Numer. Math.
DOI 10.1007/s00211-011-0386-z

Numerische Mathematik

Numerical analysis of an inverse problem for the eikonal equation

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Received: 29 July 2010 / Revised: 15 April 2011

Abstract We are concerned with the inverse problem for an eikonal equation of determining the speed function using observations of the arrival time on a fixed surface. This is formulated as an optimisation problem for a quadratic functional with the state equation being the eikonal equation coupled to the so-called Soner boundary condition. The state equation is discretised by a suitable finite difference scheme for which we obtain existence, uniqueness and an error bound. We set up an approximate optimisation problem and show that a subsequence of the discrete mimina converges to a solution of the continuous optimisation problem as the mesh size goes to zero. The derivative of the discrete functional is calculated with the help of an adjoint equation which can be solved efficiently by using fast marching techniques. Finally we describe some numerical results.

Mathematics Subject Classification (2010) 49J20 · 49L25 · 49M25 · 65M06 · 65N06

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Published online: 24 June 2011

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a Lipschitz boundary Γ and $x_0 \in \Omega$ be fixed. For a continuous, positive function $a : \bar{\Omega} \to \mathbb{R}$ and $x \in \bar{\Omega}$, $x \neq x_0$ we consider the minimisation problem

$$\inf \left\{ \int_0^1 a(\xi(r))|\xi'(r)|dr \mid \xi \in W^{1,\infty}([0,1], \bar{\Omega}),\ \xi(0) = x_0, \xi(1) = x \right\}. \tag{1.1}$$

Its optimal value, $u(x)$, gives the shortest travel time of the path connecting x_0 to x in $\bar{\Omega}$ with underlying velocity $c(x) = \frac{1}{a(x)}$. An important problem in various applications, e.g. tomography, consists in reconstructing the slowness function a from measured first arrival times on a suitable subset of Γ. A common approach to solve this inverse problem aims at minimising the misfit between the measured data and computed traveltimes, that are obtained from integrating the Euler–Lagrange equation corresponding to (1.1), see e.g. [6,19]. A more recent approach, see e.g. [13], makes use of the fact that the optimal value function for (1.1) formally is a solution of the following eikonal equation, see [12],

$$|\nabla u| = a(x), \quad x \in \Omega \setminus \{x_0\}; \tag{1.2}$$

with boundary conditions

$$u(x_0) = 0, \tag{1.3}$$

$$\nabla u(x) \cdot \nu(x) \geq 0, \quad x \in \Gamma. \tag{1.4}$$

Here, ν is the unit outer normal to Γ. The condition (1.4) is a consequence of the definition, (1.1), of the first arrival time, $u(x)$, at a point $x \in \bar{\Omega}$ which constrains paths from the source to the arrival point, x, to lie in $\bar{\Omega}$. Informally observe that the gradient of the first arrival time for an optimal path is in the tangential direction of the path and on the boundary Γ the tangent to this path, which is constrained to lie in $\bar{\Omega}$, has a non-negative component in the outward pointing normal direction.

It can be shown that the above problem has a unique Lipschitz continuous solution $u = u_a$ that satisfies (1.2), (1.3) and (1.4) in the viscosity sense, see Sect. 2. Let us return to the above-mentioned inverse problem and assume that the measured arrival times are given by a function $u_{obs} : \Gamma \to \mathbb{R}_{>0}$. Then, the misfit functional takes the form

$$\mathcal{J}(a) = \frac{1}{2} \int_\Gamma |u_a(x) - u_{obs}(x)|^2 \, do_x \tag{1.5}$$

which needs to be minimised over a suitable set K of slowness functions. The functional (1.5) may be generalised by considering several source points x_0^j, $j = 1, \ldots, S$ with first arrival times $u_{obs}^j : \Gamma \to \mathbb{R}_{>0}$ resulting in

$$\mathcal{J}(a) = \frac{1}{2} \sum_{j=1}^{S} \int_{\Gamma} |u_a^j(x) - u_{obs}^j(x)|^2 \, do_x. \tag{1.6}$$

The above approach has been studied numerically in the context of tomography in [11,17,18] using finite difference approximations of the eikonal equation and fast sweeping methods for solving the forward equation and the adjoint equation. The aim of the present work is to present a corresponding numerical analysis. Let us outline the contents of this paper and our contributions.

We begin in Sect. 2 with a brief review of the existence and uniqueness theory for (1.2)–(1.4) including a description of the viscosity formulation for (1.4), the so–called Soner boundary condition. In Sect. 3 we discretize (1.2)–(1.4) with the help of a monotone finite difference scheme. We show existence and uniqueness of the discrete solution and derive an $O(h^{\frac{1}{2}})$ error bound, which appears to be new for a Hamilton–Jacobi equation coupled to the Soner boundary condition. The discrete solution is computed with the help of the fast marching method and we show that the procedure terminates in a finite number of cycles. In Sect. 3 we address the problem of approximating the functional (1.5). We assume that the set K of admissible slowness functions consists of continuous functions that are finite linear combinations of a given partition of unity. Replacing u_a in (1.5) by the discrete solution gives rise to an approximate minimisation problem, which is shown to have a solution. We then prove that a subsequence of the discrete minima converges to a solution of the continuous minimisation problem. In practice, the discrete optimisation problem is solved by a descent method and the derivative of the discrete functional is calculated with the help of a discrete adjoint equation. In order to be able to derive this equation we require differentiability of the discrete state with respect to the slowness function which is ensured by a suitable choice of the finite difference scheme. We then show that the discrete adjoint equation has a unique solution. Furthermore, when the solution of the state equation has been computed by fast marching, the resulting ordering of grid values can be used in order to efficiently compute the adjoint solution without solving an equation. In Sect. 4 we finally present a series of numerical tests in which we consider the more general functional (1.6) and apply our discretization strategy to varying geometries and numbers of degrees of freedom for a. Let us finish this introduction by referring to [5,9], where optimal control problems for time dependent Hamilton-Jacobi equations were considered.

2 Wellposedness and approximation of the state equation

2.1 Existence and uniqueness

Definition 2.1 A function $u \in C^0(\Omega)$ is called a viscosity subsolution of (1.2) in $\Omega \setminus \{x_0\}$ if for each $\zeta \in C^\infty(\Omega)$: if $u - \zeta$ has a local maximum at a point $x \in \Omega \setminus \{x_0\}$, then

$$|\nabla \zeta(x)| \leq a(x).$$

A function $u \in C^0(\bar{\Omega})$ is called a viscosity supersolution of (1.2) in $\bar{\Omega} \setminus \{x_0\}$ if for each $\zeta \in C^\infty(\mathbb{R}^n)$: if $u - \zeta$ has a local minimum at a point $x \in \bar{\Omega} \setminus \{x_0\}$, relative to $\bar{\Omega}$, then

$$|\nabla \zeta(x)| \geq a(x).$$

A viscosity solution of (1.2), (1.3), (1.4) is then a function $u \in C^0(\bar{\Omega})$ which is a viscosity subsolution in $\Omega \setminus \{x_0\}$, a viscosity supersolution in $\bar{\Omega} \setminus \{x_0\}$ and which satisfies $u(x_0) = 0$.

Note that there is an asymmetry between the definitions of sub- and super-solutions in the above definition. The fact that u has to be a supersolution on Γ is a viscosity solution interpretation of the boundary condition (1.4). This kind of condition, also referred to as Soner boundary condition, is relevant in optimal control problems with constraints on the state variable, cf. [7,16].

In what follows we shall assume the following regularity condition on Ω, cf. [3,16], p. 278: there exists a continuous function $\eta : \bar{\Omega} \to \mathbb{R}^n$ and $\epsilon > 0$ such that

$$B_{\epsilon s}(x + s\eta(x)) \subset \Omega \quad \text{for all } x \in \bar{\Omega}, 0 < s \leq \epsilon. \tag{2.1}$$

Theorem 2.2 *Suppose that $a \in C^0(\bar{\Omega})$ is positive. Then there exists a unique viscosity solution $u \in C^0(\bar{\Omega})$ of (1.2)–(1.4). The solution is given by the formula*

$$u(x) = \inf \left\{ \int_0^1 a(\xi(r))|\xi'(r)|\, dr \mid \xi \in W^{1,\infty}([0,1], \bar{\Omega}),\ \xi(0) = x_0, \xi(1) = x \right\}.$$

Furthermore, there exists a constant $C = C(\Omega)$ such that u is Lipschitz continuous in Ω with an upper bound on the Lipschitz constant satisfying

$$lip\ (u) \leq C \max_{\bar{\Omega}} a. \tag{2.2}$$

Proof See [7,16]. □

2.2 Discretization of the state equation

The numerical solution of the boundary value problem (1.2)–(1.4) in the context of geophysical applications was considered in [1,2,4]. In particular, in [1] this problem is solved by time stepping on unstructured triangular grids. We shall use a finite difference method which is set up in such a way that the solution is differentiable with respect to the slowness function. In order to keep the exposition simple we consider the two–dimensional case although our arguments can be generalized to higher dimensions.

Let us assume that $\Omega \subset \mathbb{R}^2$ has a boundary Γ which is piecewise C^2. For $h > 0$ consider the regular grid

$$\mathbb{Z}_h^2 := \{x_\alpha = (h\alpha_1, h\alpha_2) \mid \alpha_i \in \mathbb{Z}, \quad i = 1, 2\}.$$

We suppose for simplicity that x_0 is a grid point, say $x_0 = x_{\alpha_0}$ for some $\alpha_0 \in \mathbb{Z}^2$. Next, let $\Omega_h = \Omega \cap \mathbb{Z}_h^2$ be the set of inner grid points. If for some $x_\alpha \in \Omega_h$ there are $\sigma \in \{-1, 1\}, k \in \{1, 2\}$ with $x_{\alpha+\sigma e_k} \notin \Omega$, then there exists $s \in (0, 1]$ such that $x_\alpha + s\sigma h e_k \in \Gamma$ and we set $\beta := \alpha + s\sigma e_k$ as well as $x_\beta := x_\alpha + s\sigma h e_k \in \Gamma$. We denote by $\Gamma_h \subset \Gamma$ the set of all points obtained in this way and define $G_h := \Omega_h \cup \Gamma_h$. For a point $x_\alpha \in G_h$ we let

$$\mathcal{N}_\alpha := \begin{cases} \{x_\beta \in G_h \,|\, x_\beta \text{ is a neighbour of } x_\alpha\}, & x_\alpha \in \Omega_h \\ \{x_\beta \in \Omega_h \,|\, x_\beta \text{ is a neighbour of } x_\alpha\}, & x_\alpha \in \Gamma_h. \end{cases}$$

Note that for $x_\alpha \in \Gamma_h$ the set $\mathcal{N}_\alpha$ only comprises the interior neighbours.

