Please let me (Saul) know if any of the problems are unclear or have typos. Please turn in a solution to Exercise 9.3 by noon on Friday 2024-11-29 on Moodle. If you collaborate with other students (or AI), please include their names.

Exercise 9.1. Show that a CW complex X is path-connected if and only if it is connected.

Exercise 9.2. List all surjective homomorphisms from $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$, the free group of rank two, to $\mathbb{Z}_2$, the finite group with two elements. Prove your list is complete.

Exercise 9.3. Let $X = S^1 \times I$. Let $A = S^1 \times [0, 3/4)$ and $B = S^1 \times (1/4, 1]$. Note that $\{A, B\}$ is an open cover of X satisfying the hypotheses of the Seifert–van Kampen theorem. Let $\Gamma = \pi_1(A) * \pi_1(B)$. Let $\mathcal{U}$ be the subset of Γ arising in the statement of the Seifert–van Kampen theorem. Let $N \lhd \Gamma$ be the normal closure of $\mathcal{U}$.

Give a careful description of the elements of N. Use your description to solve the *membership problem* for $N \lhd \Gamma$. That is, produce an algorithm that, given a reduced word $f \in \Gamma$, decides if f lies in N.

Exercise 9.4. Let P^2 be the real projective plane. Compute the fundamental group of $P^2 \vee P^2$ directly from the Seifert–van Kampen theorem.

Exercise 9.5. For any non-zero integers p and q we define topological spaces $B_{p,q}$ and $T_{p,q}$ as follows.

$$B_{p,q} = (S^1 \times I) \sqcup S^1 \big/ (z,0) \sim z^p,\ (z,1) \sim z^q$$

$$T_{p,q} = S^1 \times I \big/ (z,0) \sim (e^{2\pi i/p} z, 0),\ (z,1) \sim (e^{2\pi i/q} z, 1)$$

For $p = q = 1$, check that $B_{1,1} \cong T^2$ and $T_{1,1} \cong S^1 \times I$. For general p and q, find CW complex structures on $B_{p,q}$ and $T_{p,q}$. Give presentations of their fundamental groups. Provide illustrative figures. (For the completist: Suppose p or q is zero. What is $B_{p,q}$? What is the correct definition of $T_{p,q}$?)