Please let me (Saul) know if any of the problems are unclear or have typos. Please turn in a solution to Exercise 7.4 by 12:00noon on 2024-11-15 to the Moodle page. If you collaborate with other students or AI, please include their names.

For the next three problems we need the following definition. Suppose X is a topological space. We define CX to be the *cone* on X: that is,

$$CX = X \times I \Big/ (x,1) \sim (y,1) \text{ for all } x, y \in X.$$

The point $a = [(x,1)]$ is called the *apex* of the cone.

Exercise 7.1. Equip the integers $\mathbb{Z}$ with the discrete topology. Show that $C\mathbb{Z}$ is homeomorphic to the wedge sum of a countable collection of unit intervals.

Exercise 7.2. Let $I_n \subset \mathbb{R}^2$ to be the line segment connecting $(0,1)$ to $(n,0)$, for $n \in \mathbb{Z}$. Set $D = \cup_{n\in\mathbb{Z}} I_n$ and equip D with the subspace topology. Show that $C\mathbb{Z}$ is not homeomorphic to D.

Exercise 7.3. [Hard.] For any space X, show that CX is contractible. Deduce that $\pi_1(CX, a)$ is trivial.

Exercise 7.4. Suppose G and H are nontrivial groups. Show that the free product $G * H$ is not isomorphic to $\mathbb{Z}^2$.

Exercise 7.5. Suppose that $\{G_\alpha\}$ is a countable collection of countable groups. Show that $*_\alpha G_\alpha$ is countable.

For the next two problems we need the following definition. Let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ centered at $(1/n, 0) \in \mathbb{R}^2$. We define the *earring space* to be the union $E = \cup_{n=1}^{\infty} C_n$, equipped with the subspace topology (in $\mathbb{R}^2$). We take E to be a pointed space, with basepoint at $e = (0,0)$. Let $\Gamma = \pi_1(E, e)$.

Exercise 7.6.

- For all $n > 0$ give a retraction $r_n \colon E \to C_n$. Explain why r_n is continuous.
- Show that $\Gamma = \pi_1(E, e)$ is uncountable. Briefly explain why Γ is not isomorphic to

$$\pi_1\left(\bigvee_{n\in\mathbb{N}} S^1\right) \cong \mathop{*}_{n\in\mathbb{N}} \mathbb{Z}.$$

Exercise 7.7.

- Show that $E \cong E \vee E$. (Recall that we use $e = (0,0)$ as the basepoint.)
- [Medium.] Using the above or otherwise, give an injective homomorphism of $\Gamma * \Gamma$ into Γ. Determine if your homomorphism is surjective or not.