We approximate the solution of (1.2)–(1.4) by a function $U : G_h \to \mathbb{R}$ as follows:

$$U_{\alpha_0} = 0, \tag{2.3}$$

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left[\left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \right]^2 = a(x_\alpha)^2, \quad x_\alpha \in G_h \setminus \{x_{\alpha_0}\}. \tag{2.4}$$

Here we have abbreviated $U_\alpha = U(x_\alpha)$ and $h_{\alpha\beta} = |x_\alpha - x_\beta|$.

Lemma 2.3 *Suppose that $a \in C^0(\bar{\Omega})$ is positive. Then, (2.3), (2.4) has a unique solution $U : G_h \to \mathbb{R}$ and*

(a) $U_\alpha \geq 0, \; x_\alpha \in G_h$;
(b) $|U_\alpha - U_\beta| \leq C \max_{\bar{\Omega}} a \, |x_\alpha - x_\beta|, \; x_\alpha, x_\beta \in G_h$,

where the constant C is independent of h.

Proof We start by sketching the proof for the existence of a discrete solution, compare [8] for similar arguments. Note first that the function $Z : G_h \to \mathbb{R}$, $Z_\alpha := M|x_\alpha - x_{\alpha_0}|$ satisfies $Z_{\alpha_0} = 0$ as well as

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left[\left(\frac{Z_\alpha - Z_\beta}{h_{\alpha\beta}} \right)^+ \right]^2 \geq a(x_\alpha)^2, \quad x_\alpha \in G_h \setminus \{x_{\alpha_0}\}, \tag{2.5}$$

provided that M is chosen sufficiently large. We now consider the following iteration: Set $U^0 := Z$ and given $U^k : G_h \to \mathbb{R}_{\geq 0}$ define $U^{k+1} : G_h \to \mathbb{R}_{\geq 0}$ by $U^{k+1}_{\alpha_0} = 0$ and

$$U^{k+1}_\alpha := \inf \left\{ t \geq 0 \,|\, \sum_{x_\beta \in \mathcal{N}_\alpha} \left[\left(\frac{t - U^k_\beta}{h_{\alpha\beta}} \right)^+ \right]^2 \geq a(x_\alpha)^2 \right\}, \quad x_\alpha \in G_h \setminus \{x_{\alpha_0}\}.$$

An induction argument based on (2.5) and the monotonicity of the scheme shows that $(U^k_\alpha)_{k\in\mathbb{N}}$ is decreasing for all $x_\alpha \in G_h$, so that $U_\alpha := \lim_{k\to\infty} U^k_\alpha$ exists for all $x_\alpha \in G_h$. It is not difficult to verify that U is indeed a solution of (2.3), (2.4).

In order to prove uniqueness, let us suppose that $U, \tilde{U} : G_h \to \mathbb{R}$ are two solutions of (2.3), (2.4). We set

$$\mu := \max_{x_\alpha \in G_h} (U_\alpha - \tilde{U}_\alpha)$$

and suppose that $\mu > 0$. There exists a point $x_\gamma \in G_h \setminus \{x_{\alpha_0}\}$ with $U_\gamma - \tilde{U}_\gamma = \mu$. Let us introduce $W : G_h \to \mathbb{R}$ by $W_\alpha := \tilde{U}_\alpha + \mu,\ x_\alpha \in G_h$. Clearly, $W_\alpha \geq U_\alpha,\ x_\alpha \in G_h$, while $W_\gamma = U_\gamma$. Hence,

$$W_\gamma - W_\beta = U_\gamma - W_\beta \leq U_\gamma - U_\beta, \quad x_\beta \in \mathcal{N}_\gamma. \tag{2.6}$$

As a consequence,

$$\sum_{x_\beta \in \mathcal{N}_\gamma} \left[\left(\frac{\tilde{U}_\gamma - \tilde{U}_\beta}{h_{\gamma\beta}} \right)^+ \right]^2 = \sum_{x_\beta \in \mathcal{N}_\gamma} \left[\left(\frac{W_\gamma - W_\beta}{h_{\gamma\beta}} \right)^+ \right]^2 \leq \sum_{x_\beta \in \mathcal{N}_\gamma} \left[\left(\frac{U_\gamma - U_\beta}{h_{\gamma\beta}} \right)^+ \right]^2.$$

Since U and $\tilde{U}$ are both solutions we deduce from the above relation that

$$a(x_\gamma)^2 = \sum_{x_\beta \in \mathcal{N}_\gamma} \left[\left(\frac{U_\gamma - U_\beta}{h_{\gamma\beta}} \right)^+ \right]^2 = \sum_{x_\beta \in \mathcal{N}_\gamma} \left[\left(\frac{W_\gamma - W_\beta}{h_{\gamma\beta}} \right)^+ \right]^2.$$

Recalling (2.6) and observing that $a(x_\gamma)^2 > 0$ we infer that there exists $x_\beta \in \mathcal{N}_\gamma$ such that

$$0 < W_\gamma - W_\beta = U_\gamma - U_\beta,$$

and therefore

$$U_\beta - \tilde{U}_\beta = \mu \quad \text{and} \quad U_\beta < U_\gamma. \tag{2.7}$$

We can now repeat the above argument with γ replaced by β generating a sequence of points which satisfy (2.7). Since the values of U are strictly decreasing, every point appears only once, so that necessarily x_{α_0} will eventually be crossed contradicting (2.3). Hence $\mu \leq 0$, so that $U_\alpha \leq \tilde{U}_\alpha$, $x_\alpha \in G_h$. Exchanging the roles of U and $\tilde{U}$ we infer that $U_\alpha = \tilde{U}_\alpha$, $x_\alpha \in G_h$.

From the positivity of a and the definition of the scheme it is straightforward to see that the minimum of U cannot be attained at a point $x_\alpha \neq x_{\alpha_0}$, so that $\min_{x_\alpha \in G_h} U_\alpha = U_{\alpha_0} = 0$ proving (a).

In order to show (b) we first note that (2.4) together with the fact that $x_\beta \in \mathcal{N}_\alpha$ if and only if $x_\alpha \in \mathcal{N}_\beta$ implies that

$$|U_\alpha - U_\beta| \leq \max_{\bar{\Omega}} a\ |x_\alpha - x_\beta| \tag{2.8}$$

for all grid points $x_\alpha, x_\beta \in G_h$ that are neighbours of each other. In order to estimate the above difference for arbitrary pairs $x_\alpha, x_\beta \in G_h$ we first extend U to a function $\hat{U} : \bar{\Omega} \to \mathbb{R}$ as follows: The grid $\mathbb{Z}_h^2$ gives rise to a partition of $\bar{\Omega}$ into squares that are possibly truncated near the boundary. A square Q that lies entirely in $\bar{\Omega}$ is divided into two triangles along its diagonal and $\hat{U}$ is defined by linear interpolation, so that $|\nabla \hat{U}_{|Q}| \leq \max_{\bar{\Omega}} a$ in view of (2.8). A truncated square Q at the boundary can also be subdivided into triangles with possibly one curved edge and again $\hat{U}$ is defined via linear interpolation. It can be shown that this can be done in such a way that $|\nabla \hat{U}_{|Q}| \leq 3 \max_{\bar{\Omega}} a$. Thus we obtain a function $\hat{U} \in W^{1,\infty}(\Omega)$ with $|\nabla \hat{U}| \leq 3 \max_{\bar{\Omega}} a$ a.e. in Ω. Making use of the continuous embedding $W^{1,\infty}(\Omega) \hookrightarrow C^{0,1}(\bar{\Omega})$ we finally obtain for arbitrary $x_\alpha, x_\beta \in G_h$

$$|U_\alpha - U_\beta| = |\hat{U}(x_\alpha) - \hat{U}(x_\beta)| \leq C \|\nabla \hat{U}\|_{L^\infty} |x_\alpha - x_\beta| \leq C \max_{\bar{\Omega}} a \; |x_\alpha - x_\beta|,$$

which proves (b). □

2.3 Construction of a solution by the fast marching method

A solution to (2.3), (2.4) can be found efficiently, without iteration, using the fast marching procedure, see [14,15]. Recall that the idea behind this method is that the unique solution U_α of (2.3), (2.4) at a grid point x_α only depends on neighbouring values U_β such that $0 \leq U_\beta < U_\alpha$ so that the solution can be obtained in increasing order of magnitude of the grid values U_α. Solving the equation then becomes an issue of sorting the grid values.

In particular the following algorithm is used: First tag x_{α_0} as *known* and tag as $trial$ all points that are one grid point away from this *known* point. Finally tag as far all remaining points. Now cycle through the following *Fast Marching Procedure*:

- **Step 1** Compute a trial value of $\tilde{U}_\alpha$ for every $x_\alpha \in trial$ according to (2.4) assuming that it is smaller than or equal to its $trial$ neighbours.
- **Step 2** Let x_μ be any $trial$ point such that the trial values satisfy $\tilde{U}_\mu \leq \tilde{U}_\alpha$ for all $x_\alpha \in trial$.
- **Step 3** Set $U_\mu = \tilde{U}_\mu$ for all such x_μ and add x_μ to $known$ and remove from $trial$.
- **Step 4** Tag all neighbours of $known$ as $trial$ if they are not $known$.
- **Step 5** If $trial = \{\emptyset\}$ then STOP.
- **Step 6** Return to Step 1.

Lemma 2.4 *The Fast Marching Procedure terminates in* K *cycles where* K *is the number of distinct positive values taken by the solution* U *of Lemma* 2.3.

Proof Let us denote by $0 = V_0 < V_1 < V_2 < \cdots V_K$ the $K+1$ distinct values taken by U and define $E_m := \{x_\alpha \in G_h \,|\, U_\alpha = V_m\}$, $0 \leq m \leq K$. The lemma is proved once we can show that $E_0 \cup \cdots \cup E_m$ coincides with the set of *known* points after m cycles. This is certainly true for $m = 0$. Now suppose that this claim holds for some $0 \leq m < K$, so that the *known* points after m cycles are given by $E_0 \cup \cdots \cup E_m$.

