverse) is given by an element of $PSL_2(\mathbb{R})$.

Question 16. The elements $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfy $b^2 = (ba)^3 = 1$ in $\mathrm{PSL}_2(\mathbb{Z})$. Are these two relations a complete set of relations? (i.e. is $\mathrm{PSL}_2(\mathbb{Z})$ isomorphic to the free group on two generators quotiented by these two relations?)

Question 17. (a) Show (directly from the definition) that $\Delta = \frac{E_4^3 - E_6^2}{1728}$ has integer coefficients.

(b) Prove Ramanujan's congruence for the coefficients of Δ: if $\Delta = \sum a_n q^n$, then $a_n = \sigma_{11}(n) \bmod 691$ for $n \geq 1$. (Hint: $\frac{B_{12}}{24} = -\frac{691}{65520}$.)

Question 18. (a) Show that the graded ring $\mathcal{M}(\mathbb{Z})$ of modular forms whose q-expansions are in $\mathbb{Z}[[q]]$ is generated over $\mathbb{Z}$ by E_4, E_6, and Δ.

(b) Let $p \geq 5$ be prime. Let $\mathcal{I}_p$ be the kernel of the natural map $\mathcal{M}(\mathbb{Z}) \to \mathbb{F}_p[[q]]$. Show that $\mathcal{M}(\mathbb{F}_p) = \mathcal{M}(\mathbb{Z})/\mathcal{I}_p$ is generated as an $\mathbb{F}_p$-algebra by the images of E_4 and E_6 (so we no longer need Δ). Take $p = 17$; show that the images Q and R of E_4 and E_6 satisfy $8Q^4 - 7QR^2 = 1$.

(c) Show that $\mathcal{M}/\mathcal{I}_{17}$ is isomorphic to $\mathbb{F}_p[Q, R]/(8Q^4 - 7QR^2 - 1)$. Deduce that if $f \in M_{k_1} \cap \mathcal{M}$ and $g \in M_{k_2} \cap \mathcal{M}$, and $f - g \in \mathcal{I}_{17}$, then $k_1 = k_2 \bmod 16$; in other words, elements of $\mathcal{M}/\mathcal{I}_{17}$ have well-defined weights modulo 16.

Question 19. (a) Show that the weight 2 Eisenstein series

$$G_2(z) = \sum_{m,n \in \mathbb{Z}^2} \frac{1}{(m + nz)^2}$$

is not absolutely convergent, but that if we choose the order of summation to be

$$\sum_{n \in Z} \left(\sum_{m \in \mathbb{Z}}{}' \frac{1}{(m + nz)^2} \right)$$

(where the dash denotes that the $(0, 0)$ term is omitted), then it is convergent (locally uniformly in z), $G_2(z + 1) = G_2(z)$, and the q-expansion formula of proposition 1.3.4 is still valid.

(b) Show that

$$G_2(-1/z) = z^2 G_2(z) - 2\pi i z.$$

(Hint: If $\tilde{G}_2$ denotes the double sum with the opposite ordering of the variables, then $G_2(-1/z) = z^2 \tilde{G}_2(z)$. Compare this double sum with the analogous sums $H, \tilde{H}$ whose (m, n) term is $\frac{1}{(m-1+nz)(m+nz)}$, again with the two possible orderings. The difference between G_2 and H is absolutely summable, so the "error" in changing the order of the variables is the same for G_2 and for H; but the H sum telescopes.)

Question 20. Let $F(z) = q \prod_{n \geq 1} (1 - q^n)^{24}$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}z} \log F(z) = \frac{6i}{\pi} G_2(z).$$

Deduce that $\log F(-1/z) - \log z^{12} F(z)$ is constant. Set $z = i$ to deduce that the constant is 1, so F is a modular form of weight 12. Deduce that $F = \Delta$.

Question 21. (a) Show that if $E_2(z) = G_2(z)/2\zeta(2) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$, and f is any modular form of weight k, then $\partial f = 12 q \frac{\mathrm{d}f}{\mathrm{d}q} - k E_2 f$ is a modular form of weight $k + 2$. (This operator ∂ was introduced by Ramanujan.)

(b) Deduce that if $f = \sum_{n \geq 0} a_n q^n$ is a modular form of weight k with integral coefficients, and p is prime, then there exists a modular form $g = \sum_{n \geq 0} c_n q^n$ of weight $k + p + 1$ such that $c_n = n a_n \bmod p$. (So the operator $\theta = q \frac{d}{dq}$ preserves the space of mod p modular forms.)

(c) Use this to give an extremely short proof that if a_n is the nth coefficient of Δ, $a_n = n \sigma_3(n) \bmod 7$.