2024-12-05 LECTURE 30 MA3F1 SAUL SCHLEIMER

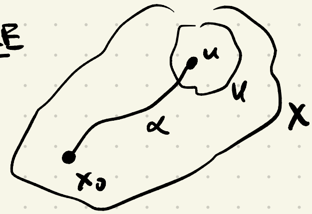
① TOPOLOGY ON $\tilde{X}$.

RECALL $\text{PATHS}(X,x_0) = \{ \alpha: I \to X \mid \alpha \text{ CONTINUOUS}, \alpha(0) = x_0 \}$

AND $\tilde{X} = \{ [\alpha] \mid \alpha \in \text{PATHS}(X, x_0) \}$

AND $\hat{x}_0 = [e]$.

SUPPOSE $U \subset X$ IS CONTRACTABLE. SUPPOSE $u \in U$ AND $\alpha: I \to X$ HAS $\alpha(0) = x_0$, $\alpha(1) = u$. PICTURE



DEFINE

$$U_\alpha = \{ [\alpha * \beta] \mid \beta \in \text{PATHS}(U, u) \}$$

CLAIM: THE COLLECTION $\{U_\alpha\}_{U,\alpha}$ IS A BASIS FOR A TOPOLOGY ON $\tilde{X}$.

PROOF: SEE HATCHER. □

② $p: (\tilde{X}, \hat{x}_0) \to (X, x_0)$ IS A COVERING MAP.

CLAIM: p CONTINUOUS.

CLAIM: $p^{-1}(U) = \bigsqcup_{[\gamma] \in \pi_1(X, x_0)} U_{\gamma * \alpha}$

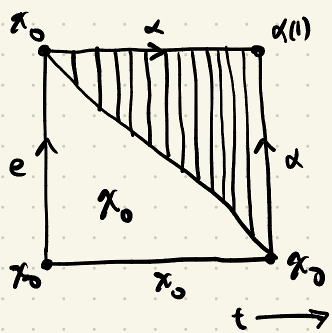
CLAIM: $p|U_\alpha : U_\alpha \to U$ IS A HOMEOMORPHISM.

$$[\gamma] \mapsto \gamma(1)$$

③ $\tilde{X}$ IS PATH-CONNECTED.

FIX $[\alpha] \in \tilde{X}$. DEFINE $F: I \times I \to X$ BY

$$F(s,t) = \begin{cases} x_0, & \text{if } s+t \le 1 \\ \alpha(s+t-1), & \text{if } 1 \le s+t \end{cases}$$



SO $f_0 = e$, $f_1 = \alpha$.

CLAIM: $\tilde{F}: I \to \tilde{X}$, $\tilde{F}(t) = [f_t]$ IS A PATH FROM $[e] = \hat{x}_0$ TO $[\alpha]$.

④ $\tilde{X}$ IS SIMPLY-CONNECTED:

LET $\tilde{e} \in \text{LOOPS}(\tilde{X}, \hat{x}_0)$ BE DEFINED BY $\tilde{e}(t) = \hat{x}_0$

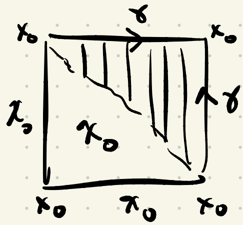
SO $\hat{e}$ IS THE CONSTANT PATH IN $\tilde{X}$. NOTE $p \circ \tilde{e} = e$.

FIX ANY $\hat{\gamma} \in \text{LOOPS}(\tilde{X}, \hat{x}_0)$.

CLAIM: $\hat{\gamma} \simeq_\partial \tilde{e}$

PROOF: DEFINE $\gamma = p \circ \tilde{\gamma}$. SO $\gamma \in \text{LOOPS}(X, x_0)$, AND $\tilde{\gamma} : I \to \tilde{X}$ IS A LIFT of γ. DEFINE $F : I \times I \to X$ AS ABOVE:

$$F(s,t) = \begin{cases} x_0, & s+t \leq 1 \\ \gamma(s+t-1), & 1 \leq s+t \end{cases}$$



SO $\hat{F} : I \to \tilde{X}$, $\hat{F}(t) = [f_t]$ IS A PATH IN $\tilde{X}$.