Let $x_\alpha \in trial$ and $\mathcal{N}_{m,\alpha} = known \cap \mathcal{N}_\alpha$. We denote by $r > \min_{x_\beta \in \mathcal{N}_{m,\alpha}} U_\beta$ the unique solution of the equation

$$\sum_{x_\beta \in \mathcal{N}_{m,\alpha}} \left[\left(\frac{r - U_\beta}{h_{\alpha\beta}} \right)^+ \right]^2 = a(x_\alpha)^2 \tag{2.9}$$

and suppose that $r \le \max_{x_\beta \in \mathcal{N}_{m,\alpha}} U_\beta$. Then there would be $x_{\beta_1}, x_{\beta_2} \in \mathcal{N}_{m,\alpha}$ with $U_{\beta_1} < r \le U_{\beta_2}$, say $x_{\beta_1} \in E_l, x_{\beta_2} \in E_k, l < k \le m$. But then the value r will have been computed as a trial value in the kth cycle. Since x_α has not been added to *known* we must have $r > V_k = U_{\beta_2}$, a contradiction. Hence $r > \max_{x_\beta \in \mathcal{N}_{m,\alpha}} U_\beta$ and in view of (2.4) the smallest of the trial values satisfying (2.9) is given by V_{m+1}. As the points x_α that take this value are added to *known* the induction step is finished. □

Remark 2.5 Observe that the unique solution of the Eq. (2.9) defining the trial values may be found by solving a quadratic equation and taking the largest root.

2.4 Error estimate

Theorem 2.6 *Suppose that* $a : \bar{\Omega} \to \mathbb{R}$ *is Lipschitz continuous and satisfies* $0 < A_m \le a(x) \le A_M$ *for all* $x \in \bar{\Omega}$. *Let* u *be the viscosity solution of* (1.2)–(1.4) *and* U *the solution of* (2.3), (2.4). *Then there exists* $h_0 > 0$ *such that for all* $0 < h \le h_0$

$$\max_{x_\alpha \in G_h} |u(x_\alpha) - U_\alpha| \le C\sqrt{h}. \tag{2.10}$$

The constants h_0 *and* C *depend on* Ω, A_m, A_M, lip (a) *and the function* η *from* (2.1).

Proof Let $\epsilon > 0$ be the constant in (2.1). Since η is uniformly continuous on $\bar{\Omega}$, there exists $\delta > 0$ such that

$$|\eta(x) - \eta(y)| < \frac{\epsilon}{2} \quad \text{for all } x, y \in \bar{\Omega} \text{ with } |x - y| < \delta. \tag{2.11}$$

Denoting by lip (u) the Lipschitz constant of u and by lip (U) the constant appearing in Lemma 2.3(b) we set

$$R := \sqrt{(\text{lip }(u))^2 + \frac{1}{2}\left(\text{lip }(u)^2 + \text{lip }(U)^2\right) + \max_{\bar{\Omega}} |\eta|^2}. \tag{2.12}$$

Let us choose $L \ge 1, M \ge 1$ so large that

$$\frac{\sqrt{2}R}{\sqrt{L}} \le \frac{\epsilon}{4}, \quad \frac{R}{\sqrt{M}} \le \frac{\delta}{2}. \tag{2.13}$$

Note that R and hence also L and M only depend on Ω and A_M in view of Theorem 2.2 and Lemma 2.3.

We first estimate $\max_{x_\alpha \in G_h} (u(x_\alpha) - U_\alpha)$. Choose $x_\gamma \in G_h$ such that

$$\max_{x_\alpha \in G_h} \Big((1-\rho\sqrt{h})u(x_\alpha) - U_\alpha \Big) = (1-\rho\sqrt{h})u(x_\gamma) - U_\gamma. \tag{2.14}$$

Here, the constant ρ will be chosen later and we take $h_0 > 0$ so small that

$$1 - \rho\sqrt{h_0} \le 1 \quad \text{and} \quad h_0 \le \epsilon^2. \tag{2.15}$$

The factor $(1-\rho\sqrt{h})$ in (2.14) is motivated by Ishii's uniqueness proof for Hamilton–Jacobi equations of eikonal type, see [10]. We now employ the usual doubling of variables technique and define $\Phi : \bar{\Omega} \times G_h \to \mathbb{R}$ by

$$\Phi(x, x_\alpha) := (1-\rho\sqrt{h})u(x) - U_\alpha - \frac{L}{\sqrt{h}}|x - x_\alpha - \sqrt{h}\eta(x_\gamma)|^2 - M\sqrt{h}|x_\alpha - x_\gamma|^2.$$

There exists $(x_h, x_{\alpha_h}) \in \bar{\Omega} \times G_h$ such that

$$\Phi(x_h, x_{\alpha_h}) = \max_{(x, x_\alpha) \in \bar{\Omega} \times G_h} \Phi(x, x_\alpha).$$

Our goal will be to show that for a suitable choice of ρ at least one of the points x_h or x_{α_h} has to coincide with the source point x_0 and then to use that $u(x_0) = U(x_0) = 0$. In order to exclude the possibility that both x_h and x_{α_h} are different from x_0 we will employ among other things the fact that u is a viscosity subsolution at x_h. However, this is only possible provided we can ensure that x_h does not belong to the boundary of Ω (compare Definition 2.1). This will be accomplished with the help of the shift $\sqrt{h}\eta(x_\gamma)$ in the third term of Φ, an idea going back to Soner [16].

Let us now carry out the proof in detail. In view of (2.1) and (2.15) we have that $x_\gamma + \sqrt{h}\,\eta(x_\gamma) \in \Omega$ for $0 < h \le h_0$ and hence

$$\Phi(x_h, x_{\alpha_h}) \ge \Phi(x_\gamma + \sqrt{h}\,\eta(x_\gamma), x_\gamma),$$

or equivalently

$$\begin{aligned}(1-\rho\sqrt{h})u(x_h) - U_{\alpha_h} - \frac{L}{\sqrt{h}}|x_h - x_{\alpha_h} - \sqrt{h}\,\eta(x_\gamma)|^2 - M\sqrt{h}|x_{\alpha_h} - x_\gamma|^2 \\ \ge (1-\rho\sqrt{h})u(x_\gamma + \sqrt{h}\,\eta(x_\gamma)) - U_\gamma.\end{aligned} \tag{2.16}$$

Using (2.14) we obtain

$$\begin{aligned}&\frac{L}{\sqrt{h}}|x_h - x_{\alpha_h} - \sqrt{h}\,\eta(x_\gamma)|^2 + M\sqrt{h}|x_{\alpha_h} - x_\gamma|^2 \\ &\quad \le (1-\rho\sqrt{h})u(x_h) - U_{\alpha_h} - (1-\rho\sqrt{h})u(x_\gamma + \sqrt{h}\,\eta(x_\gamma)) + U_\gamma \\ &\quad = (1-\rho\sqrt{h})u(x_{\alpha_h}) - U_{\alpha_h} - \Big((1-\rho\sqrt{h})u(x_\gamma) - U_\gamma\Big)\end{aligned}$$

$$
\begin{aligned}
&+(1-\rho\sqrt{h})u(x_h)-(1-\rho\sqrt{h})u(x_{\alpha_h})+(1-\rho\sqrt{h})\left(u(x_\gamma)-u(x_\gamma+\sqrt{h}\,\eta(x_\gamma))\right)\\
&\le \text{lip }(u)\left(|x_h-x_{\alpha_h}|+\sqrt{h}\max_{\bar\Omega}|\eta|\right)\\
&\le \text{lip }(u)\left(|x_h-x_{\alpha_h}-\sqrt{h}\,\eta(x_\gamma)|+2\sqrt{h}\max_{\bar\Omega}|\eta|\right)\\
&\le \frac{L}{2\sqrt{h}}|x_h-x_{\alpha_h}-\sqrt{h}\,\eta(x_\gamma)|^2+\sqrt{h}\left(\frac{3}{2}(\text{lip }(u))^2+\max_{\bar\Omega}|\eta|^2\right). \qquad (2.17)
\end{aligned}
$$

Recalling (2.12) and (2.13) we obtain that

$$
|x_h-x_{\alpha_h}-\sqrt{h}\,\eta(x_\gamma)|\le\frac{\sqrt{2}R}{\sqrt{L}}\sqrt{h}\le\frac{\epsilon}{4}\sqrt{h}<\frac{\epsilon}{2}\sqrt{h} \qquad (2.18)
$$

$$
|x_{\alpha_h}-x_\gamma|\le\frac{R}{\sqrt{M}}\le\frac{\delta}{2}<\delta. \qquad (2.19)
$$

In particular, (2.18), (2.19) and (2.11) imply that

$$
\begin{aligned}
|x_h-x_{\alpha_h}-\sqrt{h}\,\eta(x_{\alpha_h})| &\le |x_h-x_{\alpha_h}-\sqrt{h}\,\eta(x_\gamma)|+\sqrt{h}|\eta(x_{\alpha_h})-\eta(x_\gamma)|\\
&<\frac{\epsilon}{2}\sqrt{h}+\frac{\epsilon}{2}\sqrt{h}=\epsilon\sqrt{h}.
\end{aligned}
$$

Hence $x_h\in B_{\epsilon\sqrt{h}}(x_{\alpha_h}+\sqrt{h}\,\eta(x_{\alpha_h}))\subset\Omega$ for $0<h\le h_0$ by (2.1). We distinguish two cases:
Case 1: $x_h\in\Omega\setminus\{x_0\}$ and $x_{\alpha_h}\in G_h\setminus\{x_0\}$. Noting that

$$
x\mapsto u(x)-\frac{L}{(1-\rho\sqrt{h})\sqrt{h}}\left|x-x_{\alpha_h}-\sqrt{h}\eta(x_\gamma)\right|^2
$$

has a maximum at $x=x_h$ we obtain from the fact that u is a viscosity subsolution that

$$
\frac{2L}{(1-\rho\sqrt{h})\sqrt{h}}|x_h-x_{\alpha_h}-\sqrt{h}\eta(x_\gamma)|\le a(x_h). \qquad (2.20)
$$

On the other hand, observing that $\Phi(x_h,x_{\alpha_h})\ge\Phi(x_h,x_\alpha)$, $x_\alpha\in G_h$ we obtain

$$
\begin{aligned}
U_\alpha\ge U_{\alpha_h}&+\frac{L}{\sqrt{h}}\{|x_h-x_{\alpha_h}-\sqrt{h}\,\eta(x_\gamma)|^2-|x_h-x_\alpha-\sqrt{h}\,\eta(x_\gamma)|^2\}\\
&+M\sqrt{h}\{|x_{\alpha_h}-x_\gamma|^2-|x_\alpha-x_\gamma|^2\}\quad=:\quad V_\alpha.
\end{aligned}
$$

A short calculation shows that for $x_\beta\in\mathcal{N}_\alpha$

$$
V_\alpha-V_\beta=-\frac{2L}{\sqrt{h}}\left(x_h-x_\alpha-\sqrt{h}\,\eta(x_\gamma),x_\beta-x_\alpha\right)+r_{\alpha\beta}
$$

where

$$r_{\alpha\beta} = \frac{L}{\sqrt{h}}|x_\beta - x_\alpha|^2 + M\sqrt{h}\,\left(x_\beta - x_\alpha, x_\beta + x_\alpha - 2x_\gamma\right).$$

Hence,

$$\left|\frac{V_\alpha - V_\beta}{h_{\alpha\beta}}\right| \leq \frac{2L}{\sqrt{h}}\left|\left(x_h - x_\alpha - \sqrt{h}\eta(x_\gamma), \frac{x_\beta - x_\alpha}{h_{\alpha\beta}}\right)\right| + C_1\sqrt{h}, \quad x_\beta \in \mathcal{N}_\alpha \tag{2.21}$$

where the constant C_1 only depends on L, M and Ω.

Next, since $U_\alpha \geq V_\alpha$, $x_\alpha \in G_h$, $U_{\alpha_h} = V_{\alpha_h}$ we have

$$U_{\alpha_h} - U_\beta \leq V_{\alpha_h} - V_\beta, \quad x_\beta \in \mathcal{N}_{\alpha_h}.$$

We deduce from (2.4) and (2.21)

$$\begin{aligned} a(x_{\alpha_h}) &= \left(\sum_{x_\beta \in \mathcal{N}_{\alpha_h}} \left[\left(\frac{U_{\alpha_h} - U_\beta}{h_{\alpha_h\beta}}\right)^+\right]^2\right)^{\frac{1}{2}} \leq \left(\sum_{x_\beta \in \mathcal{N}_{\alpha_h}} \left[\left(\frac{V_{\alpha_h} - V_\beta}{h_{\alpha_h\beta}}\right)^+\right]^2\right)^{\frac{1}{2}} \\ &\leq \frac{2L}{\sqrt{h}}|x_h - x_{\alpha_h} - \sqrt{h}\,\eta(x_\gamma)| + C_2\sqrt{h}, \end{aligned} \tag{2.22}$$

where C_2 depends on the same quantities as C_1. Combining (2.20) and (2.22) we have

$$\begin{aligned} &\frac{2L}{\sqrt{h}}|x_h - x_{\alpha_h} - \sqrt{h}\eta(x_\gamma)| \leq (1 - \rho\sqrt{h})a(x_h) \\ &\quad \leq (a(x_h) - a(x_{\alpha_h})) + a(x_{\alpha_h}) - \rho\sqrt{h}\,a(x_h) \\ &\quad \leq \text{lip}\,(a)|x_h - x_{\alpha_h}| + \frac{2L}{\sqrt{h}}|x_h - x_{\alpha_h} - \sqrt{h}\eta(x_\gamma)| + C_2\sqrt{h} - \rho A_m\sqrt{h}. \end{aligned} \tag{2.23}$$

Furthermore, (2.18) implies that

$$|x_h - x_{\alpha_h}| \leq |x_h - x_{\alpha_h} - \sqrt{h}\,\eta(x_\gamma)| + \max_{\bar{\Omega}}|\eta|\sqrt{h} \leq \left(\frac{\epsilon}{2} + \max_{\bar{\Omega}}|\eta|\right)\sqrt{h},$$

which together with (2.23) yields

$$0 \leq \left(C_2 - \rho A_m + \text{lip}\,(a)(\frac{\epsilon}{2} + \max_{\bar{\Omega}}|\eta|)\right)\sqrt{h} < 0$$

a contradiction, if we choose for example $\rho = \frac{C_2 + \text{lip}\,(a)(\frac{\epsilon}{2} + \max_{\bar{\Omega}}|\eta|)}{A_m} + 1$. Hence this case cannot occur and we note that ρ only depends on Ω, A_m, A_M, η and lip (a).

Case 2: $x_h = x_0$ or $x_{\alpha_h} = x_0$. We obtain from (2.16) and the fact that $U_\alpha \geq 0$, $u(x_0) = 0$

$$
\begin{aligned}
(1-\rho\sqrt{h})u(x_\gamma+\sqrt{h}\,\eta(x_\gamma))-U_\gamma &\le (1-\rho\sqrt{h})u(x_h)-U_{\alpha_h}\\
&\le u(x_h)\le \min(u(x_h),u(x_{\alpha_h}))+\text{lip }(u)|x_h-x_{\alpha_h}|\\
&\le \text{lip }(u)|x_h-x_{\alpha_h}-\sqrt{h}\,\eta(x_\gamma)|+\sqrt{h}\text{ lip }(u)\max_{\bar\Omega}|\eta|\\
&\le \text{lip }(u)\left(\frac{\epsilon}{2}+\max_{\bar\Omega}|\eta|\right)\sqrt{h}
\end{aligned}
$$

by (2.18). As a consequence, recalling the definition of x_γ

$$
\begin{aligned}
\max_{x_\alpha\in G_h}(u(x_\alpha)-U_\alpha) &\le (1-\rho\sqrt{h})u(x_\gamma)-U_\gamma+\rho\sqrt{h}\max_{\bar\Omega}u\\
&\le (1-\rho\sqrt{h})u(x_\gamma+\sqrt{h}\,\eta(x_\gamma))-U_\gamma+\text{lip }(u)\max_{\bar\Omega}|\eta|\sqrt{h}\\
&\quad+\rho\sqrt{h}\max_{\bar\Omega}u\\
&\le C_3\sqrt{h}, \qquad (2.24)
\end{aligned}
$$

where C_3 depends on the same quantities as ρ. Note that the bound on $\max_{\bar\Omega} u$ follows from

$$
u(x)=u(x)-u(x_0)\le \text{lip }(u)|x-x_0|\le CA_M\text{diam }(\Omega) \qquad (2.25)
$$

in view of (1.3) and Theorem 2.2.

It remains to derive an upper bound on $\max_{x_\alpha\in G_h}(U_\alpha-u(x_\alpha))$. This will be done in a similar way as above and we will only sketch the argument. To begin, let $x_\gamma\in G_h$ be such that

$$
\max_{x_\alpha\in G_h}\Big((1-\rho\sqrt{h})U_\alpha-u(x_\alpha)\Big)=(1-\rho\sqrt{h})U_\gamma-u(x_\gamma)
$$

and define $\Phi:\bar\Omega\times G_h\to\mathbb{R}$ by

$$
\Phi(x,x_\alpha):=(1-\rho\sqrt{h})U_\alpha-u(x)-\frac{L}{\sqrt{h}}|x_\alpha-x-\sqrt{h}\eta(x_\gamma)|^2-M\sqrt{h}|x-x_\gamma|^2.
$$

There exists $(x_h,x_{\alpha_h})\in\bar\Omega\times G_h$ such that $\Phi(x_h,x_{\alpha_h})=\max_{\bar\Omega\times G_h}\Phi$. Since $x_\gamma+\sqrt{h}\,\eta(x_\gamma)\in\Omega$ for $0<h\le h_0$ there exists $x_{\tilde\gamma}\in G_h$ with

$$
|x_{\tilde\gamma}-x_\gamma-\sqrt{h}\,\eta(x_\gamma)|\le h. \qquad (2.26)
$$

The inequality $\Phi(x_h,x_{\alpha_h})\ge\Phi(x_\gamma,x_{\tilde\gamma})$ together with (2.26) implies

$$
\begin{aligned}
(1-\rho\sqrt{h})U_{\alpha_h}-u(x_h)-\frac{L}{\sqrt{h}}|x_{\alpha_h}-x_h-\sqrt{h}\,\eta(x_\gamma)|^2-M\sqrt{h}|x_h-x_\gamma|^2&\\
\ge (1-\rho\sqrt{h})U_{\tilde\gamma}-u(x_\gamma)-Lh^{\frac{3}{2}}.& \qquad (2.27)
\end{aligned}
$$

We can argue as in (2.17) to show that

$$\begin{aligned}
&\frac{L}{\sqrt{h}}|x_{\alpha_h}-x_h-\sqrt{h}\,\eta(x_\gamma)|^2+M\sqrt{h}|x_h-x_\gamma|^2\\
&\le \frac{L}{2\sqrt{h}}|x_{\alpha_h}-x_h-\sqrt{h}\,\eta(x_\gamma)|^2+\sqrt{h}\left((\text{lip }(u))^2+\frac{1}{2}\text{lip }(U)^2+\max_{\bar\Omega}|\eta|^2\right)+Lh^{\frac{3}{2}},
\end{aligned}$$

from which we conclude recalling (2.12)

$$\begin{aligned}
|x_{\alpha_h}-x_h-\sqrt{h}\,\eta(x_\gamma)| &\le \frac{\epsilon}{4}\sqrt{h}+\sqrt{2}h<\frac{\epsilon}{2}\sqrt{h},\\
|x_h-x_\gamma| &\le \frac{R}{\sqrt{M}}+\frac{L}{\sqrt{M}}h\le\frac{\delta}{2}+\frac{L}{\sqrt{M}}h<\delta
\end{aligned}$$

for $0<h\le h_0$, where h_0 is chosen smaller if necessary. Just as above we deduce that $x_{\alpha_h}\in B_{\epsilon\sqrt{h}}(x_h+\sqrt{h}\,\eta(x_h))\subset\Omega$ for $0<h\le h_1$ and then rule out that $x_h\in\bar\Omega\setminus\{x_0\}$ and $x_{\alpha_h}\in\Omega_h\setminus\{x_0\}$ by choosing ρ sufficiently large. It remains to consider the case $x_h=x_0$ or $x_{\alpha_h}=x_0$. Combining (2.27) with the fact that $x_0=x_{\alpha_0}$ and Lemma 2.3(b) we have

$$\begin{aligned}
(1-\rho\sqrt{h})U_{\tilde\gamma}-u(x_\gamma) &\le (1-\rho\sqrt{h})U_{\alpha_h}-u(x_h)+Lh^{\frac{3}{2}}\le U_{\alpha_h}+Lh^{\frac{3}{2}}\\
&\le \min(U_{\alpha_h},U_{\alpha_0})+CA_M|x_h-x_{\alpha_h}|+Lh^{\frac{3}{2}}\\
&\le CA_M|x_{\alpha_h}-x_h-\sqrt{h}\,\eta(x_\gamma)|+CA_M\max_{\bar\Omega}|\eta|\sqrt{h}+Lh^{\frac{3}{2}}\\
&\le C_4\sqrt{h}.
\end{aligned}$$

Hence, we finally have similarly as above

$$\max_{x_\alpha\in G_h}(U_\alpha-u(x_\alpha))\le C_5\sqrt{h}, \tag{2.28}$$

where C_5 again only depends on Ω, A_m, A_M, η and lip (a). The inequalities (2.24) and (2.28) imply the result. □

3 The optimal control problem

3.1 The continuous problem

Let $0<A_m<A_M<\infty$ and the positive function $u_{obs}\in C^{0,1}(\Gamma)$ be given. We introduce the set

$$K:=\{a:\bar\Omega\to\mathbb{R}\,|\,a(x)=\sum_{i=1}^{L}a_i\phi_i(x),\ A_m\le a_i\le A_M,1\le i\le L\}$$

where $\{\phi_i\}_{i=1}^L$ satisfy $\phi_i\in W^{1,\infty}(D)$, $\phi_i(x)\ge 0$, $i=1,\dots,L$ and $\sum_{i=1}^L\phi_i(x)=1$, $x\in\Omega$, $\bar\Omega\subset D=\bigcup_{i=1}^L\overline{\text{supp }(\phi_i)}$.