ALSO $\hat{F}(0) = [e]$, $p \circ \hat{F} = \gamma$ [CHECK!].

SO $\hat{F}$ IS ALSO A LIFT of γ. THUS BY UNIQUENESS of PATH LIFTING $\tilde{\gamma} = \hat{F}$. BUT $\tilde{\gamma}(1) = [e]$ AND $\hat{F}(1) = [\gamma]$ SO $[\gamma] = [e]$ AND SO $\gamma \simeq_\partial e$. THUS $\tilde{\gamma} \simeq_\partial \hat{e}$ BY HOMOTOPY LIFTING. □

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⑤ DECK GROUP ACTION:

WE DEFINE $p : \pi_1(X, x_0) \times \tilde{X} \longrightarrow \tilde{X}$ BY

$$([\gamma], [\alpha]) \longmapsto [\gamma * \alpha]$$

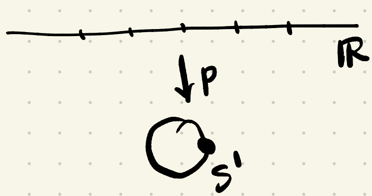
A LEFT ACTION

THIS GIVES A GROUP ACTION of $\pi_1(X, x_0)$ ON $\tilde{X}$.

ALSO: $p \circ ([\gamma] \cdot [\alpha]) = p([\gamma * \alpha]) = (\gamma * \alpha)(1) = \alpha(1) = p([\alpha])$.

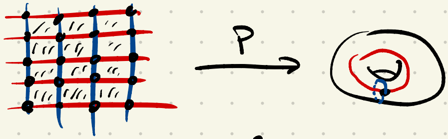
PATH LIFTING PROVES $\pi_1(X, x_0) \cong \text{DECK}(p : \tilde{X} \to X)$.

PICTURE



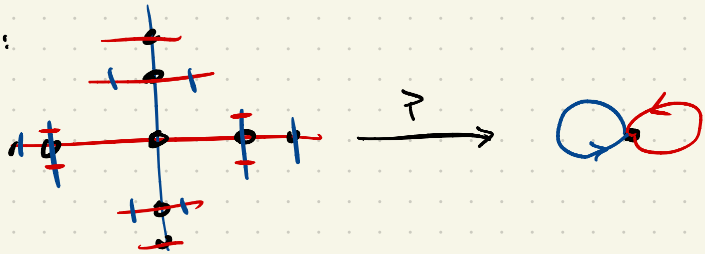
$\text{DECK}(p) \cong \mathbb{Z} \cong \pi_1(S^1)$

PICTURE:



$\mathrm{DECK}(P) \cong \mathbb{Z}^2 \cong \pi_1(T^2)$.

PICTURE:



$\mathrm{DECK}(P) \cong F_2 \cong \pi_1(R_2)$.

THEOREM: SUPPOSE (X, x_0) PATH CONN, CW-COMPLEX.
THEN $\mathrm{DECK}(p : \tilde{X} \to X) \cong \pi_1(X, x_0)$.

(6) GALOIS CORRESPONDENCE

SUPPOSE $H < \pi_1(X, x_0)$ IS A SUBGROUP.

DEFINE $X^H = {}_H\backslash^{\tilde{X}} = \tilde{X} / {[\alpha] \sim [\gamma * \alpha]}$ FOR $[\gamma] \in H$.

LET $x_0^H = [\tilde{x}_0]$ BE THE IMAGE OF $\tilde{x}_0 \in \tilde{X}$ IN X^H.

CLAIM: (1) $(X^H, x_0^H) \xrightarrow{p^H} (X, x_0)$ IS A POINTED COVER.

(2) $p^H_*(\pi_1(X^H, x_0^H)) = H$ [NOT JUST ISOMORPHIC!]

(3) IF $H \trianglelefteq \pi_1(X, x_0)$ THEN $\mathrm{DECK}(X^H \to X) \cong \pi_1(X, x_0)/H$

$p_*(\pi_1(Y)) \longleftarrow \left(Y \xrightarrow{p} X \right)$

SUBGPs of $\pi_1(X, x_0)$ $\longleftrightarrow$ POINTED COVERS of (X, x_0) *

$H \longrightarrow \left(X^H \xrightarrow{p} X \right)$

* UP TO ISOMORPHISM