Given $a \in K$ we denote by $u = u_a$ the solution of (1.2)–(1.4) given by Theorem 2.2 and consider the following optimal control problem

$$\min_{a \in K} \mathcal{J}(a) = \frac{1}{2} \int_\Gamma |u_a(x) - u_{obs}(x)|^2 \, do_x. \tag{P}$$

3.2 The discrete control problem

The aim of this section is to set up and analyze a discrete approximation of (**P**). We start by defining a suitable approximation of the integral $\int_\Gamma |u_a(x) - u_{obs}(x)|^2 \, do_x$. To this purpose we choose an embedding $\gamma : [0, 1] \to \mathbb{R}^2$, which is piecewise C^2 such that $\gamma([0, 1]) = \Gamma$, $\gamma(0) = \gamma(1)$ and $|\gamma'(t)| \geq c_0 > 0$, with the exception of finitely many $t \in [0, 1]$. For every $x_\alpha \in \Gamma_h$ there exists a unique $t_\alpha \in [0, 1]$ with $\gamma(t_\alpha) = x_\alpha$. Ordering the different preimages in the form $0 \leq t_{\alpha_1} < t_{\alpha_2} < \cdots < t_{\alpha_N} < 1$ induces an ordering of the boundary grid points $x_{\alpha_i} = \gamma(t_{\alpha_i})$, $i = 1, \ldots, N$. For each $x_\alpha = x_{\alpha_i} \in \Gamma_h$ we let

$$h_\alpha := \frac{1}{2} \left(\left| x_{\alpha_{i+1}} - x_{\alpha_i} \right| + \left| x_{\alpha_i} - x_{\alpha_{i-1}} \right| \right) \tag{3.1}$$

with the convention $t_{\alpha_{N+1}} = t_{\alpha_1}$, $t_{\alpha_0} = t_{\alpha_N}$. Furthermore, choosing a sequence $(\delta_h)_{h>0}$ with $\delta_h \geq 0$ and $\lim_{h \to 0} \delta_h = 0$ we approximate the functional $\mathcal{J}$ by

$$\mathcal{J}_h(a) = \frac{1}{2} \sum_{x_\alpha \in \Gamma_h} h_\alpha \, |U_a(x_\alpha) - u_{obs}(x_\alpha)|^2 + \frac{\delta_h}{2} \int_\Omega |\nabla a|^2 \, ,$$

where U_a is the solution of (2.3), (2.4). Let us first establish the consistency of the above approximation.

Lemma 3.1 *Let $a = \sum_{i=1}^{L} a_i \phi_i \in K$. Then*

$$\mathcal{J}_h(a) \to \mathcal{J}(a), \quad \text{as } h \to 0,$$

where the convergence is uniform in $a \in K$.

Proof The assertion is a consequence of the following estimate which we will need again later on:

$$\left| \frac{1}{2} \sum_{x_\alpha \in \Gamma_h} h_\alpha \, |U_a(x_\alpha) - u_{obs}(x_\alpha)|^2 - \frac{1}{2} \int_\Gamma |u_a(x) - u_{obs}(x)|^2 \, do_x \right| \leq C\sqrt{h}, \tag{3.2}$$

where the constant C is independent of $a \in K$. In order to prove (3.2) we write

$$\begin{aligned} &\frac{1}{2} \sum_{x_\alpha \in \Gamma_h} h_\alpha \, |U_a(x_\alpha) - u_{obs}(x_\alpha)|^2 - \frac{1}{2} \int_\Gamma |u_a(x) - u_{obs}(x)|^2 \, do_x \\ &\quad = \frac{1}{2} \sum_{i=1}^{N} h_{\alpha_i} \left\{ \left| U_a(x_{\alpha_i}) - u_{obs}(x_{\alpha_i}) \right|^2 - \left| u_a(x_{\alpha_i}) - u_{obs}(x_{\alpha_i}) \right|^2 \right\} \end{aligned}$$

$$+\frac{1}{2}\left\{\sum_{i=1}^{N} h_{\alpha_i}\left|u_a(x_{\alpha_i})-u_{obs}(x_{\alpha_i})\right|^2-\int_{\Gamma}|u_a(x)-u_{obs}(x)|^2\,do_x\right\}$$
$$\equiv\quad I+II.$$

Recalling Theorem 2.6 we have

$$|I|\le|\Gamma|\max_{x_\alpha\in G_h}|u_a(x_\alpha)-U_a(x_\alpha)|\left(\max_{\bar\Omega}u_a+\max_{G_h}U_a+2\max_{\Gamma}u_{obs}\right)\le C\sqrt{h},$$

where the constant C is independent of $a\in K$ since lip $(a)\le LA_M\max_{i=1,\ldots,L}\|\nabla\phi_i\|_{L^\infty(\Omega)}$ for all $a\in K$. Note also that $\max_{\bar\Omega}u_a$ and $\max_{G_h}U_a$ can be bounded independently of a using the estimate (2.25) and a corresponding bound for U_a.

Next, we have for $f_a(x):=\frac{1}{2}|u_a(x)-u_{obs}(x)|^2$

$$II=\sum_{i=1}^{N} f_a(x_{\alpha_i})\frac{1}{2}\left(|\gamma(t_{\alpha_{i+1}})-\gamma(t_{\alpha_i})|+|\gamma(t_{\alpha_i})-\gamma(t_{\alpha_{i-1}})|\right)-\int_0^1 f_a(\gamma(t))|\gamma'(t)|\,dt$$
$$=\sum_{i=1}^{N}\int_{t_{\alpha_i}}^{t_{\alpha_{i+1}}}\left\{\frac{1}{2}\left(f_a(\gamma(t_{\alpha_i}))+f_a(\gamma(t_{\alpha_{i+1}}))\right)-f_a(\gamma(t))\right\}|\gamma'(t)|\,dt$$
$$+\sum_{i=1}^{N}\frac{1}{2}\left(f_a(\gamma(t_{\alpha_i}))+f_a(\gamma(t_{\alpha_{i+1}}))\right)\left\{|\gamma(t_{\alpha_{i+1}})-\gamma(t_{\alpha_i})|-\int_{t_{\alpha_i}}^{t_{\alpha_{i+1}}}|\gamma'(t)|\,dt\right\}$$
$$\equiv II_1+II_2.$$

Observing that u_a and hence f_a is Lipschitz on Γ with a constant that is independent of $a\in K$ we infer that $|II_1|\le Ch$. Furthermore, since γ is piecewise C^2 we also have

$$\left|\int_{t_{\alpha_i}}^{t_{\alpha_{i+1}}}\left|\gamma'(t)\right|dt-\left|\gamma(t_{\alpha_{i+1}})-\gamma(t_{\alpha_i})\right|\right|\le Ch(t_{\alpha_{i+1}}-t_{\alpha_i}),$$

which implies that $|II_2|\le Ch$. This proves (3.2) and the result follows from the fact that $\lim_{h\to0}\delta_h=0$. □

We now consider the following discrete control problem:

$$\min_{a\in K}\mathcal{J}_h(a). \qquad (\mathbf{P_h})$$

Theorem 3.2 *The problem* ($\mathbf{P_h}$) *has at least one solution* $a_h^*\in K$. *There exists a sequence* $h\to0$ *such that* $a_h^*\to a^*$ *for some* $a^*\in K$ *and* a^* *is a solution of* (**P**).

Furthermore, if $\delta_h > 0$ *for all* $h > 0$ *and* $\lim_{h\to 0}\frac{\sqrt{h}}{\delta_h} = 0$*, then*

$$\int_\Omega |\nabla a^*|^2 \le \int_\Omega |\nabla \tilde{a}|^2 \quad \text{for every solution } \tilde{a} \text{ of } (\mathbf{P}). \tag{3.3}$$

Proof It is not difficult to see that the mapping $a \mapsto U_a$ is continuous and hence there exists a minimizer $a_h^* \in K$ of ($\mathbf{P_h}$). Furthermore, observing that $A_m \le a_{h,i}^* \le A_M, i = 1, \ldots, L$, there exists a sequence $h \to 0$ and $a^* \in K$ such that $a_h^* \to a^*$ uniformly in $\bar{\Omega}$. Let us abbreviate $u^h = u_{a_h^*}, u = u_{a^*}$. Standard stability arguments from the theory of viscosity solutions show that $u^h \to u$ uniformly in $\bar{\Omega}$ after possibly extracting a further subsequence. As a consequence,

$$\lim_{h\to 0} \mathcal{J}(a_h^*) = \mathcal{J}(a^*). \tag{3.4}$$

Now, if $a \in K$ is arbitrary we rewrite the relation $\mathcal{J}_h(a_h^*) \le \mathcal{J}_h(a)$ as

$$\mathcal{J}(a_h^*) \le \mathcal{J}_h(a) + \left(\mathcal{J}(a_h^*) - \mathcal{J}_h(a_h^*)\right).$$

Using (3.4) and Lemma 3.1 we deduce that a^* solves ($\mathbf{P}$) by passing to the limit $h \to 0$. Finally, suppose that $\delta_h > 0, h > 0$ with $\lim_{h\to 0}\frac{\sqrt{h}}{\delta_h} = 0$ and that $\tilde{a} \in K$ is an arbitrary solution of ($\mathbf{P}$). Rewriting the relation $\mathcal{J}_h(a_h^*) \le \mathcal{J}_h(\tilde{a})$ we obtain

$$\begin{aligned}
\int_\Omega \left|\nabla a_h^*\right|^2 &\le \int_\Omega |\nabla \tilde{a}|^2 + \frac{1}{\delta_h}\left(\sum_{x_\alpha\in\Gamma_h} h_\alpha \left|U_{\tilde{a}}(x_\alpha) - u_{obs}(x_\alpha)\right|^2 \right.\\
&\quad \left. - \sum_{x_\alpha\in\Gamma_h} h_\alpha \left|U_{a_h^*}(x_\alpha) - u_{obs}(x_\alpha)\right|^2\right)\\
&\le \int_\Omega |\nabla \tilde{a}|^2 + \frac{1}{\delta_h}\left(\sum_{x_\alpha\in\Gamma_h} h_\alpha \left|U_{\tilde{a}}(x_\alpha) - u_{obs}(x_\alpha)\right|^2 \right.\\
&\quad \left. - \int_\Gamma |u_{\tilde{a}}(x) - u_{obs}(x)|^2 \, do_x\right)\\
&\quad + \frac{1}{\delta_h}\left(\int_\Gamma \left|u_{a_h^*}(x) - u_{obs}(x)\right|^2 do_x - \sum_{x_\alpha\in\Gamma_h} h_\alpha \left|U_{a_h^*}(x_\alpha) - u_{obs}(x_\alpha)\right|^2\right),
\end{aligned}$$

since $\mathcal{J}(\tilde{a}) \le \mathcal{J}(a_h^*)$. Recalling (3.2) we deduce that

$$\int_\Omega |\nabla a_h^*|^2 \le \int_\Omega |\nabla \tilde{a}|^2 + C\frac{\sqrt{h}}{\delta_h}$$

so that (3.3) follows upon sending $h \to 0$. □

3.3 The discrete adjoint equation

Let $U : G_h \to \mathbb{R}$ be the solution of (2.3), (2.4). We introduce the following adjoint problem: Find $P : G_h \setminus \{x_{\alpha_0}\} \to \mathbb{R}$ such that

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left\{ \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{P_\alpha}{h_{\alpha\beta}} - \left(\frac{U_\beta - U_\alpha}{h_{\alpha\beta}} \right)^+ \frac{P_\beta}{h_{\alpha\beta}} \right\} = 0, \quad x_\alpha \in \Omega_h \setminus \{x_{\alpha_0}\}; \tag{3.5}$$

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left\{ \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{P_\alpha}{h_{\alpha\beta}} - \left(\frac{U_\beta - U_\alpha}{h_{\alpha\beta}} \right)^+ \frac{P_\beta}{h_{\alpha\beta}} \right\} = \frac{h_\alpha}{h^2} \left(u_{obs}(x_\alpha) - U_\alpha \right), \; x_\alpha \in \Gamma_h, \tag{3.6}$$

where h_α is given by (3.1). Note that the fact that P is not defined at x_{α_0} does not cause a problem in evaluating (3.5). If $x_{\alpha_0} \in \mathcal{N}_\alpha$ for some $x_\alpha \in \Omega_h$ then $\left(\frac{U_{\alpha_0} - U_\alpha}{h_{\alpha\beta}} \right)^+ = \left(\frac{-U_\alpha}{h_{\alpha\beta}} \right)^+ = 0$.

Lemma 3.3 *For a given function $Q : G_h \setminus \{x_{\alpha_0}\} \to \mathbb{R}$ there exists a unique $V : G_h \to \mathbb{R}$ with $V_{\alpha_0} = 0$ and*

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{V_\alpha - V_\beta}{h_{\alpha\beta}} = Q_\alpha, \quad x_\alpha \in G_h \setminus \{x_{\alpha_0}\}.$$

Proof It is sufficient to check that the linear problem: $V_{\alpha_0} = 0$ and

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{V_\alpha - V_\beta}{h_{\alpha\beta}} = 0, \quad x_\alpha \in G_h \setminus \{x_{\alpha_0}\}$$

only has the trivial solution. Let us enumerate the grid points in such a way that $0 = U_{\alpha_0} < U_{\alpha_1} \le U_{\alpha_2} \le \cdots \le U_{\alpha_M}$ where $M + 1 = |G_h|$. Assume that $V_{\alpha_0} = V_{\alpha_1} = \cdots = V_{\alpha_{r-1}} = 0$ already holds for some $r \in \{1, \ldots, M\}$. Then

$$0 = \sum_{x_\beta \in \mathcal{N}_{\alpha_r}} \left(\frac{U_{\alpha_r} - U_\beta}{h_{\alpha_r\beta}} \right)^+ \frac{V_{\alpha_r} - V_\beta}{h_{\alpha_r\beta}} = V_{\alpha_r} \sum_{x_\beta \in \mathcal{N}_{\alpha_r}} \left(\frac{U_{\alpha_r} - U_\beta}{h_{\alpha_r\beta}} \right)^+ \frac{1}{h_{\alpha_r\beta}}$$

since $V_\beta = 0$ if $U_\beta < U_{\alpha_r}$. Recalling (2.4) and the fact that $a(x_{\alpha_r}) > 0$ we infer that $V_{\alpha_r} = 0$. By induction we then deduce that $V \equiv 0$. ◻

Lemma 3.4 *Problem* (3.5)–(3.6) *has a unique solution* $P : G_h \setminus \{x_{\alpha_0}\} \to \mathbb{R}$.

Proof Since the problem is linear it is sufficient to check that the corresponding homogeneous problem only has the trivial solution. Hence suppose that $Q : G_h \setminus \{x_{\alpha_0}\} \to \mathbb{R}$ satisfies

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left\{ \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{Q_\alpha}{h_{\alpha\beta}} - \left(\frac{U_\beta - U_\alpha}{h_{\alpha\beta}} \right)^+ \frac{Q_\beta}{h_{\alpha\beta}} \right\} = 0, \quad x_\alpha \in G_h \setminus \{x_{\alpha_0}\}.$$

Let us denote by $V : G_h \to \mathbb{R}$ the function defined in Lemma 3.3. Rearranging the summation in the second term we obtain

$$\begin{aligned} 0 &= \sum_{x_\alpha \in G_h} V_\alpha \sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{Q_\alpha}{h_{\alpha\beta}} - \sum_{x_\alpha \in G_h} V_\alpha \sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\beta - U_\alpha}{h_{\alpha\beta}} \right)^+ \frac{Q_\beta}{h_{\alpha\beta}} \\ &= \sum_{x_\alpha \in G_h} V_\alpha \sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{Q_\alpha}{h_{\alpha\beta}} - \sum_{x_\alpha \in G_h} Q_\alpha \sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{V_\beta}{h_{\alpha\beta}} \\ &= \sum_{x_\alpha \in G_h} Q_\alpha \sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{V_\alpha - V_\beta}{h_{\alpha\beta}} \\ &= \sum_{x_\alpha \in G_h \setminus \{x_{\alpha_0}\}} Q_\alpha^2. \end{aligned}$$

Here we again used the fact that $\left(U_{\alpha_0} - U_\beta\right)^+ = 0$, $x_\beta \in G_h$. Hence $Q_\alpha = 0$ for all $x_\alpha \in G_h \setminus \{x_{\alpha_0}\}$ and the proof is complete. □

3.4 Fast solution of the discrete adjoint equation

For the efficient calculation of P the following observation is useful. Abbreviating for a point $x_\alpha \in G_h \setminus \{x_{\alpha_0}\}$

$$d_\alpha := \sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{1}{h_{\alpha\beta}} > 0$$

we can write (3.5), (3.6) as follows:

$$P_\alpha = \begin{cases} \frac{1}{d_\alpha} \sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\beta - U_\alpha}{h_{\alpha\beta}} \right)^+ \frac{P_\beta}{h_{\alpha\beta}}, & x_\alpha \in \Omega_h \setminus \{x_{\alpha_0}\}; \\ \frac{1}{d_\alpha} \left(\sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\beta - U_\alpha}{h_{\alpha\beta}} \right)^+ \frac{P_\beta}{h_{\alpha\beta}} + \frac{h_\alpha}{h^2} \left(u_{obs}(x_\alpha) - U_\alpha \right) \right), & x_\alpha \in \Gamma_h. \end{cases} \tag{3.7}$$

If P_β is known for any $x_\beta \in \mathcal{N}_\alpha$ with $U_\beta \geq U_\alpha$ then the right hand side of (3.7) is known. As a consequence, we can successively calculate the values of $P_\alpha, x_\alpha \in$

$G_h \setminus \{x_{\alpha_0}\}$ by ordering the grid points with respect to the size of $U_\alpha, x_\alpha \in G_h$. Note that such an ordering is available as a byproduct of the fast marching method.

3.5 Computation of the derivative

Let us begin by establishing the differentiability of the state with respect to the control variable.

Lemma 3.5 *Let $a = \sum_{i=1}^{L} a_i \phi_i \in K$ with corresponding solution $U = U_a$ of (2.3), (2.4). Then $\frac{\partial}{\partial a_m}[U_a(x_\alpha)]$ exists for $x_\alpha \in G_h$, $1 \le m \le L$ and $Z^{(m)} : G_h \to \mathbb{R}$ with $Z^{(m)}_\alpha = \frac{\partial}{\partial a_m}[U_a(x_\alpha)]$ satisfies $Z^{(m)}_{\alpha_0} = 0$ as well as*

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{Z^{(m)}_\alpha - Z^{(m)}_\beta}{h_{\alpha\beta}} = a(x_\alpha)\phi_m(x_\alpha), \quad x_\alpha \in G_h \setminus \{x_{\alpha_0}\}.$$

Proof Let us view (2.3), (2.4) as a system of the form $F(U,a) = 0$. In order to establish the differentiability of $a \mapsto U_a$ via the implicit function theorem we need to check that the problem: $V_{\alpha_0} = 0$ and

$$\sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{V_\alpha - V_\beta}{h_{\alpha\beta}} = 0, \quad x_\alpha \in G_h \setminus \{x_{\alpha_0}\}$$

only has the trivial solution $V \equiv 0$. This however is an immediate consequence of Lemma 3.3. The formula for $Z^{(m)}$ then follows by differentiating (2.4) with respect to a_m and recalling the structure of a. □

Let $a = \sum_{i=1}^{L} a_i \phi_i \in K$ with corresponding solution $U = U_a$ of (2.3), (2.4). Our aim is to derive a formula for $\dfrac{\partial \mathcal{J}_h}{\partial a_m}(a), m = 1, \ldots, L$, which will be used for the numerical computation within a descent method. Rather than use $Z^{(m)}$ which would involve solving an equation for each a_m we use the adjoint equation as is standard in PDE constrained optimization.

Theorem 3.6 *Let $a = \sum_{i=1}^{L} a_i \phi_i \in K$ and $m \in \{1, \ldots, L\}$. Then,*

$$\frac{\partial \mathcal{J}_h}{\partial a_m}(a) = -h^2 \sum_{x_\alpha \in G_h \setminus \{x_{\alpha_0}\}} P_\alpha \, a(x_\alpha)\phi_m(x_\alpha) + \delta_h \sum_{l=1}^{L} s_{ml} a_l,$$

where $P : G_h \setminus \{x_{\alpha_0}\} \to \mathbb{R}$ is the solution of (3.5), (3.6) and $s_{kl} = \int_\Omega \nabla\phi_k \cdot \nabla\phi_l dx, \quad k, l = 1, \ldots, L$.

Proof We deduce from the definition of $Z^{(m)}$, (3.5), (3.6) that

$$\frac{\partial \mathcal{J}_h}{\partial a_m}(a) = \sum_{x_\alpha \in \Gamma_h} h_\alpha \left(U_\alpha - u_{obs}(x_\alpha)\right) Z^{(m)}_\alpha + \delta_h \sum_{l=1}^{L} s_{ml} a_l$$

$$= -h^2 \sum_{x_\alpha \in \Gamma_h} Z_\alpha^{(m)} \sum_{x_\beta \in \mathcal{N}_\alpha} \left\{ \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{P_\alpha}{h_{\alpha\beta}} - \left(\frac{U_\beta - U_\alpha}{h_{\alpha\beta}} \right)^+ \frac{P_\beta}{h_{\alpha\beta}} \right\} + \delta_h \sum_{l=1}^{L} s_{ml} a_l$$

$$= -h^2 \sum_{x_\alpha \in G_h \setminus \{x_{\alpha_0}\}} Z_\alpha^{(m)} \sum_{x_\beta \in \mathcal{N}_\alpha} \left\{ \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{P_\alpha}{h_{\alpha\beta}} - \left(\frac{U_\beta - U_\alpha}{h_{\alpha\beta}} \right)^+ \frac{P_\beta}{h_{\alpha\beta}} \right\}$$
$$+ \delta_h \sum_{l=1}^{L} s_{ml} a_l.$$

Rearranging the summation and applying Lemma 3.5 then yields

$$\frac{\partial \mathcal{J}_h}{\partial a_m}(a) = -h^2 \sum_{x_\alpha \in G_h \setminus \{x_{\alpha_0}\}} P_\alpha \sum_{x_\beta \in \mathcal{N}_\alpha} \left(\frac{U_\alpha - U_\beta}{h_{\alpha\beta}} \right)^+ \frac{Z_\alpha^{(m)} - Z_\beta^{(m)}}{h_{\alpha\beta}} + \delta_h \sum_{l=1}^{L} s_{ml} a_l$$
$$= -h^2 \sum_{x_\alpha \in G_h \setminus \{x_{\alpha_0}\}} P_\alpha a(x_\alpha) \phi_m(x_\alpha) + \delta_h \sum_{l=1}^{L} s_{ml} a_l$$

and the proof is complete. □

4 Numerical results

Our numerical tests are carried out for an optimal control problem with multiple source points x_0^j, $j = 1, \ldots, S$ and corresponding observed data u_{obs}^j. We approximate the functional (1.6) by the discrete functional

$$\mathcal{J}_h(a) = \frac{1}{2} \sum_{j=1}^{S} \sum_{i=1}^{N} h_{\alpha_i} |U_a^j(x_{\alpha_i}) - u_{obs}^j(x_{\alpha_i})|^2 + \frac{\delta_h}{2} \int_\Omega |\nabla a|^2$$

for which the partial derivatives can be computed with the help of Theorem 3.6, so that

$$\frac{\partial \mathcal{J}_h}{\partial a_m}(a) = -h^2 \sum_{j=1}^{S} \sum_{x_\alpha \in G_h \setminus \{x_0^j\}} P_\alpha^j a(x_\alpha) \phi_m(x_\alpha) + \delta_h \sum_{l=1}^{L} s_{ml} a_l, \quad m = 1, \ldots, L.$$

Here, U_a^j and P^j are the solutions of (2.3), (2.4) and (3.5), (3.6) with $x_{\alpha_0} = x_0^j$, respectively.

4.1 Optimization algorithm

To solve the problem we use the projected gradient algorithm, which, for simplicity of notation, we present for the case of a single source point:

Step 1 Choose $\underline{a}^0 \in [A_m, A_M]^L$, $\gamma \in (0, 1)$ and tol.

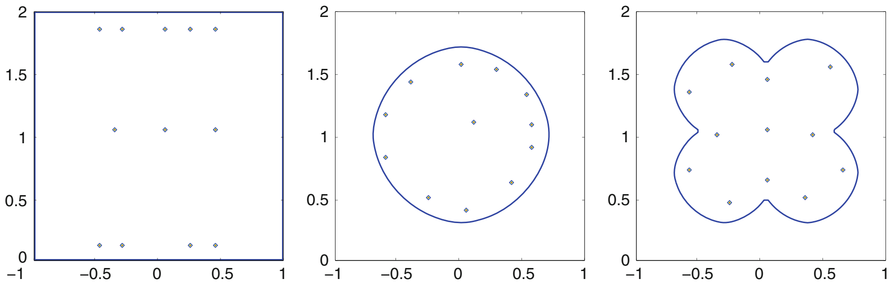

Fig. 1 The distribution of 12 source points in Ω_h^s (*left hand plot*), Ω_h^c (*centre plot*) and Ω_h^q (*right hand plot*)

Step 2 For $k = 0, 1, 2, \ldots$, do Steps 3–6.
Step 3 Set $s^k = -\nabla \mathcal{J}_h(\underline{a}^k) = -\left(\frac{\partial \mathcal{J}_h}{\partial a_1}(\underline{a}^k), \cdots, \frac{\partial \mathcal{J}_h}{\partial a_L}(\underline{a}^k)\right)$.
Step 4 Choose the maximum $\sigma_k \in \{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$ for which

$$\mathcal{J}_h(P_S(\underline{a}^k + \sigma^k s^k)) - \mathcal{J}_h(\underline{a}^k) \leq -\frac{\gamma}{\sigma_k} \| P_S(\underline{a}^k + \sigma^k s^k) - a^k \|_2^2.$$

Step 5 Set $\underline{a}^{k+1} = P_S(\underline{a}^k + \sigma_k s^k)$.
Step 6 If $\|\underline{a}^{k+1} - \underline{a}^k\|_2 <$ tol then STOP.

Here, $P_S(\underline{a})_i = \max(A_m, \min(a_i, A_M))$ and $\| \cdot \|_2$ denotes the euclidian norm in $\mathbb{R}^L$.
In the computations carried out below we found it adequate to take $\gamma = 0.01$.

4.2 Numerical experiments

For the numerical experiments we consider:

- Three domains; a square domain Ω^s, a circular domain $\Omega^c \subset \Omega^s$ and a quatrefoil domain $\Omega^q \subset \Omega^s$, see Fig. 1. For the curved domains Ω^c and Ω^q we used numerical integration to approximate $s_{kl}, k, l = 1, \ldots, L$.
- The set K is chosen in the following way. We choose the $\phi_i, i = 1, \ldots, L$ to be the basis functions associated with vertices of triangles belonging to a uniform right angled isosceles triangulation of Ω^s formed on a square grid of size $(J+1)\times(J+1)$, with triangles of diameter h_a. We set $A_m = 0.1$ and $A_M = 5$.

The observed data are generated as the exact arrival times on the boundary arising from given slowness functions. We use the three values of the slowness function $a = 1/c$ considered in [11]:

$$c(x) \equiv 1; \tag{4.8}$$
$$c(x) = 3 - 2.5\exp(-0.5x_1^2); \tag{4.9}$$
$$c(x) = 3 - 0.5\exp(-4(x_1^2 + (x_2 - 0.5)^2)) - \exp(-4(x_1^2 + (x_2 - 1.25)^2)). \tag{4.10}$$

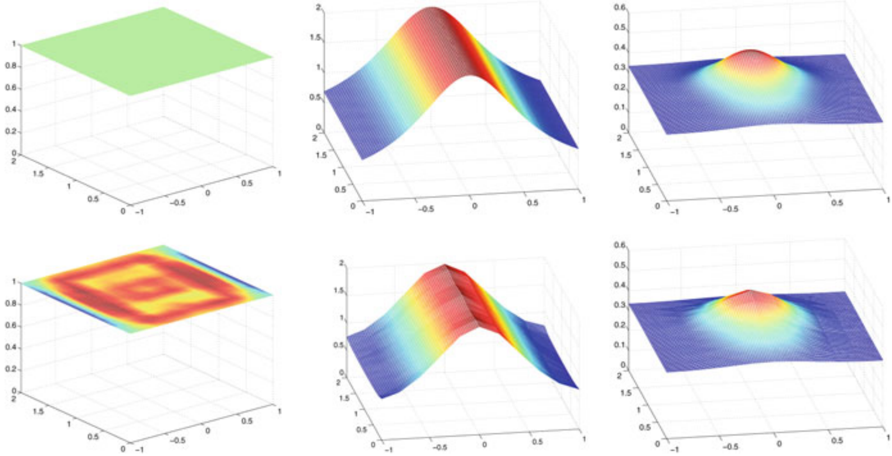

Fig. 2 $a(x)$ (*upper plots*), $a_h(x)$ with $L = 121$, $\delta_h = h$ and $S = 12$ (*lower plots*)

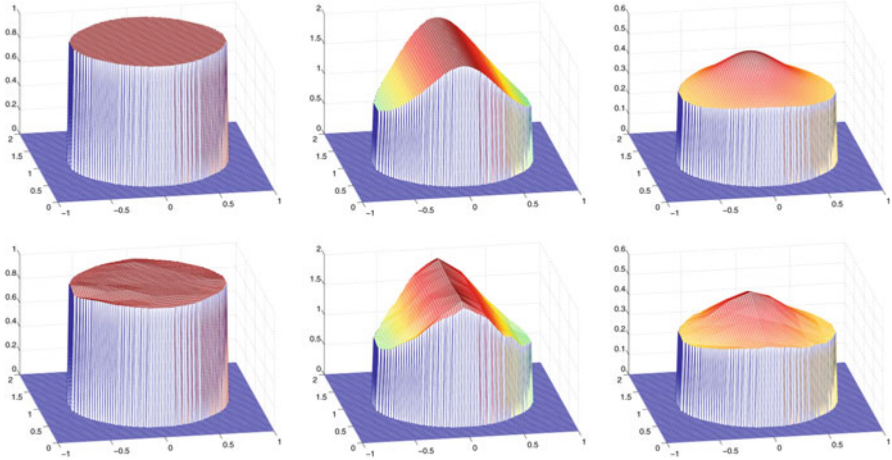

Fig. 3 $a(x)$ (*upper plots*), $a_h(x)$ with $L = 121$, $\delta_h = h$ and $S = 12$ (*lower plots*)

In Figs. 2–4 we show several examples of the recovered and exact slowness functions. They were obtained with $h = 0.02$. In Fig. 2 we take $\Omega = \Omega^s$, $L = 121$, $J = 10$, $\delta_h = h$ and $S = 12$ (see Fig. 1 for the distribution of the source points). The upper plots show $a(x)$ given by: (4.8) left hand plot, (4.9) centre plot, (4.10) right hand plot, and the three lower plots show the corresponding approximate solutions $a_h(x)$. Figures 3 and 4 take the same form as Fig. 2 but with $\Omega = \Omega^c$, $L = 65$ and $J = 10$ and $\Omega = \Omega^q$, $L = 75$ and $J = 10$, respectively.

In order to get some idea of how the parameters in the model affect the solution we include Tables 1–6. In these tables the values of $\mathcal{J}_h(a_h) = \min_K \mathcal{J}_h$ and $\|a - a_h\|_0$ are displayed. Unless otherwise specified the data in Tables 1–6 were obtained by setting $\Omega = \Omega^s$, $L = 121$, $\delta_h = h$ and $S = 12$.

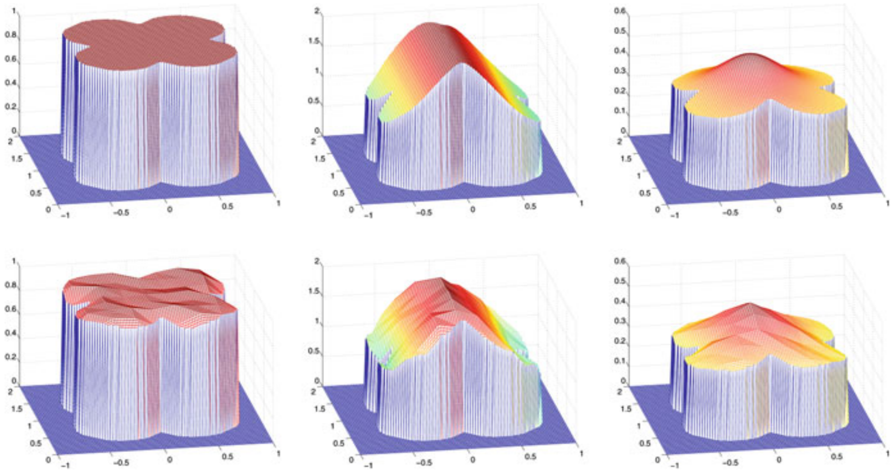

Fig. 4 $a(x)$ (*upper plots*), $a_h(x)$ with $L = 121$, $\delta_h = h$ and $S = 12$ (*lower plots*)

Table 1 $\mathcal{J}_h(a_h)$ ($\|a - a_h\|_0$) for Ω_h^s, $L = 121$, $\delta_h = h$

S	$a(x)$ given by (4.8)		$a(x)$ given by (4.9)		$a(x)$ given by (4.10)	
1	7.64×10^{-5}	(3.45×10^{-2})	3.90×10^{-2}	(5.46×10^{-1})	3.16×10^{-4}	(5.65×10^{-2})
5	4.26×10^{-6}	(2.17×10^{-3})	6.52×10^{-2}	(1.87×10^{-1})	8.26×10^{-4}	(1.94×10^{-2})
9	2.19×10^{-6}	(1.43×10^{-3})	6.79×10^{-2}	(1.61×10^{-1})	8.68×10^{-4}	(1.67×10^{-2})
12	2.22×10^{-6}	(1.30×10^{-3})	7.83×10^{-2}	(4.80×10^{-2})	1.00×10^{-3}	(5.61×10^{-3})

Table 2 $\mathcal{J}_h(a_h)$ ($\|a - a_h\|_0$) for Ω_h^c, $L = 121$, $\delta_h = h$

S	$a(x)$ given by (4.8)		$a(x)$ given by (4.9)		$a(x)$ given by (4.10)	
1	3.33×10^{-5}	(1.20×10^{-2})	1.80×10^{-2}	(2.25×10^{-1})	2.32×10^{-4}	(4.08×10^{-2})
5	1.12×10^{-4}	(7.16×10^{-3})	4.10×10^{-2}	(3.15×10^{-2})	8.63×10^{-4}	(5.16×10^{-3})
9	3.21×10^{-5}	(9.40×10^{-3})	4.59×10^{-2}	(2.66×10^{-2})	9.99×10^{-4}	(4.00×10^{-3})
12	3.40×10^{-5}	(8.40×10^{-3})	4.70×10^{-2}	(2.68×10^{-2})	1.01×10^{-3}	(4.15×10^{-3})

Table 3 $\mathcal{J}_h(a_h)$ ($\|a - a_h\|_0$) for Ω_h^q, $L = 121$, $\delta_h = h$

S	$a(x)$ given by (4.8)		$a(x)$ given by (4.9)		$a(x)$ given by (4.10)	
1	1.25×10^{-2}	(1.51×10^{-1})	4.23×10^{-2}	(2.91×10^{-1})	1.91×10^{-3}	(5.40×10^{-2})
5	4.54×10^{-3}	(3.26×10^{-2})	7.49×10^{-2}	(9.83×10^{-2})	2.13×10^{-3}	(1.57×10^{-2})
9	8.83×10^{-3}	(4.15×10^{-2})	8.34×10^{-2}	(8.16×10^{-2})	2.62×10^{-3}	(1.51×10^{-2})
12	8.23×10^{-3}	(3.55×10^{-2})	8.32×10^{-2}	(6.75×10^{-2})	2.72×10^{-3}	(1.44×10^{-2})

- In Tables 1–3 for each of the slowness functions defined above we consider four values for the number of source points S.
- In Table 4 we consider three values of L.
- In Table 5 we consider three values of δ_h.

Table 4 $\mathcal{J}_h(a_h)$ ($\|a - a_h\|_0$) for Ω_h^s, $\delta_h = h$ and $S = 12$

L	$a(x)$ given by (4.8)		$a(x)$ given by (4.9)		$a(x)$ given by (4.10)	
36	7.17×10^{-6}	(2.08×10^{-3})	8.84×10^{-2}	(8.73×10^{-2})	1.47×10^{-3}	(1.48×10^{-2})
121	2.22×10^{-6}	(1.30×10^{-3})	7.83×10^{-2}	(4.80×10^{-2})	1.00×10^{-3}	(5.61×10^{-3})
441	1.71×10^{-5}	(4.12×10^{-3})	7.64×10^{-2}	(4.33×10^{-2})	9.48×10^{-4}	(6.20×10^{-3})

Table 5 $\mathcal{J}_h(a_h)$ ($\|a - a_h\|_0$) for Ω_h^s, $L = 121$ and $S = 12$

δ	$a(x)$ given by (4.8)		$a(x)$ given by (4.9)		$a(x)$ given by (4.10)	
0	2.62×10^{-5}	(1.61×10^{-2})	1.07×10^{-4}	(2.84×10^{-2})	9.75×10^{-6}	(6.07×10^{-3})
h^2	2.92×10^{-5}	(1.45×10^{-2})	1.74×10^{-2}	(2.42×10^{-2})	3.14×10^{-5}	(5.91×10^{-3})
h	2.22×10^{-6}	(1.30×10^{-3})	7.83×10^{-2}	(4.80×10^{-2})	1.00×10^{-3}	(5.61×10^{-3})

Table 6 $\mathcal{J}_h(a_h)$ ($\|a - a_h\|_0$) for Ω_h^s, $\delta_h = h$, $S = 12$ with added noise, see (4.11)

Λ	$a(x)$ given by (4.8)		$a(x)$ given by (4.9)		$a(x)$ given by (4.10)	
0	2.22×10^{-6}	(1.30×10^{-3})	7.83×10^{-2}	(4.80×10^{-2})	1.00×10^{-3}	(5.61×10^{-3})
0.01	1.57×10^{-3}	(2.45×10^{-3})	7.99×10^{-2}	(4.79×10^{-2})	2.57×10^{-3}	(5.93×10^{-3})
0.05	3.92×10^{-2}	(1.11×10^{-2})	1.17×10^{-1}	(4.87×10^{-2})	4.02×10^{-2}	(1.24×10^{-2})
0.1	1.57×10^{-1}	(2.24×10^{-2})	2.35×10^{-1}	(5.18×10^{-2})	1.58×10^{-1}	(2.29×10^{-2})

Table 7 $a(x)$ given by (4.9), Ω_h^s, $L = 121$, $\delta_h = h$, $S = 12$

h	$\|a_{h_f} - a_h\|_0$	eoc	$\mathcal{J}_h(a_h)$	eoc
0.04	5.18×10^{-2}	–	1.45×10^{-1}	–
$0.0\dot{3}$	4.60×10^{-2}	0.699	1.21×10^{-2}	0.992
0.025	3.66×10^{-2}	0.764	9.20×10^{-2}	0.952
0.02	3.01×10^{-2}	0.876	7.41×10^{-2}	0.970
$0.01\dot{6}$	2.52×10^{-2}	0.975	6.21×10^{-2}	0.965

Table 8 $a(x)$ given by (4.10), Ω_h^s, $L = 121$, $\delta_h = h$, $S = 12$

h	$\|a_{h_f} - a_h\|_0$	eoc	$\mathcal{J}_h(a_h)$	eoc
0.04	5.69×10^{-3}	–	1.70×10^{-3}	–
$0.0\dot{3}$	5.04×10^{-3}	0.665	1.44×10^{-3}	0.910
0.025	4.29×10^{-3}	0.560	1.10×10^{-3}	0.936
0.02	3.48×10^{-3}	0.938	8.90×10^{-4}	0.949
$0.01\dot{6}$	2.91×10^{-3}	0.981	7.49×10^{-4}	0.946

- In Table 6 we add noise into the system; in particular for each of the desired speed functions, (4.8)–(4.10), we solve (2.4) to obtain $\hat{u}(x_\alpha)$ and then we set

$$u_{obs}(x_\alpha) = \hat{u}(x_\alpha) + \Lambda n(x_\alpha), \quad x_\alpha \in G_h \tag{4.11}$$

  where $n(x_\alpha) \in [-1, 1]$ is random noise and $\Lambda \in \mathbb{R}$.

We conclude our numerical results with Tables 7 and 8 which show how $\|a_{h_f} - a_h\|_0$ and $\mathcal{J}_h(a_h) = \min_K \mathcal{J}_h$ vary with h. Here we fix the convex set K by setting $L = 121$. The observed boundary data are fixed by generating them using the exact (actually computed on a fine grid) solution of the eikonal equation with the interpolations in K of the slowness functions (4.9) and (4.10). Here a_{h_f} is the approximate solution to the optimization problem computed on a fine grid with $h = 0.005$ and a_h is the approximate solution computed using $h = 0.05, 0.04, 0.0\dot{3}, 0.025, 0.02, 0.01\dot{6}$. From these tables we see that for the two desired speed functions, (4.9) and (4.10), the values of $\|a_{h_f} - a_h\|_0$ and $\mathcal{J}_h(a_h) = \min_K \mathcal{J}_h$ reduce linearly with h.

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